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SHARP INEQUALITIES FOR TRIGONOMETRIC SUMS IN TWO VARIABLES

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ABSTRACT. We prove several new inequalities for trigonometric sums in two variables. One of our results states that the double-inequality

$$-\frac{2}{3}(\sqrt{2}-1) \le \sum_{k=1}^{n} \frac{\cos((k-1/2)x)\sin((k-1/2)y)}{k-1/2} \le 2$$

holds for all integers $n \ge 1$ and real numbers $x, y \in [0, \pi]$. Both bounds are best possible.

1. Introduction

The inequalities of Fejér-Jackson and Young,

$$0 < \sum_{k=1}^{n} \frac{\sin(kx)}{k}, \quad -1 < \sum_{k=1}^{n} \frac{\cos(kx)}{k} \quad (n \in \mathbf{N};\ 0 < x < \pi), \tag{1.1}$$

are well-known examples of inequalities for trigonometric sums; see [15], [25]. Many mathematicians studied (1.1) and presented various proofs, generalizations, refinements, and numerous counterparts and analogues. Excellent accounts on this subject are given in the survey paper [6] and the monograph [23, Chapter 4]. We also refer to the research articles [1]–[4], [7]–[15], [17]–[22], [24], [25], and the references therein.

Inequalities for trigonometric sums have interesting applications: they play an important role in Fourier analysis, number theory, and the theory of univalent and p-valent functions, and they can be used to estimate the zeros of trigonometric polynomials. Moreover, they have a close connection to the theory of special functions. In fact, certain trigonometric sums are special cases of sums of Jacobi polynomials. An elegant function theoretic approach to establish extensions of (1.1) is given in [12] and [24].

Most of the known inequalities for trigonometric sums involve only one real variable. In this paper we study analogues of (1.1) in two variables. Our work

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has been inspired by a remarkable article published by L. Koschmieder [17] in 1932. A problem in heat conduction led him to the sums

$$(1.2)\qquad \begin{aligned} A_n(x,y) &= \sum_{k=1}^{n} \frac{\cos(kx)\sin(ky)}{k}, \\ A_n^*(x,y) &= \sum_{k=1}^{n} \frac{\cos((k-1/2)x)\sin((k-1/2)y)}{k-1/2}. \end{aligned}$$

He proved the following inequalities: If $0 < y - x < \pi$ and $0 < y + x < \pi$, then $A_n(x,y) > 0$. If $0 < x - y < \pi$ and $\pi < y + x < 2\pi$, then $A_n(x,y) < 0$. Moreover, if $0 \le x < y < \pi$, then $A_n^*(x,y) > 0$. This inequality is in general not true if $x > y$, as the examples $A_2^*(2\pi/3, \pi/3) = -1/6$ and $A_3^*(2\pi/3, \pi/3) = -1/15$ reveal.

In R. Askey's 'SIAM Conference Lectures' [5, p. 34] the following interesting theorem is given:

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n and order (α,β), where $\alpha \ge \beta$ and either $\beta \ge -1/2$ or $\alpha \ge -\beta$, $\beta > -1$. Further, let $\sum_{k=0}^{\infty} |a_k| < \infty$. Then

$$(1.3)\qquad F(s,t) = \sum_{k=0}^{\infty} a_k \frac{P_k^{(\alpha,\beta)}(s) P_k^{(\alpha,\beta)}(t)}{(P_k^{(\alpha,\beta)}(1))^2} \ge 0 \quad (-1 \le s, t \le 1),$$

if and only if $F(s,1) \ge 0$ $(-1 \le s \le 1)$.

The function F plays a role in the theory of partial differential equations. In fact, G. Gasper [14] proved that F is the solution of a hyperbolic boundary value problem.

We set

$$\begin{aligned} &\alpha = \beta = -1/2, \\ &a_0 = 1, \quad a_k = 1/k \quad (k = 1, \dots, n), \quad a_k = 0 \quad (k \ge n+1), \\ &s = \cos(x), \quad t = \cos(y), \end{aligned}$$

and

$$\begin{aligned} &\alpha = 1/2, \quad \beta = -1/2, \\ &a_k = 1/(k+1/2) \quad (k = 0, \dots, n-1), \quad a_k = 0 \quad (k \ge n), \\ &s = \cos(x), \quad t = \cos(y), \end{aligned}$$

respectively. Then (1.3) yields

$$(1.4)\qquad -1 \le \sum_{k=1}^{n} \frac{\cos(kx)\cos(ky)}{k} \quad (n \in \mathbf{N};\ 0 \le x, y \le \pi)$$

and

$$0 \le \sum_{k=1}^{n} \frac{\sin((k-1/2)x)\sin((k-1/2)y)}{k-1/2} \quad (n \in \mathbf{N};\ 0 \le x, y \le \pi). \tag{1.5}$$

If we replace in (1.5) x by $\pi - x$ and y by $\pi - y$, then we see that this is equivalent to

$$0 \le \sum_{k=1}^{n} \frac{\cos((k-1/2)x)\cos((k-1/2)y)}{k-1/2} \quad (n \in \mathbf{N};\ 0 \le x, y \le \pi). \tag{1.6}$$

The special case $\alpha = \beta = 1/2$ leads to a result due to L. Fejér (see [5, p. 33]): We have

$$0 \le \sum_{k=1}^{\infty} a_k \sin(kx)\sin(ky) \quad (0 \le x, y \le \pi)$$

if and only if

$$0 \le \sum_{k=1}^{\infty} k a_k \sin(kx) \quad (0 \le x \le \pi).$$

This theorem, however, gives no information about the best possible lower bound for

$$B_n(x,y) = \sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k}. \tag{1.7}$$

It is the aim of this paper to present sharp constant bounds for $A_n(x,y)$, $A_n^*(x,y)$, and $B_n(x,y)$, which hold for all $n \ge 1$ and $x, y \in [0,\pi]$. In order to prove our inequalities we need several technical lemmas, which we collect in the next section. The main results are presented in Section 3.

The numerical values have been calculated by the computer programs 'Maple V Release 5.1' and 'Maple V Release 6.01'.

2. Lemmas

The inequalities given in the first lemma are known in the literature.

LEMMA 1. *Let*

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}, \quad S_n^*(x) = \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}, \tag{2.1}$$

and

$$T_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k}. \tag{2.2}$$

(i) *For all integers $n \ge 1$ and real numbers x we have $|S_n(x)| < \mathrm{Si}(\pi)$, where $\mathrm{Si}(\pi) = \int_0^\pi \sin(t)/t\,dt = 1.8519\ldots$.*

(ii) *For all integers* $n \geq 1$ *and real numbers* x *we have* $S_n^*(x) \leq 1$, *with equality holding if and only if* $n = 1$ *and* $x = (2m + 1/2)\pi$ $(m \in \mathbf{Z})$.

