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LOCAL COMPACTNESS FOR FAMILIES OF $\mathcal{A}$-HARMONIC FUNCTIONS

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Abstract. We show that if a family of $\mathcal{A}$-harmonic functions that admits a common growth condition is closed in L^p_{loc}, then this family is locally compact on a dense open set under a family of topologies, all generated by norms. This implies that when this family of functions is a vector space, then such a vector space of $\mathcal{A}$-harmonic functions is finite dimensional if and only if it is closed in L^p_{loc}. We then apply our theorem to the family of all p-harmonic functions on the plane with polynomial growth at most d to show that this family is essentially small.

1. Introduction

A classical theorem states that the vector space of harmonic functions in $\mathbb{R}^n$ that have polynomial growth of order at most d is finite dimensional with dimension depending on d and n. Recently, Colding and Minicozzi showed in [CMI98] that the same theorem holds on a Riemannian manifold that admits a $(1, 2)$-Poincaré inequality (with a bound on the dimension depending only on d and the quantitative data of the manifold).

Often one views $\mathcal{A}$-harmonic functions (in the sense of [HKM93]) as a natural generalization of harmonic functions. However, $\mathcal{A}$-harmonicity is not in general a linear condition. We will call a family of $\mathcal{A}$-harmonic functions small if there exists a topology generated by a norm for which this family is locally compact. Note that for a vector space local compactness is equivalent to having finite dimension. We pursue a slightly weaker condition than local compactness, though in the context of a vector space it, too, is equivalent to having finite dimension.

Definition 1.1. We will say that a family S of functions on $\mathbb{R}^n$ has a *common growth condition* if there exists a non-decreasing function $g : [0, \infty) \to (0, \infty)$ so that for each $f \in S$ there exists $C_f > 0$ such that $|f(\mathbf{x})| \leq C_f g(|\mathbf{x}|)$ for all $\mathbf{x} \in \mathbb{R}^n$. In particular, we will say S is of *polynomial growth of order d*

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if $g(t) = 1 + t^d$. We will say a function f has *polynomial growth of order* d if the set $\{f\}$ is of polynomial growth of order d.

The main theorems of this paper are the following. Throughout μ is a p-admissible measure on $\mathbb{R}^n$ and $\mathcal{A}$ is p-acceptable under μ (see Section 2 for the definitions). We write $f \in L^q_{\text{loc}}(\mu)$ for a function $f : \mathbb{R}^n \to \mathbb{R}$ if for each $r > 0$, $\int_{\mathbf{B}(\mathbf{0},r)} |f|^q d\mu < \infty$. We say a net $\{f_m\}$ in $L^q_{\text{loc}}(\mu)$ converges to f in $L^q_{\text{loc}}(\mu)$ if and only if f_m converges to f in $L^q(\mathbf{B}(\mathbf{0}, r), \mu)$ for each $r > 0$. We will write $L^q_{\text{loc}}(\mathbb{R}^n)$ for $L^q_{\text{loc}}(\lambda)$, where λ is the Lebesgue n-measure on $\mathbb{R}^n$.

Theorem 1.2. *Let S be a family of $\mathcal{A}$-harmonic functions on $\mathbb{R}^n$ with a common growth condition that is closed in $L^q_{\text{loc}}(\mu)$ for some $1 \le q \le \infty$. Then there exists a family of Banach spaces, each a subset of $L^q_{\text{loc}}(\mu)$, for which S is a closed subset, such that under each of these Banach spaces there exists a relatively dense open set of S which is locally compact. Moreover, the topology generated by each of these Banach spaces is stronger than the topology generated by $L^q_{\text{loc}}(\mu)$.*

Corollary 1.3. *Let S be a vector space of $\mathcal{A}$-harmonic functions that admits a growth condition. Then S is finite dimensional if and only if S is closed in $L^q_{\text{loc}}(\mu)$ for some $1 \le q \le \infty$.*

Theorem 1.4. *Let $d > 0$, let $1 < p < \infty$ and let S be the closure in $L^p_{\text{loc}}(\mathbb{R}^2)$ of all p-harmonic functions defined on the plane with polynomial growth of order at most d. Then there exists a family of Banach spaces, each a subset of $L^p_{\text{loc}}(\mathbb{R}^2)$, for which S is a closed subset, such that under each of these Banach spaces there exists a relatively dense open set of S which is locally compact. Moreover, the topology generated by each of these Banach spaces is stronger than the topology generated by $L^p_{\text{loc}}(\mathbb{R}^2)$.*

2. Definitions

Throughout let $1 \le p < \infty$ and let μ be a measure on $\mathbb{R}^n$ that satisfies the following conditions.

(1) $d\mu(x) = \omega(x)dx$, where ω is a locally integrable a.e. positive function on $\mathbb{R}^n$.
(2) μ is a doubling measure, i.e., there exists a constant C_μ such that for every $x \in \mathbb{R}^n$ and $0 < r < \infty$ we have $\mu(\mathbf{B}(x, 2r)) \le C_\mu \mu(\mathbf{B}(x, r))$.
(3) There exists a constant C_{I} such that

$$\fint_{\mathbf{B}(x,r)} |\psi - \psi_{\mathbf{B}(x,r)}| d\mu \le C_{\text{I}}\, r \left(\fint_{\mathbf{B}(x,r)} |\nabla \psi|^p d\mu \right)^{1/p}$$

for every ball $\mathbf{B}(x, r)$ and each $\psi \in C^\infty(\mathbf{B}(x, r)) \cap L^1(\mathbf{B}(x, r), \mu)$.

Here and throughout, we set $⨍_A f d\mu := \frac{1}{\mu(A)} \int_A f d\mu$. Condition (3) is often called a $(1,p)$-Poincaré inequality. The above hypotheses imply the following three conditions.

(A) If Ω is an open set in $\mathbb{R}^n$ and $\{\psi_j\} \subset C^\infty(\Omega)$ is such that $\int_\Omega |\psi_j|^p d\mu \to 0$ and $\int_\Omega |\nabla\psi_j - \mathbf{v}|^p d\mu \to 0$ as $j \to \infty$ with $\mathbf{v}$ a Borel measurable vector field in $L^p(\Omega,\mu)$, then $\mathbf{v} = 0$ almost everywhere.

(B) There exist constants $\chi = \chi(C_{\mathrm{I}}, C_\mu) > 1$ and $C_{\mathrm{II}} = C_{\mathrm{II}}(C_{\mathrm{I}}, C_\mu)$ such that

$$\left(⨍_{\mathbf{B}(x,r)} |\psi|^{\chi p} d\mu \right)^{1/\chi p} \le C_{\mathrm{II}}\, r \left(⨍_{\mathbf{B}(x,r)} |\nabla\psi|^p d\mu \right)^{1/p}$$

for every ball $\mathbf{B}(x,r)$ and every $\psi \in C_c^\infty(\mathbf{B}(x,r))$.

