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# ON THE UNIQUENESS PROBLEM FOR CATALYTIC BRANCHING NETWORKS AND OTHER SINGULAR DIFFUSIONS

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*Dedicated to the memory of Joe Doob whose work and example inspired us both*

Abstract. Weak uniqueness is established for the martingale problem associated to a family of catalytic branching networks. This martingale problem corresponds to a stochastic differential equation with a degenerate Hölder continuous diffusion matrix. Our approach uses the semigroup perturbation method of Stroock and Varadhan and a modification of a Banach space of weighted Hölder continuous functions introduced by Bass and Perkins.

## 1. Introduction

**1.1. Catalytic branching networks.** Let $b_i, \gamma_i$, $i = 1, \ldots, d$, be Hölder continuous functions on $\mathbb{R}_+^d$ with $b_i(x) \geq 0$ if $x_i = 0$ and $\gamma_i(x) \geq 0$ for all $x$, $i = 1, \ldots, d$. We consider the operator $\mathcal{A}^{(b,\gamma)}$ on $C^2(\mathbb{R}_+^d)$ defined by

$$\mathcal{A}^{(b,\gamma)} f(x) = \sum_{i=1}^{d} \left( b_i(x) \frac{\partial f}{\partial x_i} + \gamma_i(x) x_i \frac{\partial^2 f}{\partial x_i^2} \right), \quad x \in \mathbb{R}_+^d.$$

The objective of this paper is to prove the uniqueness of solutions to the martingale problem for the operator $\mathcal{A}^{(b,\gamma)}$ under some regularity conditions on the coefficients. Uniqueness results of this type are proved in [ABBP] and [BP1] but they require the $\gamma_i(x)$ to be strictly positive in $\mathbb{R}_+^d$. The problem considered in this paper is the extension of these results to the case in which the $\gamma_i$ can degenerate on the boundary. Our work is largely motivated by models of catalytic branching networks that include catalytic branching, mutually catalytic branching and hypercyclic catalytic branching systems (see [DF] for a survey on these systems). For example, the hypercyclic catalytic

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branching model is a diffusion on $\mathbb{R}_+^d$, $d \geq 2$, solving the following system of stochastic differential equations:

$$dx_t^{(i)} = (\theta_i - x_t^{(i)})dt + \sqrt{2\gamma_i(x_t)x_t^{(i+1)}x_t^{(i)}}dB_t^i,\ i = 1, \ldots, d. \tag{1}$$

Here $x(t) = (x_t^{(1)}, \ldots, x_t^{(d)})$, addition of the superscripts is done cyclically so that $x_t^{(d+1)} = x_t^{(1)}$, $\theta_i > 0$, and $\gamma_i > 0$ satisfy some mild regularity conditions. This is a stochastic analogue of a system of ode's first proposed by Eigen and Schuster [ES] as a self-organizing system which models a macromolecular precursor to early forms of life. They noted that there is an apparent phase transition in the equilibrium behaviour of the system as you pass from 4 to 5 types (see [HS]). In (1) one has $d$ large populations in which the $(i+1)$st population catalyzes the branching of type $i$, that is, the branching rate of type $i$ is proportional to the mass of type $i+1$. Given that the original model was a precursor to a biological system consisting of a large number of self-replicating molecules, our use of Feller branching in place of an ode with the analogous catalytic structure is not at all unreasonable. For a discussion of a general class of catalytic networks based on directed graphs, see [JK].

For $d = 2$ spatially distributed systems of this type have been studied (Mytnik [M], Dawson and Perkins [DP1]) and uniqueness in law for $\gamma_i$ constant, even in infinite dimensional spatial settings, follows by a self-duality argument. Unfortunately this duality breaks down when there are $d > 2$ types and moment methods fail (cf. [DFX]) so that uniqueness was open even for $\gamma_i$ constant. Existence and some results on qualitative behaviour of solutions with more than two types in an infinite-dimensional spatial setting were derived by Fleischmann and Xiong [FX].

Even in the special case $d = 2$ mentioned above uniqueness remains open for non-constant $\gamma_i$. These diffusions arise in the renormalization analysis of Dawson, Greven, den Hollander, Sun and Swart [DGHSS] aimed at identifying the universality classes of catalytic branching and mutually catalytic branching.

