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A REGULARITY CONDITION IN SOBOLEV SPACES $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ WITH $1 \le p < n$

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ABSTRACT. Extending Malý's geometric definition of absolutely continuous functions of n variables (in a sense equivalent to that of Rado-Reichelderfer), we define classes of p-absolutely continuous functions ($1 \le p < n$) and show that this weaker notion of absolute continuity still implies differentiability almost everywhere, although it does not imply continuity or Lusin's condition (N).

1. Introduction

We investigate to what extent some basic properties that are shared by absolutely continuous functions in the sense of Rado, Reichelderfer, and Malý can be generalized to larger classes of functions. We prove that a natural extension of Malý's geometric definition of absolute continuity gives a simple regularity property (in the sense of differentiability almost everywhere) of functions belonging to the Sobolev space $W^{1,p}(\Omega)$ with $1 \le p < n$. We show that our classes AC^p, $1 \le p < n$, of p-absolutely continuous functions properly contain the classes AC^n of absolutely continuous functions of Rado, Reichelderfer and Malý. In fact, they contain even essentially discontinuous functions. However, the AC^p functions do not share all properties of the AC^n functions; in particular, Lusin's condition (N) may fail even for continuous AC^p functions when $1 \le p < n$.

Let Ω be an open subset of $\mathbb{R}^n$, and let $1 \le p < \infty$. We recall that the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$ is defined as the set of all (equivalence classes of) functions $f \in L^p(\Omega, \mathbb{R}^m)$ whose distributional partial derivatives all belong to $L^p(\Omega, \mathbb{R}^m)$.

L. Cesari [2] proved that if $p > n$, then each $f \in W^{1,p}(\Omega, \mathbb{R}^m)$ is continuous, and differentiable at almost all points $x \in \Omega$. In the same paper Cesari also gave an example of a function $f \in W^{1,n}(\Omega, \mathbb{R}^m)$ which does not have these properties.

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Regularity properties for functions belonging to $W^{1,n}(\Omega,\mathbb{R}^m)$ have been obtained by A.P. Calderon [1], T. Rado and P.V. Reichelderfer [9], E. Stein [10], and J. Malý [6].

Definition 1.1 (Malý). A function $f\colon \Omega \to \mathbb{R}^m$ is said to be *n-absolutely continuous* in Ω (briefly, $f \in AC^n(\Omega,\mathbb{R}^m)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i \omega^n(f, B_i) < \varepsilon,$$

for each disjoint system of balls $\{B_i \subset \Omega : i \in \mathbb{N}\}$ with $\sum_i \mathcal{L}^n(B_i) < \delta$. Here $\omega(f, B_i)$ denotes the oscillation of f in B_i.

Our extension of this notion is given by the following definition:

Definition 1.2. Let $1 \leq p \leq n$. We say that a function $f\colon \Omega \to \mathbb{R}^m$ is *p-absolutely continuous* in Ω (briefly, $f \in AC^p(\Omega,\mathbb{R}^m)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i \omega^p(f, B_i) r^{n-p}(B_i) < \varepsilon,$$

for each disjoint system of balls $\{B_i \subset \Omega : i \in \mathbb{N}\}$ with $\sum_i \mathcal{L}^n(B_i) < \delta$. Here $r(B_i)$ denotes the radius of the ball B_i.

Note that for $p = n$ Definition 1.2 coincides with Malý's definition.

We say that f has bounded p-variation (briefly, $f \in BV^p(\Omega,\mathbb{R}^m)$) if there exist $M > 0$ and $\eta > 0$ such that

$$\sum_i \omega^p(f, B_i) r^{n-p}(B_i) < M$$

for each disjoint system of balls $\{B_i\}$ in Ω such that $r(B_i) < \eta$.

The classes BV^p_{loc} and AC^p_{loc} are defined in the usual way: $f \in BV^p_{\text{loc}}(\Omega,\mathbb{R}^m)$ (resp. $f \in AC^p_{\text{loc}}(\Omega,\mathbb{R}^m)$) if $f \in BV^p(\Omega_0,\mathbb{R}^m)$ (resp. $f \in AC^p(\Omega_0,\mathbb{R}^m)$) for every open set Ω_0 whose closure is a compact subset of Ω.

As in Malý [6] it is easy to see that the p-absolute continuity follows from a condition analogous to one introduced by Rado and Reichelderfer:

$(R.R.)^p$ There is an absolutely continuous finite measure μ on $\mathbb{R}^n$ such that $\omega^p(f, B)\, r^{n-p}(B) \leq \mu(B)$ for each ball B in Ω

A similar remark, without the requirement that μ be absolutely continuous, holds for functions of bounded p-variation. In fact, by a deep result of Csörnyei [3] the conditions $(R.R.)^p$ and AC^p are locally equivalent, and an analogous statement holds for functions of bounded p-variation.

In this note we prove the following properties of p-absolutely continuous functions and of functions of bounded p-variation.