(iii) *For all integers* $n \geq 2$ *and real numbers* $x \in (0, \pi)$ *we have* $T_n(x) \geq -5/6$. *If* $n \neq 3$, *then the inequality is strict.*

Part (i) is proved in [15]. In [2] it is shown that $S_n^*(x) \leq 1$ for $x \in [0, \pi]$ with equality only if $n = 1$ and $x = \pi/2$. A refinement of this inequality can be found in [12] and [24]. The interpretation of these inequalities in geometric function theory is also stated there. Since S_n^* is odd and satisfies $S_n^*(x) \geq 0$ for $x \in [0, \pi]$, we obtain $S_n^*(x) \leq 0$ for $x \in [-\pi, 0]$, so that $S_n^*(x) \leq 1$ holds for all real x. A proof for (iii) is published in [8].

Lemma 2. *Let* n, ν, *and* μ *be integers such that* $1 \leq \nu < \mu \leq n/2$, *and let*

$$P_n(t) = \frac{t}{2n} \cot \frac{t}{2n} \quad (0 < t < 2n\pi). \tag{2.3}$$

Then

$$\int_{2\nu\pi}^{2\mu\pi} P_n(t) \frac{\cos(t)}{t} dt \leq \int_{2\nu\pi}^{2\nu\pi+\pi/2} P_n(t) \frac{\cos(t)}{t} dt. \tag{2.4}$$

Proof. We define

$$L_n(\nu, \mu) = \int_{2\nu\pi}^{2\mu\pi} P_n(t) \frac{\cos(t)}{t} dt,$$
$$M_n(\nu) = \int_{2\nu\pi}^{2\nu\pi+\pi/2} P_n(t) \frac{\cos(t)}{t} dt.$$

Then we get

$$\begin{aligned} L_n(\nu, \mu) - M_n(\nu) &= \int_{2\nu\pi+\pi/2}^{2\mu\pi} P_n(t) \frac{\cos(t)}{t} dt \\ &= \sum_{j=0}^{\mu-\nu-2} \Psi_n(j, \nu) + \Phi_n(\mu), \end{aligned} \tag{2.5}$$

where

$$\Psi_n(j, \nu) = \int_{2\nu\pi+(4j+1)\pi/2}^{2\nu\pi+(4j+5)\pi/2} P_n(t) \frac{\cos(t)}{t} dt,$$
$$\Phi_n(\mu) = \int_{(2\mu-3/2)\pi}^{2\mu\pi} P_n(t) \frac{\cos(t)}{t} dt.$$

Let j be an integer with $0 \le j \le \mu - \nu - 2$. Since P_n is decreasing on $(0, 2n\pi)$ and non-negative on $(0, n\pi]$, we obtain

$$\Psi_n(j,\nu) = \int_{2(\nu+j)\pi+3\pi/2}^{2(\nu+j)\pi+5\pi/2} \left(P_n(t)\frac{\cos(t)}{t} - P_n(t-\pi)\frac{\cos(t)}{t-\pi}\right) dt \tag{2.6}$$
$$\le \int_{2(\nu+j)\pi+3\pi/2}^{2(\nu+j)\pi+5\pi/2} \left(P_n(t) - P_n(t-\pi)\right) \frac{\cos(t)}{t} dt \le 0$$

and

$$\Phi_n(\mu) = \int_{(2\mu-3/2)\pi}^{(2\mu-1)\pi} \left(\frac{P_n(t)}{t} - \frac{P_n(t+\pi)}{t+\pi}\right) \cos(t)dt \tag{2.7}$$
$$+ \int_{(2\mu-1)\pi}^{(2\mu-1/2)\pi} P_n(t)\frac{\cos(t)}{t} dt \le 0.$$

From (2.5)–(2.7) we conclude that (2.4) is valid. □

Lemma 3. *Let*

$$Q_n(t) = \frac{1}{(2n+1)\sin(t/(2n+1))} \quad (0 < t < (2n+1)\pi). \tag{2.8}$$

(i) *Let n, ν^*, and μ be integers such that $2 \le \nu^* < (n+1)/2$, $1 \le \mu < n/2$, and $\nu^*/(n+1) < \mu/n < 1 - \nu^*/(n+1)$. Then we have*

$$\int_{2\nu^*\pi-\nu^*\pi/(n+1)}^{2\mu\pi+\mu\pi/n} Q_n(t)\cos(t)dt \le \int_{2\nu^*\pi-\pi/2}^{2\nu^*\pi+\pi/2} Q_n(t)\cos(t)dt. \tag{2.9}$$

(ii) *If $n \ge 5$, then*

$$\frac{2\pi}{n} Q_n\left(2\pi - \frac{\pi}{n}\right) < \frac{1}{4}. \tag{2.10}$$

(iii) *If $n \ge 7$, then*

$$\left(1 + \sin\frac{\pi}{n+1}\right) Q_n\left(2\pi - \frac{\pi}{n+1}\right) < \frac{1}{4}. \tag{2.11}$$

Proof. (i) We define

$$\rho_n(\nu^*,\mu) = \int_{2\nu^*\pi-\nu^*\pi/(n+1)}^{2\mu\pi+\mu\pi/n} Q_n(t)\cos(t)dt,$$
$$\sigma_n(\nu^*,\mu) = \int_{2\nu^*\pi-\pi/2}^{2\mu\pi+\pi/2} Q_n(t)\cos(t)dt,$$

and

$$\tau_n(\nu^*) = \int_{2\nu^*\pi-\pi/2}^{2\nu^*\pi+\pi/2} Q_n(t)\cos(t)dt.$$

If $2\nu^*\pi - \pi/2 \le t \le 2\nu^*\pi - \nu^*\pi/(n+1)$ or if $2\mu\pi + \mu\pi/n \le t \le 2\mu\pi + \pi/2$, then $Q_n(t)\cos(t)$ is positive. This implies

(2.12)
$$\sigma_n(\nu^*,\mu) - \rho_n(\nu^*,\mu) = \left(\int_{2\nu^*\pi-\pi/2}^{2\nu^*\pi-\nu^*\pi/(n+1)} + \int_{2\mu\pi+\mu\pi/n}^{2\mu\pi+\pi/2}\right) Q_n(t)\cos(t)dt \ge 0.$$

Since $\nu^* < \mu + \mu/n < \mu + 1/2$, we obtain $\nu^* \le \mu$. If $\nu^* = \mu$, then $\tau_n(\nu^*) = \sigma_n(\nu^*,\mu)$.