In this work we consider a natural class of catalytic branching networks that includes the above examples as special cases and establish the uniqueness. To describe these networks we begin with a directed graph $(V, \mathcal{E})$ with vertices $V = \{1, \ldots, d\}$ and set of directed edges $\mathcal{E} = \{e_1, \ldots, e_k\}$. We assume throughout:

Hypothesis 1. *$(i,i) \notin \mathcal{E}$ for all $i \in V$ and each vertex is the second element of at most one edge.*

Vertices denote types and an edge $(i,j) \in \mathcal{E}$ indicates that type $i$ catalyzes the type $j$ branching. Let $C$ denote the set of vertices (catalysts) which appear as the 1st element of an edge and $R$ denote the set of vertices that appear as the 2nd element (reactants). Let $c : R \to C$ be such that for $j \in R$, $c_j$ denotes the unique $i \in C$ such that $(i,j) \in \mathcal{E}$ and for $i \in C$, let $R_i = \{j : (i,j) \in \mathcal{E}\}$.

We then consider the system of stochastic differential equations:

$$
\begin{aligned}
dx_t^{(j)} &= (\theta_j - x_t^{(j)})dt + \sqrt{2\gamma_j(x_t)x_t^{(i)}x_t^{(j)}}dB_t^j, \quad \text{if } j \in R_i, \\
dx_t^{(j)} &= (\theta_j - x_t^{(j)})dt + \sqrt{2\gamma_j(x_t)x_t^{(j)}}dB_t^j, \quad \text{if } j \notin R.
\end{aligned} \tag{2}
$$

Again $x_t = (x_t^{(1)}, \ldots, x_t^{(d)}) \in \mathbb{R}_+^d$, $\theta_i > 0$, and $\gamma_i > 0$ will satisfy some mild regularity. For $\{\gamma_i\}$ constant, $\{x_t^{(j)} : j \notin R\}$ is a $|R^c|$-dimensional Feller branching immigration process and for $i \in C$, $\{x_t^{(j)} : j \in R_i\}$ is a catalytic branching process with catalyst $x_t^{(i)}$ and with immigration. As for (1), uniqueness in (2) remained open as the additional degeneracy in the diffusion coefficient prevents one from applying the results of [ABBP] and [BP1]. Nonetheless we will be using refinements and modifications of the basic ideas in [BP1] and [ABP] in our proofs. Indeed a second motivation for this work came from wanting to see if these techniques can be adapted to different sorts of degeneracies. Our conclusion here is affirmative but not without some additional work which will depend on the particular setting.

**1.2. Statement of the main result.** To complete the description of the class of diffusions we consider we now state the main conditions on the coefficients of our equations. This will be in force unless otherwise indicated. $|x|$ is the Euclidean length of $x \in \mathbb{R}^d$.

HYPOTHESIS 2. *For $i \in V$,*

$$
\begin{aligned}
\gamma_i &: \mathbb{R}_+^d \to (0, \infty), \\
b_i &: \mathbb{R}_+^d \to \mathbb{R},
\end{aligned}
$$

*are Hölder continuous on compact subsets of $\mathbb{R}_+^d$ such that $|b_i(x)| \le c(1 + |x|)$ on $\mathbb{R}_+^d$, and $b_i(x) \ge 0$ if $x_i = 0$. In addition,*

$$
b_i(x) > 0 \text{ if } i \in C \cup R \text{ and } x_i = 0. \tag{3}
$$

For $f \in C_b^2(\mathbb{R}_+^d)$, let

$$
\begin{aligned}
\mathcal{A}f(x) &= \mathcal{A}^{(b,\gamma)}f(x) \\
&= \sum_{j \in R} \gamma_j(x) x_{c_j} x_j f_{jj}(x) + \sum_{j \notin R} \gamma_j(x) x_j f_{jj}(x) + \sum_{j \in V} b_j(x) f_j(x)
\end{aligned} \tag{4}
$$

Here $f_{ij}$ (or $f_{i,j}$ if there is any ambiguity) is the $i,j$th partial derivative of $f$.

DEFINITION 3. If $\nu$ is a probability on $\mathbb{R}_+^d$, a probability $P$ on $C(\mathbb{R}_+, \mathbb{R}_+^d)$ solves the martingale problem $MP(\mathcal{A}, \nu)$ if under $P$, the law of $x_0(\omega) = \omega_0$ is

$\nu$ and for all $f \in C_b^2(\mathbb{R}_+^d)$ $(x_t(\omega) = \omega(t))$,

$$M_f(t) = f(x_t) - f(x_0) - \int_0^t \mathcal{A} f(x_s)\, ds$$

is a local martingale under $P$ with respect to the canonical right-continuous filtration $(\mathcal{F}_t)$.