Proposition 1.3. *If $f \in AC^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, then $f \in BV^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Moreover, every function $f \in BV^p(\Omega, \mathbb{R}^m)$ is locally bounded.*

Theorem 1.4. *$BV^q(\Omega, \mathbb{R}^m) \subset AC^p(\Omega, \mathbb{R}^m)$ for $1 \le p < q \le n$.*

Theorem 1.5. *Let $1 \le p \le n$ and let $f \in BV^p(\Omega, \mathbb{R}^m)$. Then f is a.e. differentiable in Ω.*

Theorem 1.6. *For $1 < p \le n$ we have*

$$BV^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \subset W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^m);$$

for $p = 1$ we have

$$BV^1(\Omega, \mathbb{R}^m) \subset BV(\Omega, \mathbb{R}^m).$$

Theorem 1.7. *For $1 \le p < q \le n$ there is a continuous function $f \in AC^p(B(0,1), \mathbb{R})$ such that $f \notin W^{1,q}(\Omega, \mathbb{R})$.*

In the following theorem we denote by $\mathcal{H}^{n-p}$ the $(n-p)$-dimensional Hausdorff measure; for $\eta > 0$ we also denote by $\mathcal{H}^{n-p}_\eta$ the corresponding η-approximating measure (see [7, Chapter 4]).

Theorem 1.8. *Let $1 \le p \le n$. Then each function $f \in AC^p(\Omega, \mathbb{R}^m)$ is continuous $\mathcal{H}^{n-p}$ almost everywhere. Moreover, for each function $f \in BV^p(\Omega, \mathbb{R}^m)$ the set of the points of discontinuity has σ-finite $\mathcal{H}^{n-p}$ measure.*

Note that if $p = n$, Theorem 1.8 says that f is continuous in Ω. This is, of course, easy to observe directly. In fact, by Definition 1.1 it follows that for any Ω_0 such that $\overline{\Omega}_0 \subset \Omega$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that $\|f(x) - f(y)\|^n < \varepsilon$ for all $x, y \in \Omega_0$ with $\|x - y\| < \delta$.

Note also that general functions from $W^{1,p}(\Omega, \mathbb{R}^n)$ satisfy only a weaker condition than that of Theorem 1.8: they are only approximately continuous except on a set of Hausdorff dimension $n - p$ (see [11, Remark 3.3.5] and [11, Theorem 3.3.3] for a stronger result), but they may be unbounded in every non-empty open subset of Ω (see [11, Exercise 3.3]).

Theorem 1.9. *For $1 \le p < n$ there is a function f belonging to $AC^p(B(0,1), \mathbb{R})$ such that for any continuous function g we have*

$$\mathcal{L}^n(\{x \in B(0,1) : g(x) \ne f(x)\}) > 0.$$

Theorem 1.10. *There is a homeomorphism $f\colon [0,1]^n \to [0,1]^n$ not satisfying Lusin's condition (N) which belongs to $AC^p(\Omega, \mathbb{R}^n)$, for $1 \le p < n$.*

2. Proof of Proposition 1.3

It suffices to consider the case when Ω is bounded and $f \in AC^p(\Omega, \mathbb{R}^m)$. Then there is $\delta > 0$ such that

$$\sum_i \omega^p(f, B_i) r^{n-p}(B_i) < 1, \tag{2.1}$$

for each disjoint system of balls $\{B_i \subset \Omega : i \in \mathbb{N}\}$ with $\sum_i \mathcal{L}^n(B_i) < \delta$.

Let

$$\eta = \frac{1}{3}\left(\frac{\delta}{\mathcal{L}^n(B(0,1))}\right)^{1/n},$$

and let $\{B_i\}$ be an arbitrary disjoint system of balls in Ω such that $r(B_i) < \eta$ for each i. Then there exist finitely many balls, say $Q_1, \ldots, Q_k$, of radius η, such that $\Omega \subset \bigcup_{j=1}^k Q_i$. For $j = 1, \ldots, k$, let $\tilde{Q}_j$ be the ball with the same center as Q_j and of radius $3\,\eta$. Thus, if a ball B of radius less than η intersects Q_j for some j, then $B \subset \tilde{Q}_j$. Hence

$$\sum_{i\,:\,B\cap Q_j \neq \emptyset} \mathcal{L}^n(B_i) < \mathcal{L}^n(\tilde{Q}_j) = \delta.$$

Thus, by (2.1), we have

$$\sum_i \omega^p(f, B_i)\, r^{n-p}(B_i) = \sum_{j=1}^k \sum_{i\,:\,B\cap Q_j \neq \emptyset} \omega^p(f, B_i)\, r^{n-p}(B_i) < k.$$

This completes the proof that $f \in BV^p(\Omega, \mathbb{R}^m)$.

Let now $f \in BV^p(\Omega, \mathbb{R}^m)$. By Definition 1.1 it follows that there are numbers $M > 0$ and $\delta > 0$ such that $\omega^p(f, B)\, r^{n-p}(B) < M$ whenever $B \subset \Omega$ and $r(B) < \delta$. If f is not locally bounded, then there is $x_0 \in \Omega$ such that f is unbounded on every open set containing x_0. Hence

$$\omega^p(f, B(x_0, \delta/2))\, r^{n-p}(B(x_0, \delta/2)) = \infty,$$

which it is a contradiction. □

3. Proof of Theorem 1.4

Let $f \in BV^q(\Omega, \mathbb{R}^m)$. By definition there are numbers $M > 0$ and $\eta > 0$ such that

$$\sum_i \omega^q(f, B_i) r^{n-q}(B_i) < M,$$

for each disjoint system $\{B_i\}$ of balls in Ω such that $r(B_i) < \eta$.