Next, let $\nu^* \le \mu - 1$. Then we get

(2.13)
$$\begin{aligned}\tau_n(\nu^*) - \sigma_n(\nu^*,\mu) &= -\int_{2\nu^*\pi+\pi/2}^{2\mu\pi+\pi/2} Q_n(t)\cos(t)dt \\ &= -\sum_{j=\nu^*}^{\mu-1}\left(\int_{(2j+1/2)\pi}^{(2j+3/2)\pi} + \int_{(2j+3/2)\pi}^{(2j+5/2)\pi}\right) Q_n(t)\cos(t)dt \\ &= \frac{1}{2n+1}\sum_{j=\nu^*}^{\mu-1}\int_{(2j+1/2)\pi}^{(2j+3/2)\pi}\cos(t)\left(\frac{1}{\sin((t+\pi)/(2n+1))}\right. \\ &\qquad \left. -\frac{1}{\sin(t/(2n+1))}\right)dt \ge 0.\end{aligned}$$

From (2.12) and (2.13) we conclude that (2.9) holds.

(ii) The function $\phi(x) = \sin(9x) - 8x$ is strictly concave on $[0,\pi/55]$ with $\phi(0) = 0$ and $\phi(\pi/55) = 0.034\ldots$. Thus, if $x = \pi/(n(2n+1))$ with $n \ge 5$, then

$$\sin\frac{(2n-1)\pi}{n(2n+1)} \ge \sin\frac{9\pi}{n(2n+1)} > \frac{8\pi}{n(2n+1)}.$$

This implies (2.10).

(iii) We set $x = 1/(n+1)$ with $0 < x \le 1/8$. Then (2.11) is equivalent to

$$0 < \frac{\sin(\pi x)}{\pi x} - \frac{4}{\pi(2-5x)} = w(x), \quad \text{say}.$$

Since w is decreasing on $(0,1/8]$, we obtain $w(x) \ge w(1/8) = 0.048\ldots$. □

Lemma 4. *Let n, ν, ν^*, μ, and μ^* be integers such that $n \ge 3$, $1 \le \nu < n/2$, $1 \le \nu^* < (n+1)/2$, $1 \le \mu \le n-1$, and $1 \le \mu^* \le n$.*

(i) *If $\nu < \mu < n - \nu$, then*

$$\frac{1}{4} + \sum_{k=1}^{n}\frac{1}{k}\left[\cos\left(\frac{2\nu\pi}{n}k\right) - \cos\left(\frac{2\mu\pi}{n}k\right)\right] > 0.$$

(ii) *If* $\nu/n < \mu^*/(n+1) < 1 - \nu/n$, *then*

$$\frac{1}{4} + \sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu\pi}{n}k\right) - \cos\left(\frac{2\mu^*\pi}{n+1}k\right)\right] > 0.$$

(iii) *If* $\nu^*/(n+1) < \mu/n < 1 - \nu^*/(n+1)$, *then*

$$\frac{1}{4} + \sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu^*\pi}{n+1}k\right) - \cos\left(\frac{2\mu\pi}{n}k\right)\right] > 0.$$

(iv) *If* $\nu^* < \mu^* < n + 1 - \nu^*$, *then*

$$\frac{1}{4} + \sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu^*\pi}{n+1}k\right) - \cos\left(\frac{2\mu^*\pi}{n+1}k\right)\right] > 0.$$

Proof. (i) Let $x = 2\mu\pi/n$, $y = 2\nu\pi/n$, and let $T_n(x)$ be the sum defined in (2.2). We have to show that

$$\frac{1}{4} + T_n(y) - T_n(x) > 0. \tag{2.14}$$

If $\mu = n/2$, then n is even and $n \geq 4$. Applying Lemma 1 (iii) we get

$$\frac{1}{4} + T_n(y) - T_n(x) = \frac{1}{4} + T_n(y) - \sum_{k=1}^{n} \frac{(-1)^k}{k} \geq \frac{5}{6} + T_n(y) > 0.$$

If $n/2 < \mu \leq n - 1$, then we set $\tilde{\mu} = n - \mu$ and $\tilde{x} = 2\tilde{\mu}\pi/n$. This leads to $1 \leq \tilde{\mu} < n/2$, $\nu < \tilde{\mu} < n - \nu$, and $T_n(x) = T_n(\tilde{x})$. Hence it suffices to assume that $1 \leq \mu < n/2$. We obtain

(2.15)

$$\begin{aligned} T_n(y) - T_n(x) &= \log(\sin(x/2)) - \log(\sin(y/2)) - \int_y^x \frac{\cos((n+1/2)t)}{2\sin(t/2)}dt \\ &\geq -\int_y^x \frac{\cos((n+1/2)t)}{2\sin(t/2)}dt = -\int_{2\nu\pi}^{2\mu\pi} P_n(t)\frac{\cos(t)}{t}dt, \end{aligned}$$

where $P_n(t)$ is defined in (2.3). Using Lemma 2 and $P_n(s) < 1$ for $0 < s < 2n\pi$ we get

$$\begin{aligned} \int_{2\nu\pi}^{2\mu\pi} P_n(t)\frac{\cos(t)}{t}dt &\leq \int_{2\nu\pi}^{2\nu\pi+\pi/2} P_n(t)\frac{\cos(t)}{t}dt \\ &= \int_0^{\pi/2} P_n(t+2\nu\pi)\frac{\cos(t)}{t+2\nu\pi}dt \\ &\leq \int_0^{\pi/2} \frac{\cos(t)}{t+2\pi}dt = 0.146\ldots < \frac{1}{4}. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we conclude that (2.14) holds.

(ii) Let $x' = 2\mu^*\pi/(n+1)$ and $y = 2\nu\pi/n$. We have to prove that

$$\frac{1}{4} + T_n(y) - T_n(x') > 0. \tag{2.17}$$

As in (i) it suffices to establish (2.17) for $1 \le \mu^* < (n+1)/2$. Let $\bar{x} = 2\mu^*\pi/n$. If $x' \le t \le \bar{x}$, then $0 < t/2 \le \pi/2$ and $2\mu^*\pi - \pi/2 \le (n+1/2)t \le 2\mu^*\pi + \pi/2$. This yields

$$\int_{x'}^{\bar{x}} \frac{\cos((n+1/2)t)}{2\sin(t/2)} dt \ge 0. \tag{2.18}$$

Using (2.18) we get

$$\begin{aligned} T_n(y) - T_n(x') &\ge -\int_y^{x'} \frac{\cos((n+1/2)t)}{2\sin(t/2)} dt \\ &= \left(-\int_y^{\bar{x}} + \int_{x'}^{\bar{x}}\right) \frac{\cos((n+1/2)t)}{2\sin(t/2)} dt \\ &\ge -\int_y^{\bar{x}} \frac{\cos((n+1/2)t)}{2\sin(t/2)} dt > -\frac{1}{4}. \end{aligned}$$