We will restrict the state space for our martingale problems. For $\mathcal{A}$ the state space will be

$$S = \{x \in \mathbb{R}_+^d : \prod_{(i,j)\in\mathcal{E}} (x_i + x_j) > 0\}.$$

We will see in Lemma 5 below that solutions to the martingale problem necessarily take values in $S$ at all positive times $t$ a.s. and so $S$ is a natural state space for $\mathcal{A}$.

THEOREM 4. *Assume Hypotheses 1 and 2 hold. Then for any probability, $\nu$, on $S$, there is exactly one solution to $MP(\mathcal{A}, \nu)$.*

**1.3. Outline of the proof.** The proof of Theorem 4 follows the Stroock-Varadhan perturbation method ([SV]) and is broken into a number of steps. Existence is proved as in Theorem 1.1 of [ABBP]. For existence, the non-degeneracy of the $\gamma_i$ assumed there is only used to ensure solutions remain in the positive orthant and here we may argue as in Lemma 5 below. The main issue is then uniqueness.

**Step 1.** A standard conditioning argument allows us to assume $\nu = \delta_x$. By using Krylov's Markov selection theorem (Theorem 12.2.4 of [SV]) and the proof of Proposition 2.1 in [ABBP], it suffices to consider uniqueness for families of strong Markov solutions.

We next observe that a solution never exits $S$.

LEMMA 5. *If $P$ is a solution of $MP(\mathcal{A}, \nu)$, where $\nu$ is a probability on $\mathbb{R}_+^d$, then $x_t \in S$ for all $t > 0$ $P$-a.s.*

The proof is deferred to Section 4.

**Step 2.** Using the localization argument of [SV] (see, e.g., the argument in [BP1]) it suffices to show that for each $x \in S$ there exists $r_0 > 0$ and coefficients $\tilde{b}_i$, $\tilde{\gamma}_i$ which agree with $b_i$ and $\gamma_i$, respectively, on $B(x, r_0) \cap \mathbb{R}_+^d$ and are such that there is at most one solution of $MP(\mathcal{A}^{(\tilde{b},\tilde{\gamma})})$.

In order to deal with the singular initial points, fix $x^0 \in S$, let $Z = Z(x^0) = \{i \in V : x_i^0 = 0\}$. Note that if $i \notin Z$, then $x_i^0 > 0$ and so $x_s^i > 0$ for small $s$ a.s.

Given $Z \subset V$, set

$$N_1 = \bigcup_{i \in C \cap Z} R_i;$$
$$\bar{N}_1 = (Z \cap C) \cup N_1;$$
$$N_2 = V \backslash \bar{N}_1.$$

Note that if $x^0 \in S$, then $N_1 \cap Z = \emptyset$.

We next recast $MP(\mathcal{A}, \delta_{x^0})$ with $x^0 \in S$ as a perturbation of a well-behaved diffusion on $S(x^0) = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i \notin N_1\}$ built from two independent families of processes associated to $\bar{N}_1$, and $N_2$, respectively.

First, for $i \in C \cap Z$, we can view $(\{x^{(j)}\}_{j \in R_i}, x^{(i)})$ near its initial point $(\{x_j^0\}_{j \in R_i}, x_i^0)$ as a perturbation of the diffusion on $\mathbb{R}^{|R_i|} \times \mathbb{R}_+$ which is given by the unique solution to the system of sde:

$$dx_t^{(j)} = b_j^0 dt + \sqrt{2\gamma_j^0 x_t^{(i)}} dB_t^{(j)}, \quad x_0^{(j)} = x_j^0, \quad \text{for } j \in R_i, \text{ and} \tag{5}$$
$$dx_t^{(i)} = b_i^0 dt + \sqrt{2\gamma_i^0 x_t^{(i)}} dB_t^{(i)}, \quad x_0^{(i)} = x_i^0,$$

where $b_j^0 = b_j(x^0) \in \mathbb{R}$, $\gamma_j^0 = \gamma_j(x^0) x_j^0 > 0$, and $b_i^0 = b_i(x^0) > 0$, $\gamma_i^0 = \gamma_i(x^0) x_{c_i}^0 > 0$ if $i \in R \cap Z$, or $b_i^0 = b_i(x^0) > 0$, $\gamma_i^0 = \gamma_i(x^0) > 0$ if $i \notin R$. Note that the non-negativity of $b_i^0$ ensures that solutions starting in $\{x_i^0 \geq 0\}$ remain there.