Given $\varepsilon > 0$, let

$$\delta = \min\left(\frac{\varepsilon^{q/(q-p)}\, \mathcal{L}^n(B(0,1))}{M^{p/(q-p)}}\, ,\, \eta^n\, \mathcal{L}^n(B(0,1))\right)$$

and let $\{B_i\}$ be a disjoint system of balls in Ω such that $\sum_i \mathcal{L}^n(B_i) < \delta$. Then, by Hölder's inequality, we have

$$\begin{aligned}
&\sum_i \omega^p(f, B_i) r^{n-p}(B_i) \\
&\quad = \sum_i \omega^p(f, B_i) r^{(n-q)p/q}(B_i) \cdot r^{n(q-p)/q}(B_i) \\
&\quad \le \left(\sum_i \omega^q(f, B_i) r^{n-q}(B_i) \right)^{p/q} \cdot \left(\sum_i r^n(B_i) \right)^{(q-p)/p} \\
&\quad \le M^{p/q} \left(\sum_i r^n(B_i) \right)^{(q-p)/q} \\
&\quad = \frac{M^{p/q}}{(\mathcal{L}^n(B(0,1)))^{(q-p)/q}} \left(\sum_i \mathcal{L}^n(B_i) \right)^{(q-p)/q} < \varepsilon.
\end{aligned}$$

This completes the proof that $f \in AC^p(\Omega, \mathbb{R}^m)$.

4. Proof of Theorem 1.5

We need the following lemma.

Lemma 4.1. *For each $\eta > 0$ and each $k \in \mathbb{N}$ there exists a disjoint system $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of balls in Ω, such that $r_i < \eta$ for $i = 1, 2, \ldots,$ and*

$$\mathcal{L}^n_e(\Omega_{(k)}) \le C\, k^{-p} \sum_i \omega^p(f, B(x_i, r_i))\; r_i^{n-p}, \tag{4.1}$$

where $C = \mathcal{L}^n(B(0,1))$ and

$$\Omega_{(k)} = \{x \in \Omega : \text{for each } \sigma > 0 \text{ there exists } y \in \mathbb{R}^n \text{ with } \|y - x\| < \sigma \text{ and } \|f(y) - f(x)\| > k\|y - x\|\,\}.$$

Proof. The family $\{B(x, r)\}$ of all balls such that $B(x, r) \subset \Omega$, $x \in \Omega_{(k)}$, $r < \eta$ and $\omega(B(x, r)) > k\, r$ is a Vitali covering of $\Omega_{(k)}$. Thus there exist $x_i \in \Omega_{(k)}$ and $r_i > 0$, $i = 1, 2, \ldots,$ with $r_i < \eta$, such that $B(x_i, r_i) \subset \Omega$, $\omega(B(x_i, r_i)) > k\, r_i$, the balls $\{B(x_i, r_i)\}$ are disjoint, and

$$\mathcal{L}^n\left(\Omega_{(k)} \setminus \bigcup_i B(x_i, r_i) \right) = 0.$$

Therefore

$$\begin{aligned}\mathcal{L}^n_e(\Omega_{(k)}) &\leq \mathcal{L}^n_e\left(\Omega_{(k)} \setminus \bigcup_i B(x_i,r_i)\right) + \mathcal{L}^n_e\left(\bigcup_i B(x_i,r_i)\right)\\ &= \mathcal{L}^n\left(\bigcup_i B(x_i,r_i)\right) = \sum_i \mathcal{L}^n\left(B(x_i,r_i)\right)\\ &= C\sum_i r_i^n \leq C\,k^{-p}\sum_i \omega^p(B(x_i,r_i))\;r_i^{n-p}. \qquad \square\end{aligned}$$

We now return to the proof of the Theorem 1.5. By Stepanoff's theorem, it is enough to prove that f is pointwise Lipschitz a.e. in Ω. Let $M, \eta > 0$ be such that

$$\sum_i \omega^p(f, B_i)\, r^{n-p}(B_i) < M$$

for each disjoint system $\{B_i\}$ of balls in Ω such that $r(B_i) < \eta$. Now it is easy to see that the set S of points at which f is not pointwise Lipschitz is contained in each set $\Omega_{(k)}$. Thus Lemma 4.1 implies

$$\mathcal{L}^n(S) \leq \mathcal{L}^n_e(\Omega_{(k)}) < C\,k^{-p}\sum_i \omega^p(f, B(x_i,r_i))\;r_i^{n-p} < C\,k^{-p}\,M,$$

and letting $k \to \infty$ we conclude that $\mathcal{L}^n(S) = 0$. $\square$

5. Proof of Theorem 1.6

Suppose that $f \in BV^p_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. As in [6] we may assume that f is supported on a compact subset of Ω and hence, by Proposition 1.3 and Theorem 1.5, that it is a bounded measurable function with compact support in $\mathbb{R}^n$. Let $M, \eta > 0$ be such that

$$\sum_i \omega^p(f, B_i) r^{n-p}(B_i) < M,$$

for each disjoint system $\{B_i\}$ of balls in $\mathbb{R}^n$ with $r(B_i) < \eta$.