(iii) We may assume that $1 \le \mu < n/2$. We show that

$$\frac{1}{4} + T_n(y') - T_n(x) > 0, \tag{2.19}$$

where $x = 2\mu\pi/n$ and $y' = 2\nu^*\pi/(n+1)$. Let $Q_n(t)$ be defined in (2.8). As above we get

$$T_n(y') - T_n(x) \ge -\int_{y'}^{x} \frac{\cos((n+1/2)t)}{2\sin(t/2)} dt = -\int_a^b Q_n(t)\cos(t) dt,$$

where $a = 2\nu^*\pi - \nu^*\pi/(n+1)$ and $b = 2\mu\pi + \mu\pi/n$. Since $\nu^*/(n+1) < \mu/n$ and $1 \le \mu < n/2$, it follows that $\nu^* < \mu + 1/2$, that is, $\nu^* \le \mu$. Next, we consider two cases.

Case 1. $\nu^* = \mu$.

If $\nu^* = 1$, then we get

$$\frac{1}{4} + T_3(y') - T_3(x) = \frac{1}{6}, \quad \frac{1}{4} + T_4(y') - T_4(x) = \frac{1}{48}(5\sqrt{5} - 1).$$

Let $n \ge 5$. Applying (2.10) we obtain

$$\begin{aligned} \int_a^b Q_n(t)\cos(t) dt &< \int_{2\pi-\pi/n}^{2\pi+\pi/n} Q_n(t)\cos(t) dt = \int_{-\pi/n}^{\pi/n} Q_n(t+2\pi)\cos(t) dt \\ &\le \int_{-\pi/n}^{\pi/n} Q_n(t+2\pi) dt \le \frac{2\pi}{n} Q_n(2\pi - \pi/n) < \frac{1}{4}. \end{aligned}$$

Thus, (2.19) holds.

Now, we assume that $\nu^* \geq 2$. Then $n \geq 5$. From (2.9) we get

$$\int_a^b Q_n(t)\cos(t)dt \leq \int_{2\nu^*\pi-\pi/2}^{2\nu^*\pi+\pi/2} Q_n(t)\cos(t)dt = \int_{-\pi/2}^{\pi/2} Q_n(t+2\nu^*\pi)\cos(t)dt$$
$$\leq 2Q_n(2\nu^*\pi - \pi/2) \leq 2Q_n(7\pi/2) < \frac{1}{4}.$$

This settles Case 1.

Case 2. $\nu^* < \mu$.

First, let $\nu^* = 1$. Then $n \geq 5$ and we obtain

$$\int_a^b Q_n(t)\cos(t)dt < \int_{2\pi-\pi/(n+1)}^{2\mu\pi+\pi/2} Q_n(t)\cos(t)dt$$
$$\leq \int_{2\pi-\pi/(n+1)}^{5\pi/2} Q_n(t)\cos(t)dt = J_n, \quad \text{say.}$$

We have $J_5 = 0.243\ldots$ and $J_6 = 0.227\ldots$. Let $n \geq 7$. Then (2.11) yields

$$J_n = \int_{-\pi/(n+1)}^{\pi/2} Q_n(t+2\pi)\cos(t)dt$$
$$\leq \left(1 + \sin\frac{\pi}{n+1}\right) Q_n\left(2\pi - \frac{\pi}{n+1}\right) < \frac{1}{4}.$$

Next, let $\nu^* \geq 2$. Using (2.9) gives

$$\int_a^b Q_n(t)\cos(t)dt \leq \int_{2\nu^*\pi-\pi/2}^{2\nu^*\pi+\pi/2} Q_n(t)\cos(t)dt,$$

so that this reduces to Case 1.

(iv) Let $y' = 2\nu^*\pi/(n+1)$ and $x' = 2\mu^*\pi/(n+1)$. Since $1 \leq \nu^* < (n+1)/2$, $1 \leq \mu^* \leq n$, and $\nu^* < \mu^* < n+1-\nu^*$, we conclude from (i):

$$\frac{1}{4} + T_n(y') - T_n(x') = \frac{1}{4} + T_{n+1}(y') - T_{n+1}(x') > 0.$$

This completes the proof of Lemma 4. □

Lemma 5. *The function*

$$f(x) = \frac{1}{\sin(x)} - \frac{1}{x} \tag{2.20}$$

is strictly absolutely monotonic on $(0,\pi)$*, that is, we have* $f^{(n)}(x) > 0$ *for all* $x \in (0,\pi)$ *and* $n = 0,1,2,\ldots$.

Proof. The series representation

$$f(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(4^k-2)B_{2k}}{(2k)!} x^{2k-1} \quad (|x| < \pi),$$

where $B_0, B_2, B_4, \dots$ are Bernoulli numbers (see [16, p. 209]), and the inequality $(-1)^{k-1}B_{2k} > 0$ for $k = 1, 2, \dots$ imply that $f^{(n)}(x) > 0$ for $0 < x < \pi$ and $n = 0, 1, 2, \dots$. □

LEMMA 6.

(i) *Let f be the function defined in* (2.20). *The sequence $n \mapsto f((1 - 2/n)\pi)/n$ is strictly increasing for $n \geq 3$ and converges to $1/(2\pi)$ as $n \to \infty$.*

(ii) *The function*

$$g(t) = \frac{\pi}{2t^2(\sin(2\pi/t))^2} + \frac{2}{\pi(2t-4)^2} + \frac{1}{t\sin(3\pi/t)} \tag{2.21}$$

is strictly decreasing on $(3, \infty)$.

Proof. (i) We define for $x \in (0, 1/2)$

$$h(x) = xf(\pi(1-2x)) = \frac{x}{\sin(2\pi x)} - \frac{x}{\pi(1-2x)}. \tag{2.22}$$

Differentiation gives

$$h'(x) = u(x) - v(x),$$

where

$$u(x) = \frac{\sin(2\pi x) - 2\pi x\cos(2\pi x)}{(\sin(2\pi x))^2},$$

$$v(x) = \frac{1}{\pi(1-2x)^2}.$$

We have

$$\frac{(\sin(2\pi x))^3}{2\pi}u'(x) = 3\pi x + \pi x\cos(4\pi x) - \sin(4\pi x).$$

A short calculation yields $3t/4 + (t/4)\cos(t) - \sin(t) > 0$ for $t \in (0, \pi]$, so that u' is positive on $(0, 1/4]$. Thus, u and v are increasing on $(0, 1/4]$. This implies for $x \in (0, 1/8]$

$$h'(x) \leq u(1/8) - v(0) = -0.014\dots.$$

Since $h(1/(k+1)) > h(1/k)$ for $k = 3, \dots, 7$, we conclude that the sequence $n \mapsto h(1/n)$ is strictly increasing for $n \geq 3$. Moreover, from (2.22) we get $\lim_{n\to\infty} h(1/n) = 1/(2\pi)$.