Secondly, for $j \in N_2$ we view this coordinate as a perturbation of the Feller branching process (with immigration)

$$dx_t^{(j)} = b_j^0 dt + \sqrt{2\gamma_j^0 x_t^{(j)}} dB_t^{(j)}, \quad x_0^{(j)} = x_j^0, \tag{6}$$

where $b_j^0 = (b_j(x^0) \vee 0)$, $\gamma_j^0 = \gamma_j(x^0) x_{c_j}^0 > 0$ if $j \in R$ or $\gamma_j^0 = \gamma_j(x^0) > 0$ if $j \notin R$.

We then view $\mathcal{A}$ as a perturbation of the generator

$$\mathcal{A}^0 = \sum_{i \in Z \cap C} \mathcal{A}_i^1 + \sum_{j \in N_2} \mathcal{A}_j^2, \tag{7}$$

where

$$\mathcal{A}_i^1 = \sum_{j \in R_i} \left\{ b_j^0 \frac{\partial}{\partial x_j} + \gamma_j^0 x_i \frac{\partial^2}{\partial x_j^2} \right\} + b_i^0 \frac{\partial}{\partial x_i} + \gamma_i^0 x_i \frac{\partial^2}{\partial x_i^2},$$
$$\mathcal{A}_j^2 = b_j^0 \frac{\partial}{\partial x_j} + \gamma_j^0 x_j \frac{\partial^2}{\partial x_j^2}.$$

One easily checks that the coefficients $b_i^0, \gamma_i^0$ given above from an $x^0 \in S$ satisfy

$$\gamma_j^0 > 0 \text{ all } j, \; b_j^0 \geq 0 \text{ if } j \notin N_1 = \bigcup_{i \in Z \cap C} R_i, b_j^0 > 0 \text{ if } j \in Z \cap (R \cup C), \tag{8}$$

where $Z \subset V$ satisfies

$$N_1 \cap Z = \emptyset. \tag{9}$$

In general we will always assume the above conditions when dealing with $\mathcal{A}^0$ whether or not it arises from a particular $x^0 \in S$ as above. The $\mathcal{A}^0$ martingale problem is then well-posed and the solution is a diffusion on

$$S_0 \equiv S(x^0) = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i \in V \backslash N_1\} \tag{10}$$

with semigroup $P_t$ and resolvent

$$R_\lambda f = \int_0^\infty e^{-\lambda s} P_s f ds. \tag{11}$$

The law of this diffusion is a product of independent catalytic processes and Feller branching processes (with immigration). More precisely we may write

$$P_t = \prod_{i \in Z \cap C} P_t^i \prod_{j \in N_2} P_t^j, \tag{12}$$

where for $i \in Z \cap C$, $P_t^i$ is the semigroup of solutions to (5) on bounded measurable functions on $\mathbb{R}^{|R_i|} \times \mathbb{R}_+$—we refer to this as the $m = (|R_i| + 1)$-dimensional catalytic semigroup. For $j \in N_2$, $P_t^j$ is the one-dimensional Feller branching semigroup of solutions to (6) on bounded measurable functions on $\mathbb{R}_+^{\{j\}}$.

**Step 3: A Key Estimate.** Set

$$\begin{aligned} \mathcal{B}f &:= (\mathcal{A} - \mathcal{A}^0) f \\ &= \sum_{j \in V} (\tilde{b}_j(x) - b_j^0) \frac{\partial f}{\partial x_j} \\ &\quad + \sum_{i \in Z \cap C} \left\{ \sum_{j \in R_i} \left[ (\tilde{\gamma}_j(x) - \gamma_j^0) x_i \frac{\partial^2 f}{\partial x_j^2} \right] + (\tilde{\gamma}_i(x) - \gamma_i^0) x_i \frac{\partial^2 f}{\partial x_i^2} \right\} \\ &\quad + \sum_{i \in N_2} \left\{ (\tilde{\gamma}_i(x) - \gamma_i^0) x_i \frac{\partial^2 f}{\partial x_i^2} \right\}, \end{aligned} \tag{13}$$

where for $j \in V$, $\tilde{b}_j(x) = b_j(x)$ for $j \in N_1$, $\tilde{\gamma}_j(x) = \gamma_j(x) x_j$, and for $i \in (Z \cap C) \cup N_2$, $\tilde{\gamma}_i(x) = 1_{i \in R} \gamma_i(x) x_{c_i} + 1_{i \notin R} \gamma_i(x)$.