Fix a function ψ_1 in $\mathcal{C}_\infty(\mathbb{R}^n, \mathbb{R})$ such that ψ_1 has its support in $B(0, 1/6)$, $\psi_1 \geq 0$, and $\int_{\mathbb{R}^n} \psi_1 = 1$, and set $\psi_k(x) = k^n\,\psi_1(kx)$, $k = 1, 2, \dots$. Define $\mathbf{Z}_k$ as the set of all points $x \in \mathbb{R}^n$ such that kx has integer coordinates. Then the family $\{B(x+y, 1/(3k)) : x \in \mathbf{Z}_k\}$ is a system of disjoint balls in $\mathbb{R}^n$, for each $y \in B(0, 2n/k)$. Thus, for each $k > (2\,n)/\eta$ we have

$$\sum_{x\in\mathbf{Z}_k} \omega^q(f, B(x+y, 1/(3k))\, r^{n-q}(B(x+y, 1/(3k))) < M. \tag{5.1}$$

Let $\psi_k * f$ denote the convolution of ψ_k and f and set $C = \sup \|\nabla \psi_1\|$. Noting that $\int_{B(0,1/6k)} \nabla\psi_k(t)\,dt = 0$, we infer

$$\begin{aligned}
&\int_{B(x+y,1/6k)} \|\nabla(\psi_k * f)\|^p(s)\,ds \\
&\quad = \int_{B(x+y,1/6k)} \left\| \int_{B(0,1/6k)} \nabla\psi_k(t) f(s-t)\,dt \right\|^p ds \\
&\quad = \int_{B(x+y,1/6k)} \left\| \int_{B(0,1/6k)} \nabla\psi_k(t)(f(s-t)-f(s))\,dt \right\|^p ds \\
&\quad = \int_{B(x+y,1/6k)} \left\| \int_{B(0,1/6k)} k^{n+1}\nabla\psi_1(kt)(f(s-t)-f(s)) \right\|^p ds \\
&\quad \le C^p \int_{B(x+y,1/6k)} \left(k^{n+1} \int_{B(0,1/6k)} \|(f(s-t)-f(s))\|\,dt \right)^p ds \\
&\le C^p \omega^p(f, B(x+y,1/3k)) \int_{B(x+y,1/6k)} \left(\frac{k^{n+1}}{6^n\,k^n}\,\mathcal{L}^n(B(0,1)) \right)^p ds \\
&\le C^p \omega^p(f, B(x+y,1/3k)) \frac{1}{6^n k^n}\,\frac{k^p}{6^{np}}\,(\mathcal{L}^n(B(0,1)))^{p+1} \\
&< C_1 \cdot \omega^p(f, B(x+y,1/3k)) \left(\frac{1}{3k}\right)^{n-p} \\
&= C_1 \cdot \omega^p(f, B(x+y,1/3k))\, r^{n-p}(B(x+y,1/3k)),
\end{aligned}$$

where

$$C_1 = \left(\frac{C\,\mathcal{L}^n(B(0,1))}{6^n}\right)^p \frac{\mathcal{L}^n(B(0,1))}{3^p\,2^n}.$$

Hence, by (5.1), we have

$$\begin{aligned}
\int_\Omega \|\nabla(\psi_k * f)\|^p(s)\,ds &\le \sum_{x\in \mathbf{Z_k}} \int_{B(x,n/k)} \|\nabla(\psi_k * f)\|^p(s)\,ds \\
&\le C_2 k^n \sum_{x\in \mathbf{Z_k}} \int_{B(x,2n/k)} \left(\int_{B(x+y,1/6k)} \|\nabla(\psi_k * f)\|^p(s)\,ds \right) dy \\
&\le C_2 k^n \int_{B(x,2n/k)} \sum_{x\in \mathbf{Z_k}} \left(\int_{B(x+y,1/6k)} \|\nabla(\psi_k * f)\|^p(s)\,ds \right) dy \\
&\le C_2 k^n \int_{B(x,2n/k)} C_1\,M\,dy \\
&\le C_1\,C_2 \cdot (2n)^n\,\mathcal{L}^n(B(0,1))\,M.
\end{aligned}$$

This implies that the sequence $\{\psi_k * f\}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^m)$. Also, from standard properties of the convolution it follows that $\psi_k * f$ converges to f in L^p. If $p > 1$, then $W^{1,p}(\Omega, \mathbb{R}^m)$ is reflexive, and hence a subsequence of $\psi_k * f$ converges weakly to a function $g \in W^{1,p}(\Omega, \mathbb{R}^m)$; since it converges to f in L^p, we have $f = g$ and so $f \in W^{1,p}(\Omega, \mathbb{R}^m)$. If $p = 1$, the same argument works if we use instead of the reflexivity the compactness of the unit ball of BV functions in the L^1 norm (see [11, Corollary 5.5.4]). $\square$

6. Proof of Theorems 1.7 and 1.9

Let $0 < c < 1$ and let a_k be a sequence such that

$$1 \geq a_1 \geq a_2 \geq \cdots > 0.$$

Moreover, given $1 \leq p < q \leq n$, let $n_k \geq 1$ be such that

$$a_k n_k \leq a_{k+1} n_{k+1}, \quad k = 1, 2, \ldots, \tag{6.1}$$

and

$$\sum_{k=1}^{\infty} (a_{k+1}\, n_{k+1})^p\, c^{k(n-p)} < \infty, \tag{6.2}$$

$$\sum_{k=1}^{\infty} (a_{k+1}\, n_{k+1})^q\, c^{k(n-q)} = \infty. \tag{6.3}$$

The numbers c and a_k and the integers n_k will be specified later.