(C) There exists a constant $C_{\mathrm{III}} = C_{\mathrm{III}}(C_{\mathrm{I}}, C_\mu)$ such that

$$\int_{\mathbf{B}(x,r)} |\psi - \psi_{\mathbf{B}(x,r)}|^p d\mu \le C_{\mathrm{III}}\, r^p \int_{\mathbf{B}(x,r)} |\nabla\psi|^p d\mu$$

for every ball $\mathbf{B}(x,r)$ and every $\psi \in C^\infty(\mathbf{B}(x,r)) \cap L^1(\mathbf{B}(x,r),\mu)$.

Hajłasz, Koskela and Franchi showed (A) in Theorem 10 of [FHK99] under much more general conditions. Heinonen and Koskela showed (C) in Lemma 5.15 of [HK98] (see also Theorem 4.18 of [Hei01]) in the context of geodesic metric spaces. Hajłasz and Koskela showed (B) in Theorem 5.1 of [HK00], also in a much more general setting than a manifold. We follow [HKM93] and call a measure μ satisfying (1)–(3) above p-admissible. Hölder's inequality shows that if μ is p-admissible, then for all $q > p$, μ is q-admissible.

If μ is a p-admissible measure, for each open set Ω of $\mathbb{R}^n$ we can form the Sobolev space $H^{1,p}(\Omega,\mu)$; see Chapter 1 of [HKM93] for its properties. We also define $H^{1,p}_{\mathrm{loc}}(\mu)$ as the set of measurable functions f defined on all of $\mathbb{R}^n$ such that for each bounded open set Ω the function f is an element of $H^{1,p}(\Omega,\mu)$.

We say a function $f \in H^{1,p}_{\mathrm{loc}}(\mu)$ is $\mathcal{A}$-harmonic if it weakly satisfies the equation

$$\mathrm{div}(\mathcal{A}(x, \nabla f)) = 0,$$

i.e., if for each $\phi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \langle \mathcal{A}(x,\nabla f), \nabla\phi \rangle \, dx = 0 \ ,$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions.

(1) $\mathcal{A}$ is Borel.
(2) For a.e. $x \in \mathbb{R}^n$, the mapping $\mathbf{v} \to \mathcal{A}(x,\mathbf{v})$ is continuous.
(3) There exists $C_{\mathrm{i}} > 0$ such that $|\mathcal{A}(x,\mathbf{v})| \le C_{\mathrm{i}}\, |\mathbf{v}|^{p-1}\, \omega(x)$.

(4) If $\lambda \neq 0$, then for a.e. $x \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ we have

$$\mathcal{A}(x, \lambda \mathbf{v}) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \mathbf{v}) \ .$$

(5) There exists $C_{\mathrm{ii}} > 0$ such that for a.e. x and for all $\mathbf{v}$ in $\mathbb{R}^n$ we have

$$\langle \mathcal{A}(x, \mathbf{v}), \mathbf{v} \rangle \geq C_{\mathrm{ii}} |\mathbf{v}|^p \omega(x) \ .$$

(6) For a.e. x in $\mathbb{R}^n$ and every $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{w}$ we have

$$\langle \mathcal{A}(x, \mathbf{v}) - \mathcal{A}(x, \mathbf{w}), \mathbf{v} - \mathbf{w} \rangle > 0 \ .$$

We will write $C_{\mathcal{A}}$ to represent the constants C_{i} and C_{ii}. We will call $\mathcal{A}$ p-acceptable with constants C_{i} and C_{ii} under μ whenever it satisfies conditions (1)–(6) above. For each $1 < p < \infty$, $\mathcal{A}_p(x, \mathbf{v}) := |\mathbf{v}|^{p-2}\mathbf{v}$ is p-acceptable under the Lebesgue measure. Functions which are $\mathcal{A}_p$-harmonic are called p-harmonic functions.

The following can be found in Chapters 3 and 6 of [HKM93].

Proposition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{A}$-harmonic. Then the following hold.*

(1) *The function f has a continuous representative.*
(2) *There exist constants $\alpha = \alpha(C_\mu, C_I, C_{\mathcal{A}}) > 0$ and $C = C(C_\mu, C_I, C_{\mathcal{A}}) > 0$ such that for each $x \in \mathbb{R}^n$ and $0 < r < R < \infty$ we have*

$$\operatorname*{osc}_{\mathbf{B}(x,r)} f \leq C (r/R)^\alpha \operatorname*{osc}_{\mathbf{B}(x,R)} f \ .$$

(3) *There exists a constant $C = C(C_\mu, C_I, C_{\mathcal{A}})$ such that*

$$r^p \int_{\mathbf{B}(x,r)} |\nabla f|^p d\mu \leq C \int_{\mathbf{B}(x,2r)} |f|^p d\mu$$

for each $r > 0$ and x in $\mathbb{R}^n$.

(4) *For each $0 < q < \infty$ and each $\tau > 1$ there exists a constant $C = C(C_\mu, C_I, C_{\mathcal{A}}, \tau, q)$ such that*

$$\sup_{\mathbf{B}(x,r)} |f| \leq C \left(⨍_{\mathbf{B}(x,\tau r)} |f|^q d\mu \right)^{1/q}$$

for each $r > 0$ and x in $\mathbb{R}^n$.

When we refer to an $\mathcal{A}$-harmonic function we will always refer to the pointwise defined continuous representative of f.

Remark 2.2. Using the Arzela-Ascoli theorem, properties (2) and (4) immediately imply that if $\{f_n\}$ is a sequence of $\mathcal{A}$-harmonic functions that converges to a function f in $L^1_{\mathrm{loc}}(\mu)$, then this sequence also converges to f in $L^q_{\mathrm{loc}}(\mu)$ for every $1 \leq q \leq \infty$. Moreover, Theorem 6.13 of [HKM93] shows that the convergence in $L^\infty_{\mathrm{loc}}(\mu)$ implies that f is also $\mathcal{A}$-harmonic. Hence a set S of $\mathcal{A}$-harmonic functions is closed in $L^q_{\mathrm{loc}}(\mu)$ for some $1 \leq q \leq \infty$ if and only if it is closed in $L^s_{\text{loc}}(\mu)$ for each $1 \le s \le \infty$. Property (4) implies that if S is a set of $\mathcal{A}$-harmonic functions, then S is bounded in $L^1_{\text{loc}}(\mu)$ if and only if S is bounded in $L^q_{\text{loc}}(\mu)$ for every $1 \le q \le \infty$. Additionally, property (3) implies that S is bounded in $H^{1,p}_{\text{loc}}(\mu)$ if and only if S is bounded in $L^p_{\text{loc}}(\mu)$. Moreover, the Arzela-Ascoli theorem also gives the following result.