(ii) Let $t > 3$ be a real number. We set $s = \pi/t$. Differentiation gives

$$g'(t) = \alpha(t) + \beta(t),$$

where

$$\alpha(t) = -\left(\frac{s}{\pi\sin(3s)}\right)^2 (\sin(3s) - 3s\cos(3s)),$$

$$\beta(t) = -\frac{s^3}{\pi^2(\sin(2s))^3}\left(\sin(2s) - 2s\cos(2s) + \pi\left(\frac{\sin(2s)}{\pi - 2s}\right)^3\right).$$

Since $\sin(x) - x\cos(x) > 0$ for $0 < x < \pi$, we get $g'(t) < 0$. □

LEMMA 7. *Let ν and μ be positive integers with $\nu < \mu$ and let*

$$I(\nu,\mu) = \int_{\nu\pi}^{\mu\pi} \frac{\sin(t)}{t} dt. \tag{2.23}$$

(i) *If ν is even and μ is odd, then $I(\nu,\mu) > 0$.*
(ii) *If ν and μ are odd, then $I(\nu,\mu) > \int_{\nu\pi}^{(\nu+1)\pi} \sin(t)/t\, dt$.*
(iii) *If ν and μ are even, then $I(\nu,\mu) > 0$.*

Proof. (i) Let $\nu = 2K$, $\mu = 2M+1$, where $K \le M$. Then we obtain

$$I(\nu,\mu) = \sum_{j=K}^{M-1}\left(\int_{2j\pi}^{(2j+1)\pi} + \int_{(2j+1)\pi}^{(2j+2)\pi}\right)\frac{\sin(t)}{t}dt + \int_{2M\pi}^{(2M+1)\pi}\frac{\sin(t)}{t}dt.$$

Since

$$\int_{(2j+1)\pi}^{(2j+2)\pi}\frac{\sin(t)}{t}dt = -\int_{2j\pi}^{(2j+1)\pi}\frac{\sin(t)}{\pi+t}dt,$$

we get

$$I(\nu,\mu) = \pi\sum_{j=K}^{M-1}\int_{2j\pi}^{(2j+1)\pi}\frac{\sin(t)}{t(\pi+t)}dt + \int_{2M\pi}^{(2M+1)\pi}\frac{\sin(t)}{t}dt > 0.$$

(ii) Applying part (i) we obtain

$$I(\nu,\mu) - \int_{\nu\pi}^{(\nu+1)\pi}\frac{\sin(t)}{t}dt = I(\nu+1,\mu) > 0.$$

(iii) Let $\nu = 2K$, $\mu = 2M$, where $K+1 \le M$. Then we get

$$I(\nu,\mu) = \pi\sum_{j=K}^{M-1}\int_{2j\pi}^{(2j+1)\pi}\frac{\sin(t)}{t(\pi+t)}dt > 0.$$ □

LEMMA 8. *Let n, ν, and μ be integers such that $n \ge 3$ and $1 \le \nu < \mu < 2n$. Further, let*

$$R_n(\nu,\mu) = \frac{1}{2n}\int_{\nu\pi}^{\mu\pi} f(t/(2n))\sin(t)dt, \tag{2.24}$$

where f is defined in (2.20).

(i) *If ν is even and μ is odd, then $R_n(\nu,\mu) > 0$.*
(ii) *If ν and μ are odd, then $R_n(\nu,\mu) > 0$.*
(iii) *If ν and μ are even, then*

$$R_n(\nu,\mu) > \frac{1}{2n}\int_{(\mu-1)\pi}^{\mu\pi} f(t/(2n))\sin(t)dt.$$

Proof. (i) Let $\nu = 2K$, $\mu = 2M+1$, where $K \le M$. Then we obtain

(2.25)
$$\begin{aligned} 2nR_n(\nu,\mu) &= \int_{2K\pi}^{(2K+1)\pi} f(t/(2n))\sin(t)dt \\ &\quad + \sum_{j=K}^{M-1}\left(\int_{(2j+1)\pi}^{(2j+2)\pi} + \int_{(2j+2)\pi}^{(2j+3)\pi}\right) f(t/(2n))\sin(t)dt \\ &= \int_{2K\pi}^{(2K+1)\pi} f(t/(2n))\sin(t)dt \\ &\quad + \sum_{j=K}^{M-1}\int_{(2j+1)\pi}^{(2j+2)\pi}[f(t/(2n)) - f((\pi+t)/(2n))]\sin(t)dt. \end{aligned}$$

Applying Lemma 5 we conclude from (2.25) that $R_n(\nu,\mu)$ is positive.

(ii) Let $\nu = 2K+1$, $\mu = 2M+1$, where $K+1 \le M$. From Lemma 5 we get

$$2nR_n(\nu,\mu) = \sum_{j=K}^{M-1}\int_{(2j+1)\pi}^{(2j+2)\pi}[f(t/(2n)) - f((\pi+t)/(2n))]\sin(t)dt > 0.$$

(iii) Applying part (i) we obtain

$$2nR_n(\nu,\mu) - \int_{(\mu-1)\pi}^{\mu\pi} f(t/(2n))\sin(t)dt = 2nR_n(\nu,\mu-1) > 0. \qquad \square$$

Lemma 9. *Let n, ν, and μ be integers such that $n \ge 3$ and $1 \le \nu < \mu < 2n-\nu$. Then we have*

(2.26)
$$\frac{2}{3}(\sqrt{2}-1) + \sum_{k=1}^{n}\frac{1}{2k-1}\left[\sin\left((2k-1)\frac{\mu\pi}{2n}\right) - \sin\left((2k-1)\frac{\nu\pi}{2n}\right)\right] > 0.$$

Proof. We denote the sum in (2.26) by $\Delta_n(\nu,\mu)$. We let f, I, and R_n be the functions defined in (2.20), (2.23), and (2.24), respectively, and set $x = \mu\pi/(2n)$ and $y = \nu\pi/(2n)$. Using the identity

$$\sum_{k=1}^{n}\frac{\sin((2k-1)x)}{2k-1} = \frac{1}{2}\int_0^x \frac{\sin(2nt)}{\sin(t)}dt \quad (x \in \mathbf{R})$$

we obtain

$$\Delta_n(\nu,\mu) = \frac{1}{2}\int_y^x \frac{\sin(2nt)}{\sin(t)}dt = \frac{1}{2}\int_y^x \frac{\sin(2nt)}{t}dt + \frac{1}{2}\int_y^x f(t)\sin(2nt)dt$$
$$= \frac{1}{2}\int_{\nu\pi}^{\mu\pi} \frac{\sin(t)}{t}dt + \frac{1}{4n}\int_{\nu\pi}^{\mu\pi} f(t/(2n))\sin(t)dt.$$

Hence we have

$$\Delta_n(\nu,\mu) = \frac{1}{2}(I(\nu,\mu) + R_n(\nu,\mu)). \tag{2.27}$$

In order to prove (2.26) we distinguish four cases.