By localization and continuity of the above coefficients we may assume that the coefficients preceding the derivatives of $f$ in the above operator are small, say less than $\eta$ in absolute value. The key step (see Proposition 36) will be to find a Banach space of continuous functions, depending on $\mathcal{A}^0$, with norm $\| \ \|$ so that for $\eta$ small enough and $\lambda_0 > 0$ large enough,

$$\|\mathcal{B} R_\lambda f\| \leq \frac{1}{2} \|f\| \quad \forall \ \lambda > \lambda_0. \tag{14}$$

Once this inequality is established the uniqueness of the resolvent of our strong Markov solution will follow as in [SV] and [BP1]. In the next section we describe the Banach space which will be used in (14).

**1.4. Weighted Hölder norms and semigroup norms.** In this subsection we introduce the basic Banach spaces of functions. Given the subsets $Z, C$ and $\{R_i, i \in Z \cap C\}$ of $V$ we define $N_1, N_2$ as above and the generator $\mathcal{A}^0$ as in (7). The state space for the diffusion with generator $\mathcal{A}^0$ is $S_0 := \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i \notin N_1\}$ and the corresponding semigroup is defined on $C_b(S_0)$. We next define the Banach subspace $C_w^\alpha(S_0)$ of $C_b(S_0)$ and some related norms. Note that in the localization argument used in the proof in Section 4 of Theorem 4, the set $Z = Z(x^0)$ will depend on a point $x^0 \in S$ and $S_0 = S(x^0)$.

Let $f : S_0 \to \mathbb{R}$ be bounded and measurable and $\alpha \in (0,1)$. For $i \in V$ let $\check{e}_i$ denote the unit vector in the $i$th direction, and

$$|f|_{C^\alpha,i} = \sup\left\{|f(x + h\check{e}_i) - f(x)||h|^{-\alpha} :\ h \neq 0,\ x \in S_0\right\},$$

and for $i \in Z \cap C$, let

$$\begin{aligned}|f|_{\alpha,i} = \sup\{&|f(x+h) - f(x)|(|h|^{-\alpha} x_i^{\alpha/2} \vee |h|^{-\alpha/2}) : \\ &h_i > 0,\ h_k = 0 \ \text{ if } k \notin \{i\} \cup R_i,\ x \in S_0\}.\end{aligned}$$

For $j \in N_2$, let

$$|f|_{\alpha,j} = \sup\{|f(x+h) - f(x)||h|^{-\alpha} x_j^{\alpha/2} : h_j > 0, h_k = 0 \text{ if } k \neq j, x \in S_0\}.$$

Set $I = (Z \cap C) \cup N_2$. Then let

$$|f|_{C_w^\alpha} = \max_{j \in I} |f|_{\alpha,j}, \quad |f|_{C^\alpha} = \sup_{i \leq d} |f|_{C^\alpha,i},$$
$$\|f\|_{C_w^\alpha} = |f|_{C_w^\alpha} + \|f\|_\infty, \quad \|f\|_{C^\alpha} = |f|_{C^\alpha} + \|f\|_\infty,$$

where $\|f\|_\infty$ is the supremum norm of $f$. We let

$$C_w^\alpha(S_0) = \{f \in C_b(S_0) : \|f\|_{C_w^\alpha} < \infty\}$$

denote the Banach space of weighted $\alpha$-Hölder continuous functions on $S_0$. We also denote by

$$C^\alpha(S_0) = \{f \in C_b(S_0) : \|f\|_{C^\alpha} < \infty\}$$

the classical Banach space of $\alpha$-Hölder continuous functions on $S_0$.

$C_w^\alpha(S_0)$ will be the Banach space we use in (14) above. It is a modification of the weighted Hölder norm used in [BP1].

In proving (14) most of the work will go into analyzing the semigroups $P_t^i$ in (12) for $i \in Z \cap C$ on its state space $\mathbb{R}^{|R_i|} \times \mathbb{R}_+$. In this context a special role will be played by another norm which we first define in a general context.