PROPOSITION 2.3. *Let $\{f_n\}$ be a sequence of $\mathcal{A}$-harmonic functions that is bounded in $L^q_{\text{loc}}(\mu)$ for some $1 \le q \le \infty$. Then there exists a subsequence that converges in $L^r_{\text{loc}}(\mu)$ for every $1 \le r \le \infty$ to an $\mathcal{A}$-harmonic function f.*

3. Norms

DEFINITION 3.1. We call a non-decreasing continuous function $h : [1, \infty) \to (0, \infty)$ with $\int_1^\infty \frac{1}{h(t)} dt < \infty$ a *growth condition.*

DEFINITION 3.2. Let $h, k : [1, \infty) \to (0, \infty)$ be growth conditions. We say *k dominates h* if

$$\lim_{t\to\infty} \frac{h(t)}{k(t)} = 0 \ .$$

DEFINITION 3.3. Let $h : [1, \infty) \to (0, \infty)$ be a growth condition, $1 \le q, r < \infty$, and $f \in L^q_{\text{loc}}(\mu)$. Set

$$\|f\|_{(q,r,h)} := \left(\int_1^\infty \frac{1}{h(t)} \left(⨍_{\mathbf{B(0},t)} |f|^q d\mu\right)^{r/q} dt\right)^{1/r}$$

and

$$\|f\|_{(\infty,r,h)} := \left(\int_1^\infty \frac{1}{h(t)} \left(\operatorname{ess\,sup}_{\mathbf{B(0},t)} |f|\right)^r dt\right)^{1/r} .$$

The following proposition can easily be proved by mimicking a common proof of Minokowski's inequality.

PROPOSITION 3.4. *Let h be a growth condition, let $1 \le r < \infty$ and let $1 \le q \le \infty$. Then on the set*

$$L^{(q,r,h)}(\mu) := \{f \in L^q_{\text{loc}}(\mu) \mid \|f\|_{(q,r,h)} < \infty\},$$

$\|\cdot\|_{(q,r,h)}$ is a norm.

PROPOSITION 3.5. *Let h be a growth condition, $1 \le q \le \infty$ and $1 \le r < \infty$. We then have the following for each sequence $\{f_j\} \subset L^{(q,r,h)}(\mu)$.*

(1) *If $f \in L^{(q,r,h)}(\mu)$ with $\lim_{j\to\infty} \|f - f_j\|_{(q,r,h)} = 0$, then $f_j \to f$ in $L^q_{\text{loc}}(\mu)$.*
(2) *If $f_j \to f$ in $L^q_{\text{loc}}(\mu)$, then $\|f\|_{(q,r,h)} \le \liminf_{j\to\infty} \|f_j\|_{(q,r,h)}$.*

(3) *If k is another growth condition with $k \geq Ch$ for some constant C, then $L^{(q,r,h)}(\mu) \subset L^{(q,r,k)}(\mu)$ and $\|\cdot\|_{(q,r,k)} \leq C'\|\cdot\|_{(q,r,h)}$ with $C' = C'(C,r)$.*
(4) *If $S \subset L^{(q,r,h)}(\mu)$ is bounded in $L^{(q,r,h)}(\mu)$, then S is bounded in $L^q_{\mathrm{loc}}(\mu)$.*
(5) *If $f_j \to f$ in $L^q_{\mathrm{loc}}(\mu)$ and $\sup_j \|f_j\|_{(q,r,h)} < \infty$ and if k is another growth condition that dominates h, then $\lim_{j\to\infty} \|f - f_j\|_{(q,r,k)} = 0$.*

Proof. Items (1), (3) and (4) are immediate consequences of the definition. Item (2) follows from Fatou's lemma. To prove item (5), note that by items (2) and (3) we have $f \in L^{(q,r,h)}(\mu) \subset L^{(q,r,k)}(\mu)$ with $\|f\|_{(q,r,h)} \leq M$, where $M = \sup_j \|f_j\|_{(q,r,h)}$. Now, for each $m > 1$ we have

$$\begin{aligned}
\lim_{j\to\infty} \|f - f_j\|^r_{(q,r,k)} &= \lim_{j\to\infty} \int_1^\infty \frac{1}{k(t)} \left(\fint_{\mathbf{B}(\mathbf{0},t)} |f - f_j|^q d\mu \right)^{r/q} dt \\
&= \lim_{j\to\infty} \int_1^m \frac{1}{k(t)} \left(\fint_{\mathbf{B}(\mathbf{0},t)} |f - f_j|^q d\mu \right)^{r/q} dt \\
&\quad + \lim_{j\to\infty} \int_m^\infty \frac{1}{k(t)} \left(\fint_{\mathbf{B}(\mathbf{0},t)} |f - f_j|^q d\mu \right)^{r/q} dt \\
&\leq 0 + \sup_{t\geq m} \frac{h(t)}{k(t)} \lim_{j\to\infty} \|f - f_j\|^r_{(q,r,h)} \\
&\leq (2M)^r \sup_{t\geq m} \frac{h(t)}{k(t)},
\end{aligned}$$

which goes to zero as $m \to \infty$. □

PROPOSITION 3.6. *For each growth condition h, each $1 \leq q \leq \infty$ and each $1 \leq r < \infty$, the normed space $L^{(q,r,h)}(\mu)$ is a Banach space.*

Proof. Since $L^{(q,r,h)}(\mu)$ is a normed space, we only need to show that if $\{f_n\} \subset L^{(q,r,h)}(\mu)$ is a sequence such that $\sum_{n=1}^\infty \|f_n\|_{(q,r,h)} \leq N < \infty$, then there exists an $f \in L^{(q,r,h)}(\mu)$ such that $\lim_{n\to\infty} \|f - S_n\|_{(q,r,h)} = 0$, where $S_n = \sum_{k=1}^n f_k$. Let $T_n = \sum_{k=1}^n |f_k|$. Then for each n, $\|T_n\|_{(q,r,h)} \leq N$. Hence $\{T_n\}$ is bounded in $L^{(q,r,h)}(\mu)$. Applying Proposition 3.5(4) yields that for every $R > 0$, $\{T_n\}$ is bounded in $L^q(\mathbf{B}(\mathbf{0},R),\mu)$. Since $T_n = \sum_{k=1}^n |f_k|$, and for each ball $\mathbf{B}(\mathbf{0},R)$, $L^q(\mathbf{B}(\mathbf{0},R),\mu)$ is a Banach space, there exists $f^{(R)}$ such that $S_n \to f^{(R)}$ in $L^q(\mathbf{B}(\mathbf{0},R),\mu)$. Since limits are unique, we have $f^{(R)} = f^{(r)}$ almost everywhere on $\mathbf{B}(\mathbf{0},r)$ whenever $r < R$. Hence we can define f to be equal to f^R on $\mathbf{B}(\mathbf{0},R)$. We have that $S_n \to f$ in $L^q_{\mathrm{loc}}(\mu)$.