Case 1. *ν is even and μ is odd.*

Using Lemma 7 (i) and Lemma 8 (i) we obtain that $I(\nu,\mu)$ and $R_n(\nu,\mu)$ are positive. Thus, (2.27) implies $\Delta_n(\nu,\mu) > 0$.

Case 2. *ν and μ are odd.*

Applying Lemma 7 (ii), Lemma 8 (ii), and (2.27) we get

$$\Delta_n(\nu,\mu) > \frac{1}{2}I(\nu,\mu) > -\frac{1}{2}\int_0^\pi \frac{\sin(t)}{\nu\pi + t}dt \geq -\frac{1}{2}\int_0^\pi \frac{\sin(t)}{\pi + t}dt$$
$$= -0.216\ldots > -0.276\ldots = -\frac{2}{3}(\sqrt{2}-1).$$

Case 3. *ν and μ are even.*

From Lemma 8 (iii) and Lemma 5 we obtain

$$R_n(\nu,\mu) > -\frac{1}{2n}\int_0^\pi f\left(\frac{t+(\mu-1)\pi}{2n}\right)\sin(t)dt \tag{2.28}$$
$$\geq -\frac{1}{n}f\left(\frac{\mu\pi}{2n}\right) \geq -\frac{1}{n}f\left(\left(1-\frac{2}{\pi}\right)\pi\right).$$

Applying Lemma 6 (i) and (2.28) we get

$$R_n(\nu,\mu) > -\frac{1}{2\pi}. \tag{2.29}$$

Using Lemma 7 (iii), (2.27), and (2.29) we have

$$\Delta_n(\nu,\mu) > -\frac{1}{4\pi} = -0.079\ldots > -\frac{2}{3}(\sqrt{2}-1).$$

Case 4. *ν is odd and μ is even.*

First, let $n = 3$. Then $\nu = 1$ and $\mu = 2$ or $\mu = 4$. A direct computation yields

$$\Delta_3(1,2) = \Delta_3(1,4) = -0.240\ldots > -\frac{2}{3}(\sqrt{2}-1).$$

Next, we assume that $n \geq 4$ and we consider two subcases.

Case 4.1. *ν is odd and $\mu = \nu + 1$.*

We have

$$I(\nu,\mu) = -\int_0^\pi \frac{\sin(t)}{\nu\pi + t}dt \geq -\int_0^\pi \frac{\sin(t)}{\pi + t}dt = -0.433\ldots \tag{2.30}$$

and

$$\begin{aligned} R_n(\nu,\mu) &= -\frac{1}{2n}\int_0^\pi f\left(\frac{\nu\pi + t}{2n}\right)\sin(t)dt \\ &\geq -\frac{1}{n}f\left(\frac{(\nu+1)\pi}{2n}\right) \geq -\frac{1}{4}f(\pi/2) = -0.090\ldots. \end{aligned} \tag{2.31}$$

From (2.27), (2.30), and (2.31) we get

$$\Delta_n(\nu,\mu) \geq -0.2625 > -\frac{2}{3}(\sqrt{2}-1).$$

Case 4.2. *ν is odd and μ is even with $\mu \geq \nu + 3$.*

We have $\mu \leq 2n-(\nu+1)$. If $\mu = 2n-(\nu+1)$, then $\Delta_n(\nu,\mu) = \Delta_n(\nu,\nu+1)$, so that this case reduces to Case 4.1. Thus, it suffices to consider the case when $\mu \leq 2n-(\nu+3) \leq 2n-4$. Then we obtain

$$\begin{aligned} I(\nu,\mu) &\geq \int_{\nu\pi}^{(\nu+3)\pi} \frac{\sin(t)}{t}dt \\ &= \int_0^\pi \left(-\frac{1}{\nu\pi + t} + \frac{1}{(\nu+1)\pi + t} - \frac{1}{(\nu+2)\pi + t}\right)\sin(t)dt \\ &\geq \int_0^\pi \left(-\frac{1}{\pi + t} + \frac{1}{2\pi + t} - \frac{1}{3\pi + t}\right)\sin(t)dt = -0.359\ldots \end{aligned} \tag{2.32}$$

and

$$R_n(\nu,\mu) \geq \frac{1}{2n}\int_{(\mu-3)\pi}^{\mu\pi} f(t/(2n))\sin(t)dt = -\frac{1}{2n}\int_0^\pi K_n(t,\mu)\sin(t)dt, \tag{2.33}$$

where

$$\begin{aligned} K_n(t,\mu) = f((t+(\mu-3)\pi)/(2n)) &- f((t+(\mu-2)\pi)/(2n)) \\ &+ f((t+(\mu-1)\pi)/(2n)). \end{aligned}$$

Applying Lemma 5 we conclude that $K_n(t,\mu)$ is positive and strictly increasing with respect to t and μ. Hence (2.33) yields

(2.34)

$$\begin{aligned} R_n(\nu,\mu) &\geq -\frac{1}{n}K_n(\pi,\mu) \geq -\frac{1}{n}K_n(\pi, 2n-4) \\ &= -\frac{1}{n}\left[f\left(\frac{(2n-6)\pi}{2n}\right) - f\left(\frac{(2n-5)\pi}{2n}\right) + f\left(\frac{(2n-4)\pi}{2n}\right)\right]. \end{aligned}$$

By the mean-value theorem and Lemma 5 we get

$$f\left(\frac{(2n-4)\pi}{2n}\right) - f\left(\frac{(2n-5)\pi}{2n}\right) \le \frac{\pi}{2n} f'\left(\frac{(2n-4)\pi}{2n}\right).$$

Next, we define

$$a_n = \frac{\pi}{2n^2} f'\left(\frac{(2n-4)\pi}{2n}\right) = \frac{\pi\cos(2\pi/n)}{2n^2(\sin(2\pi/n))^2} + \frac{2}{\pi(2n-4)^2}$$

and

$$b_n = \frac{1}{n} f\left(\frac{(2n-6)\pi}{2n}\right) = \frac{1}{n\sin(3\pi/n)} - \frac{2}{\pi(2n-6)}.$$

Using this notation we obtain from (2.34)

$$R_n(\nu,\mu) \ge -(a_n+b_n) = -c_n, \quad \text{say.} \tag{2.35}$$

A direct calculation gives

$$c_4 = 0.075\ldots, \quad c_5 = 0.090\ldots, \quad c_6 = 0.099\ldots, \quad c_7 = 0.106\ldots. \tag{2.36}$$

From (2.27), (2.32), (2.35), and (2.36) we obtain that $\Delta_n(\nu,\mu) > -(2/3)(\sqrt{2} - 1)$ for $4 \le n \le 7$.