For each $k \in \mathbb{N}$ let

$$\alpha_{0,k} = c^{k+1} < \alpha_{1,k} < \cdots < \alpha_{n_k,k} = c^k$$

be a division of $[c^{k+1}, c^k]$ into n_k intervals, each of length $(c^k - c^{k+1})/n_k$, and for $i = 1, \ldots, n_k$ let $d_{i,k}$ be the midpoint of the interval $[\alpha_{i-1,k}, \alpha_{i,k}]$. Define $\psi\colon [0,1] \longrightarrow \mathbb{R}^+$ to be linear on the intervals $[\alpha_{i-1,k}, d_{i,k}]$ and $[d_{i,k}, \alpha_{i,k}]$, and such that $\psi(\alpha_{i,k}) = 0$ and $\psi(d_{i,k}) = a_k$, for $i = 1, \ldots, n_k$ and $k = 1, 2, \ldots$. We will show that the function $f(x) = \psi(\|x\|)$ is $AC^p(B(0,1), \mathbb{R})$ and that f is not $W^{1,q}(B(0,1), \mathbb{R})$.

To this end define, for a Lebesgue measurable set $E \subset \mathbb{R}^n$,

$$\mu(E) = \left(\frac{4}{c(1-c)}\right)^n \frac{1}{\mathcal{L}^n(B(0,1))} \sum_k \left(\frac{a_{k+1}\, n_{k+1}}{c^k}\right)^p \mathcal{L}^n(E \cap B(0, c^k)).$$

By (6.2) we have $\mu(B(0,1)) < +\infty$, so μ is a finite measure that is absolutely continuous with respect to the Lebesgue measure. Therefore, to prove that $f \in AC^p(B(0,1), \mathbb{R})$ it is enough to verify that, for each ball $B \subset B(0,1)$,

$$\omega^p(f, B)\, r^{n-p}(B) \leq \mu(B).$$

Let k be maximal such that $B \subset B(0, c^k)$. We consider two cases, $B \cap B(0, c^{k+2}) = \emptyset$ and $B \cap B(0, c^{k+2}) \neq \emptyset$. In the first case, by (6.1) and since $0 < c < 1$, it follows

$$\omega(f, B) \leq \frac{4}{c(1-c)} \frac{a_{k+1}\, n_{k+1}}{c^k}\, r(B).$$

Thus

$$\begin{aligned}
\omega^p(f, B) r^{n-p}(B) &\leq \left(\frac{4}{c(1-c)}\right)^p \left(\frac{a_{k+1}\, n_{k+1}}{c^k}\right)^p r^n(B) \\
&= \left(\frac{4}{c(1-c)}\right)^p \left(\frac{a_{k+1}\, n_{k+1}}{c^k}\right)^p \frac{\mathcal{L}^n(B \cap B(0, c^k))}{\mathcal{L}^n(B(0,1))} \\
&\leq \mu(B).
\end{aligned}$$

In the second case, since $B \cap B(0, c^{k+2}) \neq \emptyset$, we have

$$r(B) \geq \frac{c^k\, c\,(1-c)}{2}.$$

Moreover, since $B \subset B(0, c^k)$, it follows that $\omega(f; B) \leq a_k$. Therefore

$$\begin{aligned}
\omega^p(f, B)\, r^{n-p}(B) &\leq a_k^p\, r^{n-p}(B) \\
&\leq \left(\frac{a_k}{c^k}\right)^p \left(\frac{2}{c(1-c)}\right)^p \frac{\mathcal{L}^n(B \cap B(0, c^k))}{\mathcal{L}^n(B(0,1))} \\
&\leq \left(\frac{4}{c(1-c)}\right)^n \frac{1}{\mathcal{L}^n(B(0,1))} \left(\frac{a_k\, n_k}{c^k}\right)^p \mathcal{L}^n(B \cap B(0, c^k)) \\
&\leq \mu(B).
\end{aligned}$$

This completes the proof that $f \in AC^p(B(0,1), \mathbb{R})$.

Now, since

$$\|\nabla f\| = \frac{2\, a_k\, n_k}{c^k - c^{k+1}}$$

almost everywhere in $B(0, c^k) \setminus B(0, c^{k+1})$, we have

$$\begin{aligned}
\int_{B(0,1)} \|\nabla f\|^q &= \left(\frac{2}{1-c}\right)^q c^n\,(1-c^n)\,\mathcal{L}^n(B(0,1)) \sum_k (a_{k+1}\, n_{k+1})^q\, c^{k(n-q)} \\
&= \infty.
\end{aligned}$$

Hence f is not in $W^{1,q}(B(0,1), \mathbb{R})$.

It remains to specify the numbers c, a_k, and n_k. To prove Theorem 1.7, we take $a_k = (1/k)^{1/q}$, $c = 2^{-q}$, and $n_k = 2^{k(n-q)}$. Then the conditions (6.1), (6.2), and (6.3) are satisfied, and f is continuous. Thus the proof of Theorem 1.7 is complete.