Thus, by Proposition 3.5(2),

$$\begin{aligned}\|f\|_{(q,r,h)} &\le \liminf_{n\to\infty} \|S_n\|_{(q,r,h)} \\ &\le \liminf_{n\to\infty} \|T_n\|_{(q,r,h)} \le N \ .\end{aligned}$$

Hence, $f \in L^{(q,r,h)}(\mu)$. Also, for almost every y, we have $f(y) = \sum_{n=1}^{\infty} f_n(y)$. Using that

$$\sum_{k=1}^{\infty} \|f_k\|_{(q,r,h)} \le N < \infty \ ,$$

we have

$$\begin{aligned}\lim_{n\to\infty} \|f - S_n\|_{(q,r,h)} &= \lim_{n\to\infty} \left\| \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{n} f_k \right\|_{(q,r,h)} \\ &\le \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \|f_k\|_{(q,r,h)} = 0 \ .\end{aligned}$$

□

We are now ready to prove our first main theorem.

THEOREM 3.7. *Let $S \subset L^{(q,r,h)}(\mu)$, with h a growth condition, $1 \le q \le \infty$ and $1 \le r < \infty$, be a family of $\mathcal{A}$-harmonic functions that is closed in $L^q_{\mathrm{loc}}(\mu)$. Then for every growth function k that dominates h there exists a set $G_{(q,h),(q,k)} \subset S$ which is locally compact in $L^{(q,r,k)}(\mu)$, open and dense in S under the topology of $L^{(q,r,k)}(\mu)$. Also, if k_2 is another growth function that dominates k, then $G_{(q,h),(q,k_2)} \subset G_{(q,h),(q,k)}$ and the topologies of $L^{(q,r,k)}(\mu)$ and $L^{(q,r,k_2)}(\mu)$ agree on $G_{(q,h),(q,k_2)}$. Moreover, the set $G_{(q,h),(q,k)}$ is canonically defined by q, r, h and k.*

Proof. Let

$$\begin{aligned}\bar{G}_{(q,h),(q,k)} = \Big\{ f \in S \mid \exists R > 0, \exists \delta > 0 \text{ s.t.} \\ \mathrm{Cl}_{(q,r,k)}(\mathbf{B}^S_{(q,r,h)}(0,R)) \supseteq \mathbf{B}^S_{(q,r,k)}(f,\delta) \Big\}\end{aligned}$$

and

$$\begin{aligned}G_{(q,h),(q,k)} = \Big\{ f \in S \mid \exists R > 0, \exists \delta > 0 \text{ s.t.} \\ \mathbf{B}^S_{(q,r,h)}(0,R) \supseteq \mathbf{B}^S_{(q,r,k)}(f,\delta) \Big\},\end{aligned}$$

where, for a growth function l,

$$\mathbf{B}^S_{(q,r,l)}(f,\delta) = \{ g \in S \mid \|f-g\|_{(q,r,l)} < \delta \}$$

and for every $T \subseteq S$,

$$\mathrm{Cl}_{(q,r,k)}(T) = \text{Closure of T under } \|\cdot\|_{(q,r,k)} \ .$$

We define

$$\overline{\mathbf{B}}^S_{(q,r,k)}(f,s) = \{g \in S \mid \|f-g\|_{(q,r,k)} \le s\}$$

and define $\overline{\mathbf{B}}^S_{(q,r,h)}(f,s)$ similarly. Because S is closed in $L^q_{\mathrm{loc}}(\mu)$ and convergence in $\|\cdot\|_{(q,r,k)}$ implies convergence in $L^q_{\mathrm{loc}}(\mu)$ for sequences, we have that for every subset T of S, $\mathrm{Cl}_{(q,r,k)}(T) \subseteq S$. We first claim that $\bar{G}_{(q,h),(q,k)} = G_{(q,h),(q,k)}$. Clearly,

$$\mathrm{Cl}_{(q,r,k)}(\mathbf{B}^S_{(q,r,h)}(0,R)) \supseteq \mathbf{B}^S_{(q,r,h)}(0,R),$$

which implies that $G_{(q,h),(q,k)} \subseteq \bar{G}_{(q,h),(q,k)}$. For the other set inclusion, note that by Proposition 3.5(2), $\|g\|_{(q,r,h)} \le \liminf_{n\to\infty} \|g_n\|_{(q,r,h)}$ whenever $\{g_n\}_{n=1}^\infty$ is a sequence in $L^q_{\mathrm{loc}}(\mu)$ that converges to g in $L^q_{\mathrm{loc}}(\mu)$. Hence

$$\mathrm{Cl}_{(q,r,k)}(\mathbf{B}^S_{(q,r,h)}(0,R)) \subseteq \overline{\mathbf{B}}^S_{(q,r,h)}(0,R) \subset \mathbf{B}^S_{(q,r,h)}(0,R+1),$$

which implies that $G_{(q,h),(q,k)} \supseteq \bar{G}_{(q,h),(q,k)}$. We will now show that $G_{(q,h),(q,k)}$ is a relatively open subset of S in the topology generated by $L^{(q,r,k)}(\mu)$. For each $f \in G_{(q,h),(q,k)}$, there exists an $R>0$ and a $\delta>0$ such that

$$\mathbf{B}^S_{(q,r,h)}(0,R) \supseteq \mathbf{B}^S_{(q,r,k)}(f,\delta) \ .$$

Let $g \in \mathbf{B}^S_{(q,r,k)}(f,\delta/2)$. We then have

$$\mathbf{B}^S_{(q,r,h)}(0,R) \supseteq \mathbf{B}^S_{(q,r,k)}(f,\delta) \supseteq \mathbf{B}^S_{(q,r,k)}(g,\delta/2) \ .$$

We conclude that $\mathbf{B}^S_{(q,r,k)}(f,\delta/2) \subseteq G_{(q,h),(q,k)}$.

We will now show that $G_{(q,h),(q,k)}$ is locally compact in $L^{(q,r,k)}(\mu)$. Indeed, fix an $f \in G_{(q,h),(q,k)}$ and let $R>0$ and $\delta>0$ be as in the definition of $G_{(q,h),(q,k)}$. It suffices to show that $\overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2)$ is compact in $L^{(q,r,k)}(\mu)$. Let $\{f_n\}_{n=1}^\infty \subset \overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2)$. Then for all n, $\|f_n\|_{(q,r,h)} \le R$. Hence by Proposition 3.5(4) the sequence is bounded in $L^q_{\mathrm{loc}}(\mu)$. Applying Proposition 2.3 creates a subsequence $\{f_{n_m}\}$ that converges in $L^q_{\mathrm{loc}}(\mu)$ to a function g. Moreover, g will be in S because S is closed in $L^q_{\mathrm{loc}}(\mu)$. We have that for all m, $\|f_{n_m}\|_{(q,r,h)} \le R$, and k dominates h. Applying Proposition 3.5(5) we conclude that $f_{n_m} \to g$ in $\|\cdot\|_{(q,r,k)}$ and $\|f-g\|_{(q,r,k)} \le \delta/2$. Hence $g \in \overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2)$.