Suppose now that $n \ge 8$. Then we get $c_n \le g(n)$, where g is defined in (2.21). Applying Lemma 6 (ii) we obtain $c_n \le g(8) = 0.188\ldots$. This result combined with (2.27), (2.32), and (2.35) leads to

$$\Delta_n(\nu,\mu) \ge -0.2745 > -\frac{2}{3}(\sqrt{2}-1).$$

This completes the proof of Lemma 9. □

3. Main results

With the help of the lemmas proved in the previous section we are now in a position to present sharp bounds for the trigonometric sums given in (1.2) and (1.7). First, we complement Koschmieder's inequalities for $A_n(x,y)$ mentioned in Section 1.

THEOREM 1. *For all natural numbers n and real numbers $x, y \in [0,\pi]$ we have*

$$-\operatorname{Si}(\pi) < \sum_{k=1}^{n} \frac{\cos(kx)\sin(ky)}{k} < \operatorname{Si}(\pi). \tag{3.1}$$

Both bounds are best possible.

Proof. We have

$$A_n(x,y) = \frac{1}{2}\left(S_n(y-x) + S_n(x+y)\right),$$

where $S_n(x)$ is defined in (2.1). Applying Lemma 1 (i) it follows that (3.1) is valid. Furthermore, the limit relation

$$\lim_{n\to\infty} S_n\left(\frac{\pi}{n+1}\right) = \mathrm{Si}(\pi)$$

and the formulas

$$A_n\left(0, \frac{\pi}{n+1}\right) = S_n\left(\frac{\pi}{n+1}\right), \quad A_n\left(\pi, \frac{n\pi}{n+1}\right) = -S_n\left(\frac{\pi}{n+1}\right)$$

yield that the bounds in (3.1) are sharp. □

REMARK 1. The proof of Theorem 1 reveals that (3.1) holds for all real numbers x, y.

Now, we study the sum given in (1.7). The following analogue of (1.4) and (3.1) holds.

THEOREM 2. *For all natural numbers n and real numbers $x, y \in [0, \pi]$ we have*

$$-\frac{1}{8} \le \sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k}. \tag{3.2}$$

The equality sign holds in (3.2) *if and only if $n = 2$, $x = 5\pi/6$, $y = \pi/6$ or $n = 2$, $x = \pi/6$, $y = 5\pi/6$.*

Proof. We have $B_1(x, y) = \sin(x)\sin(y) \ge 0$. Let $n \ge 2$ and $M = \{(x, y) \in \mathbf{R}^2 |\, 0 \le y \le x \le \pi\}$. Since $B_n(t, t) \ge 0$ and $B_n(t, 0) = B_n(\pi, t) = 0$ for $t \in [0, \pi]$, we conclude that B_n attains only non-negative values on the boundary of M. Next, we assume that B_n attains its absolute minimum at (x_0, y_0), where $0 < y_0 < x_0 < \pi$. Let $\alpha = x_0 - y_0$ and $\beta = x_0 + y_0$. Then we obtain

$$0 = -\frac{\partial B_n(x,y)}{\partial x}\Big|_{(x,y)=(x_0,y_0)} + \frac{\partial B_n(x,y)}{\partial y}\Big|_{(x,y)=(x_0,y_0)} = \sum_{k=1}^{n} \sin(k\alpha) \tag{3.3}$$

and

$$0 = \frac{\partial B_n(x,y)}{\partial x}\Big|_{(x,y)=(x_0,y_0)} + \frac{\partial B_n(x,y)}{\partial y}\Big|_{(x,y)=(x_0,y_0)} = \sum_{k=1}^{n} \sin(k\beta). \tag{3.4}$$

Using the identity

$$\sum_{k=1}^{n} \sin(a + kb) = -\sin(a) + \frac{\sin(a + nb/2)\sin((n+1)b/2)}{\sin(b/2)} \tag{3.5}$$

as well as (3.3) and (3.4) we get

$$\alpha = \frac{2\nu\pi}{n} \quad (\nu \in \mathbf{N};\ 1 \le \nu < n/2) \text{ or}$$
$$\alpha = \frac{2\nu^*\pi}{n+1} \quad (\nu^* \in \mathbf{N};\ 1 \le \nu^* < (n+1)/2)$$

and

$$\beta = \frac{2\mu\pi}{n} \quad (\mu \in \mathbf{N};\ 1 \le \mu \le n-1) \text{ or}$$
$$\beta = \frac{2\mu^*\pi}{n+1} \quad (\mu^* \in \mathbf{N};\ 1 \le \mu^* \le n).$$

This leads to the following four cases:

(i) $x_0 = \frac{(\nu+\mu)\pi}{n}, \quad y_0 = \frac{(\mu-\nu)\pi}{n} \quad (\nu < \mu < n-\nu),$

(ii) $x_0 = \left(\frac{\nu}{n} + \frac{\mu^*}{n+1}\right)\pi, \quad y_0 = \left(\frac{\mu^*}{n+1} - \frac{\nu}{n}\right)\pi$
$(\nu/n < \mu^*/(n+1) < 1 - \nu/n),$

(iii) $x_0 = \left(\frac{\nu^*}{n+1} + \frac{\mu}{n}\right)\pi, \quad y_0 = \left(\frac{\mu}{n} - \frac{\nu^*}{n+1}\right)\pi$
$(\nu^*/(n+1) < \mu/n < 1 - \nu^*/(n+1)),$

(iv) $x_0 = \frac{(\nu^*+\mu^*)\pi}{n+1}, \quad y_0 = \frac{(\mu^*-\nu^*)\pi}{n+1} \quad (\nu^* < \mu^* < n+1-\nu^*).$