Given a Markov semigroup $\{P_t\}$ on the bounded Borel functions on $D$, where $D \subset \mathbb{R}^d$ and $\alpha \in (0,1)$ the *semigroup norm* (cf. [ABP]) is defined via

$$|f|_\alpha = \sup_{t>0} \frac{\|P_t f - f\|_\infty}{t^{\alpha/2}},$$
$$\|f\|_\alpha = |f|_\alpha + \|f\|_\infty.$$

The associated Banach space of functions is

$$S^\alpha = \{f : D \to \mathbb{R} : f \text{ Borel},\ \|f\|_\alpha < \infty\}.$$

We will use this norm for the catalytic semigroups $P_t^i$, $i \in Z \cap C$, from (12) and in fact show (Theorem 19 below) that it is equivalent to the weighted Hölder norm $\|\ \|_{C_w^\alpha}$ in this $(|R_i|+1)$-dimensional context. This equivalence, which plays an important role in our proofs, is patterned after a similar result in [ABP], where the semigroup in question is a product of independent Feller branching processes.

We first obtain bounds on the supnorm of the appropriate first and second order differential operators applied to $P_t^i f$. These bounds are singular and non-integrable in $t$ as $t \downarrow 0$ (see the first sets of inequalities in Propositions 16 and 17). The semigroup norm allows us to easily obtain bounds on the same quantities in terms of $|f|_\alpha$, now with an improved and integrable singularity at $t = 0$ (see the second set of inequalities in the same Propositions). The simple proof of this improvement, given in Proposition 16 below, is taken from [ABP]. A similar reduction of singularities holds for the Hölder norms of the same functions—see again the improvements from the first set of inequalities to the second set in Propositions 22 and 23 below.

Convention 1. *Throughout this paper all constants appearing in statements of results may depend on a fixed parameter $\alpha \in (0,1)$ and $\{b_j^0, \gamma_j^0 : j \in V\}$. By* (8)

$$M^0 = M^0(\gamma^0, b^0) \equiv \max_{i \in V} (\gamma_i^0 \vee (\gamma_i^0)^{-1} \vee |b_i^0|) \vee \max_{i \in Z \cap (R \cup C)} (b_i^0)^{-1} < \infty. \tag{15}$$

*Given $\alpha \in (0,1)$ and $0 < M < \infty$, we can, and shall, choose the constants to hold uniformly for all coefficients satisfying $M^0 \le M$.*

In order to simplify the notation, in most of Section 2 we will work in the special case

$$d = 2,\ |R_i| = 1,\ Z = \{2\}, N_2 = \emptyset, \text{ and } |f|_{C_w^\alpha} = |f|_{\alpha,2}. \tag{16}$$

**1.5. Outline of the paper.** In Section 2 we establish properties of the basic semigroups that are used to verify (14) in the norm $\|\cdot\|_{C_w^\alpha}$. In Subsection 2.1 we review representations of the catalytic and branching semigroups which play an important role in the proofs. In Subsection 2.2 we show that the semigroup takes bounded Borel functions to $C^2$ functions, in Subsection 2.3

we obtain $L^\infty$ bounds on the first and second order partial derivatives of the semigroup and prove the equivalence of norms mentioned above. In Subsection 2.4 weighted Hölder bounds on the $(m+1)$-dimensional catalytic semigroups are derived (in the case $m=1$) and the corresponding bounds from [BP1] for the one-dimensional branching semigroups are noted in Remark 29. In Section 3 the required $L^\infty$ and weighted Hölder norms on the multidimensional resolvent are obtained and then these bounds are used in Section 4 to complete the proof of the uniqueness.

## 2. Properties of the basic semigroups

**2.1. Representations of the catalytic semigroups and branching semigroups.** In this section, until otherwise indicated, we work with the catalytic generator

$$\mathcal{A}^1 = \sum_{j=1}^{m} \left\{ b_j^0 \frac{\partial}{\partial x_j} + \gamma_j^0 x_{m+1} \frac{\partial^2}{\partial x_j^2} \right\} + b_{m+1}^0 \frac{\partial}{\partial x_{m+1}} + \gamma_{m+1}^0 x_{m+1} \frac{\partial^2}{\partial x_{m+1}^2}$$

with semigroup $P_t$ on the state space $\mathbb{R}^m \times \mathbb{R}_+$. We assume the coefficients satisfy (cf. (8))

$$\gamma_j^0 > 0 \text{ all } j \leq m+1; \quad b_j^0 \in \mathbb{R} \text{ if } j \leq m,\ b_{m+1}^0 > 0,$$

and

(17) Convention 1 applies with $M^0 = \big[\max_{i \leq m+1} \gamma_i^0 \vee (\gamma_i^0)^{-1} \vee |b_i^0|\big] \vee (b_{m+1}^0)^{-1}$.