In the proof that $G_{(q,h),(q,k)}$ is dense in S under the topology of $L^{(q,r,k)}(\mu)$ we will slightly mirror the proof of the Open Mapping Theorem by using the Baire Category Theorem. Suppose $G_{(q,h),(q,k)}$ is not dense in S under $L^{(q,r,k)}(\mu)$. Then there exists an $f \in S$ and $\delta>0$ such that $\mathbf{B}^S_{(q,r,k)}(f,\delta) \cap G_{(q,h),(q,k)} = \emptyset$. Now, S is closed in $L^q_{\mathrm{loc}}(\mu)$ and hence in the Banach space

$L^{(q,r,k)}(\mu)$. Because S is closed in $L^{(q,r,k)}(\mu)$ we have that $\overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2)$ is closed in the Banach space $L^{(q,r,k)}(\mu)$ and hence complete under $\|\centerdot\|_{(q,r,k)}$. Since $S \subset L^{(q,r,h)}(\mu)$, we have

$$\overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2) = \bigcup_{R=1}^{\infty} \left(\overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2) \cap \mathbf{B}^S_{(q,r,h)}(0,R)\right) . \tag{1}$$

Let $A_R = \mathrm{Cl}_{(q,r,k)}(\mathbf{B}^S_{(q,r,h)}(0,R))$. Then by Baire Category Theorem there exists an $R \in \mathbb{N}$ such that the set A_R has non-empty relative interior as a subset of $\overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2)$ under $\|\centerdot\|_{(q,r,k)}$. Hence there exists a function $g \in \overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2)$ and an $\epsilon > 0$ such that

$$\mathrm{Cl}_{(q,r,k)}(\mathbf{B}^S_{(q,r,h)}(0,R)) \supset \mathbf{B}^S_{(q,r,k)}(g,\epsilon) \cap \overline{\mathbf{B}}^S_{(q,r,k)}(f,\delta/2) . \tag{2}$$

Thus there exist $\eta > 0$ and $g' \in \mathbf{B}^S_{(q,r,k)}(f,\delta/2)$ such that

$$\mathrm{Cl}_{(q,r,k)}(\mathbf{B}^S_{(q,r,h)}(0,R)) \supset \mathbf{B}^S_{(q,r,k)}(g',\eta) . \tag{3}$$

Hence $g' \in \bar{G}_{(q,h),(q,k)} = G_{(q,h),(q,k)}$, contradicting that $g' \in \mathbf{B}^S_{(q,r,k)}(f,\delta/2)$ and $\mathbf{B}^S_{(q,r,k)}(f,\delta) \cap G_{(q,h),(q,k)} = \emptyset$.

We now show that if k_2 is a growth function that dominates k, then $G_{(q,h),(q,k_2)} \subset G_{(q,h),(q,k)}$ and the topologies generated by $\|\centerdot\|_{(q,r,k)}$ and $\|\centerdot\|_{(q,r,k_2)}$ on $G_{(q,h),(q,k_2)}$ are identical. By Proposition 3.5(3) there exists a constant $C = C(k,k_2)$ such that for every f and $\delta > 0$,

$$\mathbf{B}^S_{(q,r,k_2)}(f,\delta) \supset \mathbf{B}^S_{(q,r,k)}(f,\delta/C) .$$

Hence we immediately see that $G_{(q,h),(q,k_2)} \subset G_{(q,h),(q,k)}$. Also, by Proposition 3.5(3), if $f_n \to f$ in $\|\centerdot\|_{(q,r,k)}$, then $f_n \to f$ in $\|\centerdot\|_{(q,r,k_2)}$. Conversely, if $f_n \to f$ in $\|\centerdot\|_{(q,r,k_2)}$ with $f \in G_{(q,h),(q,k_2)}$, then $f_n \to f$ in $L^q_{\mathrm{loc}}(\mu)$. We have that $f \in G_{(q,h),(q,k_2)}$. Hence there exist $R > 0$ and $\delta > 0$ such that

$$\mathbf{B}^S_{(q,r,h)}(0,R) \supset \mathbf{B}^S_{(q,r,k_2)}(f,\delta) . \tag{4}$$

Since $f_n \to f$ in $\|\centerdot\|_{(q,r,k_2)}$, by (4) we may assume that for all n, $f_n \in \mathbf{B}^S_{(q,r,k_2)}(f,\delta)$. Thus for all n, $\|f_n\|_{(q,r,h)} \leq R$. Since k dominates h and $f_n \to f$ in $L^q_{\mathrm{loc}}(\mu)$, using Proposition 3.5(5) we conclude that $f_n \to f$ in $\|\centerdot\|_{(q,r,k)}$. □

The above theorem states that changing the growth condition k does not affect the topology strongly. A similar result is true when we change q. However, care must be taken when changing q. For each $\tau \in \mathbb{R}^+$ and each growth condition h, we set $h_\tau(t) = h(\tau t)$. Note that Proposition 2.1 gives the following result.

Proposition 3.8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{A}$-harmonic. Then for each growth condition h, each $1 \le r < \infty$, each $1 \le q \le \infty$ and each $\tau > 1$ we have $\|f\|_{(\infty,r,h_\tau)} \le C(C_{\mathcal{A}}, \tau, r)\|f\|_{(q,r,h)}$.*

Theorem 3.9. *Let $S \subset L^{(q,r,h)}(\mu)$, with h a growth condition, $1 \le q \le \infty$ and $1 \le r < \infty$, be a family of $\mathcal{A}$-harmonic functions that is closed in $L^q_{\text{loc}}(\mu)$. Then for every growth function k that dominates h_τ with $\tau > 1$ and every $1 \le s \le \infty$ there exists a set $G_{(q,h),(s,k)} \subset S$ which is locally compact in $L^{(q,r,k)}(\mu)$, open and dense in S under the topology of $L^{(s,r,k)}(\mu)$. Also, if $s \le s_2 \le \infty$, then $G_{(q,h),(s,k)} \subset G_{(q,h),(s_2,k)}$ and the topologies of $L^{(s,r,k)}(\mu)$ and $L^{(s_2,r,k)}(\mu)$ agree on $G_{(q,h),(s,k)}$.*

Proof. We only sketch the proof, as it closely mirrors the proof of Theorem 3.7. By Proposition 3.8 and Remark 2.2, we have that $S \subset L^{(\infty,r,h_\tau)}(\mu)$ and S is closed in $L^u_{\text{loc}}(\mu)$ for all $1 \le u \le \infty$. We set

$$G_{(q,h),(s,k)} = \left\{ f \in S \mid \exists R > 0, \exists \delta > 0 \text{ s.t. } \mathbf{B}^S_{(q,r,h)}(0,R) \supseteq \mathbf{B}^S_{(s,r,k)}(f,\delta) \right\} .$$