If $n = 2$, then only case (iii) holds and we get $x_0 = 5\pi/6$, $y_0 = \pi/6$ with $B_2(x_0, y_0) = -1/8$. Thus, it remains to show that if $n \ge 3$, then $1/8 + B_n(x_0, y_0) > 0$, where (x_0, y_0) is given in (i)–(iv). This means we have to establish that

$$\frac{1}{8} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu\pi}{n}k\right) - \cos\left(\frac{2\mu\pi}{n}k\right)\right] > 0, \tag{3.6}$$

$$\frac{1}{8} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu\pi}{n}k\right) - \cos\left(\frac{2\mu^*\pi}{n+1}k\right)\right] > 0, \tag{3.7}$$

$$\frac{1}{8} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu^*\pi}{n+1}k\right) - \cos\left(\frac{2\mu\pi}{n}k\right)\right] > 0, \tag{3.8}$$

and

$$\frac{1}{8} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\left[\cos\left(\frac{2\nu^*\pi}{n+1}k\right) - \cos\left(\frac{2\mu^*\pi}{n+1}k\right)\right] > 0. \tag{3.9}$$

Applying Lemma 4 we conclude that the inequalities (3.6)–(3.9) are valid. This completes the proof of Theorem 2. □

REMARK 2. Since $B_{2m+1}(\pi/2,\pi/2)=\sum_{k=0}^{m} 1/(2k+1)$, we conclude that there is no upper bound for $B_n(x,y)$, which is independent of n, x, and y.

Our third theorem presents sharp upper and lower bounds for Koschmieder's sum $A_n^*(x,y)$ defined in (1.2).

THEOREM 3. *For all natural numbers n and real numbers $x,y\in[0,\pi]$ we have*

$$-\frac{2}{3}(\sqrt{2}-1)\le \sum_{k=1}^{n}\frac{\cos((k-1/2)x)\sin((k-1/2)y)}{k-1/2}\le 2. \tag{3.10}$$

The equality sign holds in the left-hand inequality of (3.10) *if and only if $n=2$, $x=3\pi/4$, $y=\pi/4$, and in the right-hand inequality if and only if $n=1$, $x=0$, $y=\pi$.*

Proof. Let $S_n^*(x)$ be the expression defined in (2.1). Applying Lemma 1 (ii) we get

$$A_n^*(x,y)=S_n^*\left(\frac{y-x}{2}\right)+S_n^*\left(\frac{x+y}{2}\right)\le 2,$$

with equality if and only if $S_n^*((y-x)/2)=S_n^*((x+y)/2)=1$, which is true if and only if $n=1$, $x=0$, $y=\pi$.

Next, we establish the left-hand side of (3.10). We have

$$A_1^*(x,y)=2\cos(x/2)\sin(y/2)\ge 0 \quad (x,y\in[0,\pi]).$$

In [17] it is proved that $A_n^*(x,y)\ge 0$ if $0\le x\le y\le\pi$. We have $A_n^*(t,t)\ge 0$ and $A_n^*(t,0)=A_n^*(\pi,t)=0$ for $t\in[0,\pi]$. This implies that A_n^* attains only non-negative values on the boundary of $M=\{(x,y)\in\mathbf{R}^2 \,|\, 0\le y\le x\le\pi\}$. We assume that there exist numbers x_1,y_1 with $0<y_1<x_1<\pi$ such that $A_n^*(x,y)\ge A_n^*(x_1,y_1)$ for all $(x,y)\in M$. Let $\alpha^*=x_1-y_1$ and $\beta^*=x_1+y_1$. Then we get

$$\begin{aligned}0&=-\frac{\partial A_n^*(x,y)}{\partial x}\Big|_{(x,y)=(x_1,y_1)}+\frac{\partial A_n^*(x,y)}{\partial y}\Big|_{(x,y)=(x_1,y_1)}\\&=\sum_{k=1}^{n}\cos((k-1/2)\alpha^*)\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}0&=\frac{\partial A_n^*(x,y)}{\partial x}\Big|_{(x,y)=(x_1,y_1)}+\frac{\partial A_n^*(x,y)}{\partial y}\Big|_{(x,y)=(x_1,y_1)}\\&=\sum_{k=1}^{n}\cos((k-1/2)\beta^*).\end{aligned} \tag{3.12}$$

Applying

$$\sum_{k=1}^{n} \cos(a+kb) = -\cos(a) + \frac{\cos(a+nb/2)\sin((n+1)b/2)}{\sin(b/2)}$$

and

$$\cos((n-1)a)\sin((n+1)a) - \cos(a)\sin(a) = \frac{1}{2}\sin(2na)$$

we obtain from (3.11) and (3.12)

$$\sin(n\alpha^*) = 0, \quad \sin(n\beta^*) = 0.$$

Thus, we have

$$x_1 = (\nu+\mu)\frac{\pi}{2n}, \quad y_1 = (\mu-\nu)\frac{\pi}{2n},$$

where

$$\nu, \mu \in \mathbf{N}, \quad 1 \le \nu < \mu < 2n-\nu.$$

If $n = 2$, then we get $x_1 = 3\pi/4$, $y_1 = \pi/4$, and $A_2^*(x_1, y_1) = -(2/3)(\sqrt{2}-1)$. Applying Lemma 9 we obtain for $n \ge 3$

$$\begin{aligned} A_n^*(x_1, y_1) &= \sum_{k=1}^{n} \frac{1}{2k-1}\left[\sin\left((2k-1)\frac{\mu\pi}{2n}\right) - \sin\left((2k-1)\frac{\nu\pi}{2n}\right)\right] \\ &> -\frac{2}{3}(\sqrt{2}-1). \end{aligned}$$

This completes the proof of Theorem 3. □

Remark 3. The method of proof we have applied to establish Theorems 2 and 3 can also be used to prove the inequalities (1.4)–(1.6) and to cover all cases of equality. We obtain that the equality sign holds in (1.4) if and only if $n = 1$, $x = \pi$, $y = 0$ or $n = 1$, $x = 0$, $y = \pi$; in (1.5) if and only if $x = 0$ or $y = 0$; and in (1.6) if and only if $x = \pi$ or $y = \pi$. This reveals that the given lower bounds are sharp. We note that there are no constant upper bounds for the sums in (1.4)–(1.6).

Remark 4. If we replace in (3.10) the variable x by $\pi - x$, then we obtain a counterpart of (1.5):

For all natural numbers n and real numbers $x, y \in [0, \pi]$ we have

$$-\frac{2}{3}(\sqrt{2}-1) \le \sum_{k=1}^{n} (-1)^{k+1} \frac{\sin((k-1/2)x)\sin((k-1/2)y)}{k-1/2} \le 2. \tag{3.13}$$

The equality sign holds in the left-hand inequality of (3.13) *if and only if $n = 2$, $x = y = \pi/4$, and in the right-hand inequality if and only if $n = 1$, $x = y = \pi$.*

Similarly, from (1.4)–(1.6), (3.1), (3.2) we get further inequalities for alternating sums.

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