To prove Theorem 1.9, we take $a_k = 1$, $c = 2^{-q}$, and $n_k = 2^{k(n-q)}$. Then conditions (6.1), (6.2), and (6.3) are satisfied, and f is discontinuous. To complete the proof of Theorem 1.9, it suffices to note that a function that is almost everywhere equal to f cannot be continuous at the origin. □

7. Proof of Theorem 1.8

Let $f \in AC^p(\Omega, \mathbb{R}^m)$, and set

$$D = \{x \in \Omega : f \text{ is not continuous at } x\}.$$

We have to show that $\mathcal{H}^{n-p}(D) = 0$. To this end, for each $k \in N$ let

$$D_k = \{x \in D : \omega(f, x) > 1/k\}.$$

We will show that $\mathcal{L}^n(D_k) = 0$. Taking $\varepsilon = 1$ in Definition 1.2, let $\delta = \delta(1)$ and let $U \subset \Omega$ be an open set with $\mathcal{L}^n(U) < \delta$ and $D_k \cap U \neq \emptyset$. Then, given $\sigma > 0$ and a disjoint system of balls $\{B_i\}$ such that $r(B_i) < \sigma$, $B_i \subset U$, $i = 1, 2, \ldots$, and $D_k \cap U \subset \bigcup_i B_i$, we have

$$\begin{aligned} \mathcal{L}^n(D_k \cap U) &\leq \sum_i r^n(B_i) < \sigma^p \sum_i r^{n-p}(B_i) \\ &< \sigma^p k^p \sum_i \omega^p(f, B_i) r^{n-p}(B_i) < \sigma^p k^p. \end{aligned}$$

Letting $\sigma \to 0$ we obtain $\mathcal{L}^n(D_k \cap U) = 0$, and since U is arbitrary, it follows that $\mathcal{L}^n(D_k) = 0$.

Let $\varepsilon > 0$, and let $\delta = \delta(\varepsilon)$ be chosen according to Definition 1.2. Since $\mathcal{L}^n(D_k) = 0$ we can find an open set $G \subset \Omega$ such that $D_k \subset G$ and $\mathcal{L}^n(G) < \delta$. Now, for $\eta > 0$ let $\{B_i = B(x_i, r_i)\}_{i \in I}$ be a system of balls such that $B_i \subset G$ and $r(B_i) < \eta/10$ for each $i \in I$, and $D_k \subset \bigcup_i B_i$. By the "5r-covering Theorem" (see [7, Theorem 2.1]), there is a subset $J \subset I$ such that the balls $\{B_j\}_{j \in J}$ are disjoint, and for each $i \in I$ there is $j \in J$ with $B_i \subset B(x_j, 5r_j)$. Hence $D_k \subset \bigcup_{j \in J} B(x_j, 5r_j)$, and $\sum_{j \in J} \mathcal{L}^n(B_j) \leq \mathcal{L}^n(G) < \delta$. Therefore

$$\begin{aligned} \mathcal{H}^{n-p}_\eta(D_k) &\leq \sum_{j \in J} (2r(B(x_j, 5r_j)))^{n-p} = 10^{n-p} \sum_{j \in J} r^{n-p}(B_j) \\ &< 10^{n-p} k^p \sum_{j \in J} \omega^p(f, B_j) r^{n-p}(B_j) \\ &< 10^{n-p} k^p \varepsilon. \end{aligned}$$

Since η is arbitrary, it follows that $\mathcal{H}^{n-p}(D_k) \leq 10^{n-p} k^p \varepsilon$, and since ε is arbitrary, we obtain $\mathcal{H}^{n-p}(D_k) = 0$. Thus we have $\mathcal{H}^{n-p}(D) = \lim_k \mathcal{H}^{n-p}(D_k) = 0$.

Assume now that $f \in BV^p(\Omega, \mathbb{R}^m)$ and define D and D_k as above. Since $D = \bigcup_k D_k$, it is enough to prove that $\mathcal{H}^{n-p}(D_k) < +\infty$ for each $k \in \mathbb{N}$. Assume that there is $k \in \mathbb{N}$ with $\mathcal{H}^{n-p}(D_k) = \infty$. Then for each positive M there is $\tau > 0$ such that $\mathcal{H}^{n-p}_\tau(D_k) > k^p \, 2^{n-p} \, M$. Let $0 < \eta < \tau$ and let

$\{B_i\}$ be a disjoint system of balls such that $B_i \subset \Omega$ and $r(B_i) < \eta/2$ for $i = 1, 2, \ldots$, and $D_k \subset \bigcup_i B_i$. Then

$$\sum_i (2\, r(B_i))^{n-p} \geq \mathcal{H}^{n-p}_\eta(D_k) > k^p\, 2^{n-p}\, M,$$

and hence

$$\sum_i \omega^p(f, B_i)\, r^{n-p}(B_i) > \frac{1}{k^p} \sum_i r^{n-p}(B_i) > M,$$

in contradiction to the hypothesis $f \in BV^p(\Omega, \mathbb{R}^m)$. $\square$

8. Proof of Theorem 1.10

We use an improvement of Ponomarev's example [8] due to J. Kauhanen, P. Koskela, and J. Malý [5]. Let Q be a closed cube with side s, and let $\{c_k\}$ be a sequence of positive numbers such that $c_k < 1$ for each k. Divide Q into 2^n nonoverlapping cubes P_i, $i = 1, \ldots, 2^n$, such that $\mathcal{L}^n(P_i) = \mathcal{L}^n(Q)/2^n$ for each i. Inside each P_i take a closed cube Q_i with side $s\,(c_1/2)$, such that P_i and Q_i are concentric. Then apply the above algorithm to Q_i, for each i. We thus obtain 4^n new closed cubes $Q_{i,j}$, $i, j = 1, \ldots, 2^n$, with sides $s\,(c_1\, c_2/2^2)$. Continuing this process, we obtain a system of cubes $\{Q_\alpha\}$ with $\alpha \in \Sigma_k = \{1, 2, \ldots, 2^n\}^k$, $k = 0, 1, \ldots$ (where we let $\Sigma_0 = \{\emptyset\}$, $Q_\emptyset = Q$), such that the side of Q_α is $s(c_1\, c_2 \ldots c_k)/2^k$. Let $E = \bigcap_{k=1}^\infty \bigcup_{\alpha \in \Sigma_k} Q_\alpha$. Then E is a nonempty closed set.