Using the same argument as for Theorem 3.7 and the fact that S is closed, we obtain that $G_{(q,h),(s,k)}$ is relatively open and dense in S under $\|\cdot\|_{(s,r,k)}$. To see that $G_{(q,h),(s,k)}$ is locally compact in $\|\cdot\|_{(s,r,k)}$, let $f \in G_{(q,h),(s,k)}$ and let $R > 0$ and $\delta > 0$ be as in the definition of $G_{(q,h),(s,k)}$. We will show that $\mathbf{B}^S_{(s,r,k)}(f,\delta/2)$ is pre-compact. Indeed, let $\{f_n\}_{n=1}^\infty$ be any sequence in $\mathbf{B}^S_{(s,r,k)}(f,\delta/2)$. Then for all n we have, by Proposition 3.8,

$$\|f_n\|_{(s,r,h_\tau)} \le \|f_n\|_{(\infty,r,h_\tau)} \le C\|f_n\|_{(q,r,h)} \le CR \ . \tag{5}$$

Hence the sequence is bounded in $L^s_{\text{loc}}(\mu)$. As before, apply Proposition 2.3 to extract a subsequence that converges is $L^s_{\text{loc}}(\mu)$. By (5), this subsequence is bounded in $\|\cdot\|_{(s,r,h_\tau)}$, and because k dominates h_τ, Proposition 3.5(5) implies that it converges in $\|\cdot\|_{(s,r,k)}$, as needed.

That $G_{(q,h),(s,k)} \subset G_{(q,h),(s_2,k)}$ follows directly from the inequality

$$\|\cdot\|_{(s,r,k)} \le \|\cdot\|_{(s_2,r,k)}, \tag{6}$$

which follows from Hölder's inequality. To show that the topologies are equivalent, note that by (6) the topology generated by $\|\cdot\|_{(s,r,k)}$ is coarser than the topology generated by $\|\cdot\|_{(s_2,r,k)}$. For the other inclusion, let $f_n \to f$ under $\|\cdot\|_{(s,r,k)}$ with $f \in G_{(q,h),(s,k)}$. As before, we may assume that there exists $R > 0$ such that for all n, $\|f_n\|_{(q,r,h)} \le R$. Applying (5), we have for all n that $\|f_n\|_{(s_2,r,h_\tau)} \le CR$. Since $f_n \to f$ in $\|\cdot\|_{(s,r,k)}$, we also have that $f_n \to f$ in $L^s_{\text{loc}}(\mu)$, and applying Remark 2.2 yields that $f_n \to f$ in $L^{s_2}_{\text{loc}}(\mu)$. Using the assumption that k dominates h_τ and applying Proposition 3.5(5) gives that $f_n \to f$ in $\|\cdot\|_{(s_2,r,k)}$. $\square$

REMARK 3.10. Although we have presented our results in $\mathbb{R}^n$, the only really necessary tools for the proof are the inequalities of Proposition 2.1. The proofs presented in [HKM93] can be adapted to the manifold setting or beyond with great ease, as they do not use the specific properties of $\mathbb{R}^n$. Rather, these proofs require only that the measure satisfies a $(1, p)$-Poincaré inequality and that it is doubling.

4. p-harmonic functions in the plane

Here we extend the fact that the space of all harmonic functions on the plane with growth of order at most d is finite dimensional to the space of p-harmonic functions. Our key technique is a result of Iwaniec and Manfredi found in [IM89] stating that the complex derivative of p-harmonic function is a quasiregular mapping. We combine this result with one found in [HK95] and [Väi72] to show that if a sequence of p-harmonic functions, all with polynomial growth of order bounded by a fixed number d, converges locally uniformly, then the limit function also satisfies a bound on its growth. We now begin to describe the details, which involve the theory of quasiregular mappings. We refer the reader to the monograph by Rickman [Ric93].

We use the notation of [Ric93, Chapter 1]: For an open discrete continuous mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, B_f refers to the branch set of f, $\#S$ refers to the cardinality of a set S. We define $N(y, f, U)$ as the cardinality of $f^{-1}(y) \cap U$ and set $N(f, U) := \sup_{y \in U} N(y, f, U)$. We say a point $y \in \mathbb{R}^n$ is (U, f)-admissible if $y \notin f(\partial U)$, and in this case we denote the local degree by $\mu(y, f, U)$.

PROPOSITION 4.1. *Let $\{f_j\}$ be a sequence that converges locally uniformly in $\mathbb{R}^n$ to a function f with each f_j K-quasiregular and $N(f_j, \mathbb{R}^n) \leq m$. Then either f is a constant or $N(f, \mathbb{R}^n) \leq m$.*

Proof. Since each f_n is K-quasiregular, f is also K-quasiregular. If f is a constant, then we are done. Otherwise f is a continuous, open, discrete mapping. Let $S = f(B_f) \cup \bigcup_{j=1}^{\infty} f_j(B_{f_j})$. Then S has Lebesgue measure zero. Thus $T = \mathbb{R}^n - S$ has full measure and hence is dense. We first show that for $y \in T$, $N(y, f, \mathbb{R}^n) \leq m$. Indeed, suppose there exists a y in T so that $N(y, f, \mathbb{R}^n) \geq m + 1$. Then there exist at least $m + 1$ distinct points in $\mathbb{R}^n$, $\{x_i\}_{i=1}^{m+1}$ such that $f(x_i) = y$. Now, f is discrete and open. Hence there exists an $R > 0$ and an $\epsilon > 0$ such that for all i, $|x_i| < R$ and

$$f^{-1}(y) \cap \mathbf{B}(0, R + \epsilon) \subset \mathbf{B}(0, R - \epsilon) \ . \tag{7}$$

Note that (7) implies that $\operatorname{dist}(y, f(\partial B_R)) > 0$. Now, $f_j \to f$ locally uniformly in $\mathbb{R}^n$. Hence there exists a j such that

$$\sup_{\mathbf{B}(0,2R+2\epsilon)} |f - f_j| < \frac{1}{10} \operatorname{dist}(y, f(\partial B_R)), \tag{8}$$

which implies that y is $(f_j, \mathbf{B}(0,R))$-admissible. Since $y \notin f_j(B_{f_j})$, applying [Ric93, I.4.10] we conclude that $N(y, f_j, \mathbf{B}(0,R)) = \mu(y, f_j, \mathbf{B}(0,R))$. Let $h_t(x) = tf(x) + (1-t)f_j(x)$. Then $h_1 = f$, $h_0 = f_j$ and h_t maps f homotopically to f_j. Also, by (7) and (8), $y \notin h_t(\partial\mathbf{B}(0,R))$ for each $0 \le t \le 1$. We thus have

$$\begin{aligned} m+1 &\le N(y, f, \mathbf{B}(0,R)) \le \mu(y, f, \mathbf{B}(0,R)) \\ &= \mu(y, f_j, \mathbf{B}(0,R)) = N(y, f_j, \mathbf{B}(0,R)) \le m, \end{aligned}$$