We first apply this construction with $Q = [0,1]^n$ and the constant sequence $\{c_k = b\}$. We obtain a system of cubes $\{Q_\alpha\}$ with $\alpha \in \Sigma_k$, $k = 0, 1, \ldots$, such that the side of Q_α is $(b/2)^k$ for each k and the set $E = \bigcap_{k=1}^\infty \bigcup_{\alpha \in \Sigma_k} Q_\alpha$ is a null set. In fact, we have

$$\mathcal{L}^n(E) = \lim_{k \to \infty} \mathcal{L}^n \left(\bigcup_{\alpha \in \Sigma_k} Q_\alpha \right) = \lim_{k \to \infty} b^{2\,k} = 0.$$

Now let $\{d_k\}$ be a sequence of positive numbers such that $d_k < 1$ for each k, and $\prod_{k=1}^\infty d_k > 0$. We apply the same construction with $Q = [0,1]^n$ and the sequence $\{c_k = d_k\}$. We then obtain a system of cubes $\{\tilde{Q}_\alpha\}$ with $\alpha \in \Sigma_k$, $k = 0, 1, \ldots$, such that the side of $\tilde{Q}_\alpha$ is $(d_1\, d_2 \ldots d_k)/2^k$ for each k. Let $\tilde{E} = \bigcap_{k=1}^\infty \bigcup_{\alpha \in \Sigma_k} \tilde{Q}_\alpha$. Then

$$\mathcal{L}^n(\tilde{E}) = \lim_{h \to \infty} \mathcal{L}^n \left(\bigcup_{|\alpha| = h} \tilde{Q}_\alpha \right) = \prod_{h=1}^\infty d_h > 0.$$

For $k = 1, 2, \dots$ and $\alpha \in \Sigma_k$ let x_α be the center of Q_α and let

$$s_k = \frac{d_1 d_2 \cdots d_{k-1}(1-d_k)}{b^{k-1}(1-b)}, \quad t_k = \frac{d_1 d_2 \cdots d_{k-1}}{2^{k+1}} \left(1 - \frac{1-d_k}{1-b}\right),$$
$$v_k = \frac{d_1 d_2 \cdots d_k}{b^k}.$$

It is easy to see that

$$(8.1) \qquad \begin{aligned} s_k \frac{1}{2}\left(\frac{b}{2}\right)^k + t_k &= \frac{1}{2}\left(\frac{d_1 d_2 \cdots d_k}{2^k}\right), \\ s_k \frac{1}{4}\left(\frac{b}{2}\right)^{k-1} + t_k &= \frac{1}{4}\left(\frac{d_1 d_2 \cdots d_{k-1}}{2^{k-1}}\right), \\ v_k \frac{1}{2}\left(\frac{b}{2}\right)^k &= \frac{1}{2}\left(\frac{d_1 d_2 \cdots d_k}{2^k}\right). \end{aligned}$$

Denote by f_0 the identity function on $[0,1]^n$ and, for $k = 1, 2, \dots$, define

$$f_k(x) = \begin{cases} f_{k-1}(x), & \text{if } x \notin \bigcup_{\alpha \in \Sigma_k} P_\alpha, \\ f_{k-1}(x_\alpha) + s_k(x - x_\alpha) + t_k \frac{x - x_\alpha}{\|x - x_\alpha\|}, & \text{if } x \in P_\alpha \setminus Q_\alpha,\ \alpha \in \Sigma_k, \\ f_{k-1}(x_\alpha) + v_k(x - x_\alpha), & \text{if } x \in Q_\alpha,\ \alpha \in \Sigma_k. \end{cases}$$

By (8.1) it follows that f_k is continuous and maps P_α onto $\tilde{P}_\alpha$, Q_α onto $\tilde{Q}_\alpha$, the boundary of P_α onto the boundary of $\tilde{P}_\alpha$, and the boundary of Q_α onto the boundary of $\tilde{Q}_\alpha$.

If $k > h$ then $f_k(x) = f_h(x)$ for $x \notin \bigcup_{\alpha \in \Sigma_k} P_\alpha$, and

$$|f_k(x) - f_h(x)| \leq \frac{(d_1\, d_2 \dots d_{k-1})\sqrt{2}}{2^k} < \frac{1}{2^{k-1}},$$

for $x \in \bigcup_{\alpha \in \Sigma_k} P_\alpha$. Therefore the sequence $\{f_k\}$ is uniformly convergent to a continuous function f. It is easily seen that f is one-to-one, $f([0,1]^n) = [0,1]^n$, and $f(E) = \tilde{E}$. Moreover, by the compactness of $[0,1]^n$, f maps closed sets into closed sets. Thus f is an homeomorphism.