a contradiction. Hence for $y \notin f(B_f) \cup \bigcup_{j=1}^{\infty} f_j(B_{f_j})$, $N(y, f, \mathbb{R}^n) \le m$. For $y \notin T$, suppose that $\# f^{-1}(y) \ge m+1$. As before, let $R > 0$ be such that $\# f^{-1}(y) \cap \mathbf{B}(0,R) \ge m+1$ and $f^{-1}(y) \cap \partial\mathbf{B}(0,R) = \emptyset$. Since $\# f^{-1}(y) \ge m+1$, we can use [Ric93, I.4.10] to conclude that $m+1 \le \mu(y, f, \mathbf{B}(0,R))$. Now, f is quasiregular. Hence $f(\partial\mathbf{B}(0,R))$ has Lebesgue n-measure zero. Let U be the component of $\mathbb{R}^n - f(\partial\mathbf{B}(0,R))$ containing y. As T and the complement of $f(\partial\mathbf{B}(0,R))$ have full measure, we know there exists an element $y' \in U \cap T$ that is not an element of $f(\partial\mathbf{B}(0,R))$. Hence y' is $(f, \mathbf{B}(0,R))$ admissible. The preceding argument showed that $\mu(y', f, \mathbf{B}(0,R)) \le m$. Since y and y' are both in the same component of $\mathbb{R}^n - f(\partial\mathbf{B}(0,R))$, [Ric93, I.4.4] implies that

$$m+1 \le \mu(y, f, \mathbf{B}(0,R)) = \mu(y', f, \mathbf{B}(0,R)) \le m,$$

a contradiction. Hence, for all y, $N(y, f, \mathbb{R}^n) \le m$. □

We now quote and paraphrase a portion of Theorem 1.5 of [HK95]. Actually, Koskela and Heinonen show quite more than the following, but this is all that we need here. Additionally, the first implication of the following theorem was first shown by Väisälä; see [Väi72].

Theorem 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a non-constant K-quasiregular mapping. If there exist constants $C > 0$ and $d > 0$ such that $|f(x)| \le C(1 + |x|^d)$, then $N(f, \mathbb{R}^n) \le m = m(n, K, d)$. Also, if $N(f, \mathbb{R}^n) < \infty$, then there exist constants $C > 0$ and $d = d(n, K, N(f, \mathbb{R}^n))$ such that $|f(x)| \le C(1 + |x|^d)$.*

Combining Theorem 4.2 and Proposition 4.1 gives the following result.

Corollary 4.3. *Let $\{f_n\}$ be a sequence of K-quasiregular mappings of $\mathbb{R}^n$ with polynomial growth of order at most d that converges locally uniformly to a function f. Then f is a K-quasiregular mapping with polynomial growth of order at most $D = D(d, K, n)$.*

We also need the following result, which was also first proved by Reshetnyak; we cite [HKM93, pp. 269–273] for the proof.

Theorem 4.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be K-quasiregular. Then each of the coordinate functions of f is $\mathcal{A}_f$-harmonic for some n-acceptable family $\mathcal{A}_f$*

under the Lebesgue n-measure, with $C_{\mathcal{A}_f}$ depending only on K and n. In particular, if $\{f_j\}$ is a sequence of K-quasiregular mappings that converge to a mapping f in $L^q_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for some $1 \le q \le \infty$, then it also converges to f in $L^s_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ for each $1 \le s \le \infty$.

One can easily adapt the norms and proofs of Theorems 3.7 and 3.9 to obtain the following result.

Corollary 4.5. *Let $Q(K,n,m)$ be the set of all K-quasiregular mappings $f:\mathbb{R}^n \to \mathbb{R}^n$ with f constant or $N(f,\mathbb{R}^n) \le m$. Then for each $1 \le q \le \infty$ there exists a family of Banach spaces, each a subset of $L^q_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, for which $Q(K,n,m)$ is a closed subset, such that under each of these Banach spaces there exists a relatively dense open subset of $Q(K,n,m)$ which is locally compact. Moreover, the topology generated by each of these Banach spaces is stronger than the topology generated by $L^q_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^n)$.*

The following Caccioppoli estimate shows that a sequence of p-harmonic functions converges in $L^p_{\text{loc}}(\mathbb{R}^n)$ if and only if it converges in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

Proposition 4.6. *Let f and g be p-harmonic functions defined on an open set Ω. Then for each $\psi \in C^\infty_c(\Omega)$ we have for $p \ge 2$,*

$$\int_\Omega |\psi|^p|\nabla f - \nabla g|^p dx \le C(p)\left(\int_\Omega |\nabla\psi|^p|f-g|^p dx\right)^{1/p} \times \left(\int_\Omega |\psi|^p|\nabla f|^p dx + \int_\Omega |\psi|^p|\nabla g|^p dx\right)^{\frac{p-1}{p}},$$

and for $1 < p < 2$,

$$\int_\Omega |\psi|^p|\nabla f - \nabla g|^p dx \le C(p)\left(\int_\Omega |\nabla\psi|^p|f-g|^p dx\right)^{1/2} \times \left(\int_\Omega |\psi|^p|\nabla f|^p dx + \int_\Omega |\psi|^p|\nabla g|^p dx\right)^{1/2}.$$

Proof. Note that for $\mathcal{A}_p(x,\mathbf{v}) := |\mathbf{v}|^{p-2}\mathbf{v}$ we have

$$\langle \mathcal{A}(x,\mathbf{v}) - \mathcal{A}(x,\mathbf{w}), \mathbf{v}-\mathbf{w}\rangle \ge \frac{1}{C(p)}(|\mathbf{v}|+|\mathbf{w}|)^{p-2}|\mathbf{v}-\mathbf{w}|^2 . \tag{9}$$

Let $h = |\psi|^p(f-g)$. Then $h \in W^{1,p}_0(\Omega)$. Hence,

$$0 = \int_\Omega \langle \mathcal{A}(x,\nabla f), \nabla h\rangle\, dx \ = \int_\Omega \langle \mathcal{A}(x,\nabla g), \nabla h\rangle\, dx$$

and by calculation,

$$\nabla h = p\psi|\psi|^{p-2}(f-g)\nabla\psi + |\psi|^p(\nabla f - \nabla g),$$

Hence,

$$\int_\Omega \langle \mathcal{A}(x,\nabla f) - \mathcal{A}(x,\nabla g), |\psi|^p(\nabla f - \nabla g)\rangle\, dx$$
$$= -\int_\Omega \left\langle \mathcal{A}(x,\nabla f) - \mathcal{A}(x,\nabla g), p\psi(f-g)|\psi|^{p-2}\nabla\psi\right\rangle dx.$$

Taking absolute values, and applying Hölder's inequality yields

$$\begin{aligned}
&\int_\Omega \Big\langle \mathcal{A}(x,\nabla f) - \mathcal{A}(x,\nabla g), |\psi|^p(\nabla f - \nabla g)\Big\rangle dx \\
&\qquad \le C(p)\int_\Omega |\psi|^{p-1}|f-g|\,|\nabla\psi|\,|\mathcal{A}(x,\nabla f) - \mathcal{A}(x,\nabla g)|\, dx \\
&\qquad \le C(p)\int_\Omega |\psi|^{p-1}|f-g|\,|\nabla\psi|\,(|\nabla f|^{p-1} + |\nabla g|^{p-1})\, dx \\
&\qquad \le C(p)\left(\int_\Omega |\nabla\psi|^p|f-g|^p dx\right)^{1/p} \\
&\qquad\qquad \times\left(\int_\Omega |\psi|^p|\nabla f|^p dx + \int_\Omega |\psi|^p|\nabla g|^p dx\right)^{\frac{p-1}{p}}.
\end{aligned}$$

For $p \ge 2$ we have by using (9)

$$\begin{aligned}
&\int_\Omega |\psi|^p|\nabla f - \nabla g|^p dx \\
&\qquad \le C(p)\int_\Omega |\psi|^p\,\langle \mathcal{A}(x,\nabla f) - \mathcal{A}(x,\nabla g), \nabla f - \nabla g\rangle\, dx \\
&\qquad \le C(p)\left(\int_\Omega |\nabla\psi|^p|f-g|^p dx\right)^{1/p} \\
&\qquad\qquad \times\left(\int_\Omega |\psi|^p|\nabla f|^p dx + \int_\Omega |\psi|^p|\nabla g|^p dx\right)^{\frac{p-1}{p}}
\end{aligned}$$

and for $1 < p < 2$, letting $q = 2/p > 1$, we have, again by using (9),

$$\begin{aligned}
&\int_\Omega |\psi|^p|\nabla f - \nabla g|^p dx \\
&\qquad = \int_\Omega |\psi|^p|\nabla f - \nabla g|^p(|\nabla f| + |\nabla g|)^{\frac{p-2}{q}}(|\nabla f| + |\nabla g|)^{\frac{2-p}{q}}\, dx \\
&\qquad \le \left(\int_\Omega |\psi|^p|\nabla f - \nabla g|^2(|\nabla f| + |\nabla g|)^{p-2} dx\right)^{p/2} \\
&\qquad\qquad \times\left(\int_\Omega |\psi|^p(|\nabla f| + |\nabla g|)^p dx\right)^{\frac{2-p}{2}}
\end{aligned}$$

$$\begin{aligned}
&\le C(p)\left(\int_\Omega \langle \mathcal{A}(x,\nabla f)-\mathcal{A}(x,\nabla g), |\psi|^p(\nabla f-\nabla g)\rangle\, dx\right)^{p/2}\\
&\qquad\times\left(\int_\Omega |\psi|^p(|\nabla f|+|\nabla g|)^p dx\right)^{\frac{2-p}{2}}\\
&\le C(p)\left(\int_\Omega |\nabla\psi|^p|f-g|^p dx\right)^{1/2}\\
&\qquad\times\left(\int_\Omega |\psi|^p|\nabla f|^p dx+\int_\Omega |\psi|^p|\nabla g|^p dx\right)^{\frac{p-1}{2}}\\
&\qquad\times\left(\int_\Omega |\psi|^p(|\nabla f|+|\nabla g|)^p dx\right)^{\frac{2-p}{2}}\\
&\le C(p)\left(\int_\Omega |\nabla\psi|^p|f-g|^p dx\right)^{1/2}\\
&\qquad\times\left(\int_\Omega |\psi|^p|\nabla f|^p dx+\int_\Omega |\psi|^p|\nabla g|^p dx\right)^{1/2}. \qquad \square
\end{aligned}$$

We now quote a remarkable result stated in [IM89, p. 4].

THEOREM 4.7. *Let $u:\mathbb{R}^2\to\mathbb{R}$ be a p-harmonic function. Then $f=\frac{\partial u}{\partial z}:\mathbb{R}^2\to\mathbb{R}^2$ is K-quasiregular with $K\le\max(p-1,1/(p-1))$. Here,*

$$\frac{\partial u}{\partial z}:=\left(\frac{1}{2}\frac{\partial u}{\partial x},-\frac{1}{2}\frac{\partial u}{\partial y}\right)\ .$$

We are now ready to prove that for each $d>0$ the set of all p-harmonic functions on $\mathbb{R}^2$ with growth of order at most d is essentially small. We let $T_d(p)$ be the set of all p-harmonic functions defined on the plane with growth of order at most d. We also define $S_d(p)$ as the closure of $T_d(p)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$.

PROPOSITION 4.8. *For each $d>0$ there exists $m=m(d,p)$ such that $S_d(p)\subset T_m(p)$.*

Proof. Let $\{u_j\}$ be a sequence in $T_d(p)$ that converges in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ to a function u. Then, by Remark 2.2, Proposition 2.3, and Proposition 4.6, u is also p-harmonic and $u_j\to u$ in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$. Let $f_j=\frac{\partial u_j}{\partial z}$ and $f=\frac{\partial u}{\partial z}$. Then $f_j\to f$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$. Moreover, by Theorem 4.7, for each j, f_j is K-quasiregular with $K\le\max(p-1,1/(p-1))$. Hence, by Theorem 4.4, $f_j\to f$ locally uniformly and f is also K-quasiregular. Now, for each j and $r>0$ we have

$$r^p\int_{\mathbf{B}(0,r)}|\nabla u_j|^p dx\le C\int_{\mathbf{B}(0,2r)}|u_j|^p dx\ .$$

Hence, for each j and $r > 0$ we have

$$\int_{\mathbf{B}(0,r)} |f_j|^p dx \le Cr^{-p} \int_{\mathbf{B}(0,2r)} |u_j|^p dx \le C(1 + r^{(d-1)p+2})$$

with $C = C(p, u_j)$. Now each f_j is K-quasiregular. Thus, by Theorem 4.4, for each j, the coordinate functions of f_j are $\mathcal{A}_j$-harmonic for some 2-acceptable family $\mathcal{A}_j$ with $C_{\mathcal{A}_j} = C(K)$. Applying Proposition 2.1(4) yields for each j and $r > 0$,

$$\sup_{\mathbf{B}(0,r)} |f_j| \le C \left(\fint_{\mathbf{B}(0,2r)} |f_j|^p \right)^{1/p} \le C(1 + r^{d-1})$$

with $C = C(f_j, p)$. Hence, by Theorem 4.3, there exists $N = N(d, p)$ such that f has polynomial growth of order at most N. Now, $|\nabla u(x)| = 2|f(x)|$. Hence $|\nabla u(x)|$ also has polynomial growth of order at most N. Integration gives that $|u(x)| \le C(1 + |x|^m)$, where $m = m(d, p)$. Hence $u \in T_m(p)$, as needed. $\square$

Proposition 4.8 and Theorem 3.7 give our main result.

Theorem 4.9. *Let $S_d(p)$ be the closure in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ of all p-harmonic functions defined on the plane with polynomial growth of order at most d. Then there exists a family of Banach spaces, each a subset of $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, for which $S_d(p)$ is a closed subset, such that under each of these Banach spaces there exists a relatively dense open set of S which is locally compact.*

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