To complete the proof, we show that $f \in AC^p((0,1)^n, \mathbb{R}^n)$, for $1 \leq p < n$. To this end define, for a measurable set $E \in \mathbb{R}^n$,

$$\mu(E) = \left(\frac{8}{1-b}\right)^p \frac{1}{\mathcal{L}^n(B(0,1))} \sum_{k=0}^{\infty} \frac{1}{b^{(k+2)p}} \mathcal{L}^n\left(E \cap \bigcup_{\alpha \in \Sigma_k} Q_\alpha\right).$$

It is easy to see that, since $b < 1$,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{b^{(k+2)p}} \mathcal{L}^n\left(\bigcup_{\alpha \in \Sigma_k} Q_\alpha\right) &= \sum_k \frac{1}{b^{(k+2)p}} 2^{kn} \left(\frac{b}{2}\right)^{kn} \\ &= \frac{1}{b^{2p}} \sum_k b^{k(n-p)} < \infty. \end{aligned}$$

Hence μ is a finite measure that is absolutely continuous with respect to the Lebesgue measure. Thus, in order to prove that $f \in AC^p((0,1)^n, \mathbb{R}^n)$ it is enough to verify that, for each ball $B \subset (0,1)^n$,

$$\omega^p(f,B)\, r^{n-p}(B) \le \mu(B).$$

If $x, y \in P_\alpha \setminus Q_\alpha$ with $\alpha \in \Sigma_k$, $k = 1, 2, \ldots$, then we have

$$\|x - x_\alpha\| \ge \frac{1}{2}\left(\frac{b}{2}\right)^k, \quad \|y - x_\alpha\| \ge \frac{1}{2}\left(\frac{b}{2}\right)^k.$$

Since

$$\left\| \frac{x - x_\alpha}{\|x - x_\alpha\|} \right\| \le \left\| \frac{2(x - x_\alpha)}{(b/2)^k} \right\|, \quad \left\| \frac{y - x_\alpha}{\|y - x_\alpha\|} \right\| \le \left\| \frac{2(y - x_\alpha)}{(b/2)^k} \right\|,$$

it follows that

$$\begin{aligned} &\left\| \frac{x - x_\alpha}{\|x - x_\alpha\|} - \frac{y - x_\alpha}{\|y - x_\alpha\|} \right\| \\ &\qquad \le \left\| \frac{2(x - x_\alpha)}{(b/2)^k} - \frac{2(y - x_\alpha)}{(b/2)^k} \right\| = 2\left(\frac{2}{b}\right)^k \|x - y\|. \end{aligned}$$

Therefore, since $f = f_k$ on $P_\alpha \setminus Q_\alpha$, we have

$$\begin{aligned} \|f(x) - f(y)\| &\le s_k\, \|x - y\| + 2\, t_k \left(\frac{2}{b}\right)^k \|x - y\| \\ &\le \frac{d_1 d_2 \cdots d_k}{b^k}\, \|x - y\| \\ &< \frac{1}{b^k}\, \|x - y\| \end{aligned} \tag{8.2}$$

for all $x, y \in P_\alpha \setminus Q_\alpha$.

Suppose now that B is a ball contained in $(0,1)^n$, and let k be maximal such that $B \subset Q_\alpha$ for some $\alpha \in \Sigma_k$. We consider the two cases, $B \cap \bigcup_{\beta \in \Sigma_{k+2}} Q_\beta = \emptyset$ and $B \cap \bigcup_{\beta \in \Sigma_{k+2}} Q_\beta \ne \emptyset$. In the first case, by (8.2) applied to $x, y \in B$ we have

$$\omega(f,B) \le \frac{1}{b^{k+2}}\; 2\, r(B).$$

Therefore

$$\begin{aligned} \omega^p(f,B) r^{n-p}(B) &\le \left(\frac{2}{b^{k+2}}\right)^p r^n(B) \\ &= \left(\frac{2}{b^{k+2}}\right)^p \frac{\mathcal{L}^n(B \cap \bigcup_{\alpha \in \Sigma_k} Q_\alpha)}{\mathcal{L}^n(B(0,1))} \\ &\le \mu(B). \end{aligned}$$

In the second case, since $B \cap \bigcup_{\beta \in \Sigma_{k+2}} Q_\beta \neq \emptyset$, and since B is not completely contained in Q_γ with $\gamma \in \Sigma_{k+1}$, it follows that

$$r(B) \geq \frac{1-b}{4} \left(\frac{b}{2}\right)^{k+1}.$$

Since $B \subset Q_\alpha$ with $\alpha \in \Sigma_k$, we have

$$\omega(f, B) \leq \frac{d_1\, d_2\, \cdot\, d_k}{2^k} \leq \frac{1}{2^k}.$$

Therefore

$$\begin{aligned}\omega^p(f,B)\, r^{n-p}(B) &\leq \frac{1}{2^{kp}}\, r^{n-p}(B) \\ &< \left(\frac{8}{(1-b)}\right)^p \frac{1}{b^{(k+1)p}} \frac{\mathcal{L}^n(B \cap \bigcup_{\alpha \in \Sigma_k} Q_\alpha)}{\mathcal{L}^n(B(0,1))} \\ &\leq \mu(B).\end{aligned}$$

This completes the proof that $f \in AC^p((0,1)^n, \mathbb{R}^n)$.

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