If the associated process is denoted by $x_t = (\{x_t^{(j)}\}_{j=1}^m, x_t^{(m+1)})$, this semigroup has the explicit representation

$$(18) \quad P_t f(x_1, \ldots, x_m, x_{m+1}) \\ = E_{x_{m+1}} \left[ \int_{\mathbb{R}^m} f(z_1, \ldots, z_m, x_t^{(m+1)}) \prod_{j=1}^{m} p_{\gamma_j^0 2I_t}(z_j - x_j - b_j^0 t)\, dz_j \right],$$

where $P_{x_{m+1}}$ is the law of the Feller branching immigration process $x^{(m+1)}$ on $C(\mathbb{R}_+, \mathbb{R}_+)$, with generator

$$\mathcal{A}'_0 = b_{m+1}^0 \frac{\partial}{\partial x_{m+1}} + \gamma_{m+1}^0 x_{m+1} \frac{\partial^2}{\partial x_{m+1}^2},$$

$$I_t = \int_0^t x_s^{(m+1)} ds,$$

and for $y \in (0, \infty)$

$$p_y(z) := \frac{e^{-\frac{z^2}{2y}}}{(2\pi y)^{1/2}}.$$

For $(y_1, y_2) \in (0,\infty) \times [0,\infty)$ and $x = (x_1, \ldots, x_m)$, let

(19)
$$G(y_1, y_2) = G_{t,x}(y_1, y_2) = \int_{\mathbb{R}^m} f(z_1, \ldots, z_m, y_2) \prod_{j=1}^{m} p_{\gamma_j^0 2y_1}(z_j - x_j - b_j^0 t) dz_j.$$

Then (18) can be rewritten as

$$P_t f(x_1, \ldots, x_m, x_{m+1}) = E_{x_{m+1}}(G(I_t, x_t^{(m+1)})). \tag{20}$$

Proposition 6. *The joint Laplace functional of* $(x_t^{(m+1)}, \int_0^t x_s^{(m+1)} ds)$ *is given by*

$$\begin{aligned} L(\lambda_1, \lambda_2) &= E_{x_{m+1}}\left[\exp\left(-\lambda_1 x_t^{(m+1)} - \frac{\lambda_2}{2}\int_0^t x_s^{(m+1)} ds\right)\right] \\ &= \left(\cosh\left(\sqrt{\frac{\lambda_2 \gamma_{m+1}^0}{2}}\ t\right) + \sqrt{2}\frac{\lambda_1}{\sqrt{\lambda_2/\gamma_{m+1}^0}} \sinh\left(\sqrt{\frac{\lambda_2\gamma_{m+1}^0}{2}}\ t\right)\right)^{-\frac{b_{m+1}^0}{\gamma_{m+1}^0}} \\ &\quad \cdot \exp\left(-x_{m+1}\sqrt{\lambda_2/2\gamma_{m+1}^0}\, \frac{\left(1 + \frac{\sqrt{2}\lambda_1}{\sqrt{\lambda_2/\gamma_{m+1}^0}} \coth\left(\sqrt{\frac{\lambda_2\gamma_{m+1}^0}{2}}\ t\right)\right)}{\left(\coth\left(\sqrt{\frac{\lambda_2\gamma_{m+1}^0}{2}}\ t\right) + \frac{\sqrt{2}\lambda_1}{\sqrt{\lambda_2/\gamma_{m+1}^0}}\right)}\right). \end{aligned}$$

*Proof.* See [Y1], Equation (2.1), page 16 (with $\gamma_{m+1}^0 = 2$). □

Lemma 7.

(a)

$$\begin{aligned} &E_{x_{m+1}}(x_t^{(m+1)}) = x_{m+1} + b_{m+1}^0 t, \\ &E_{x_{m+1}}((x_t^{(m+1)})^2) = x_{m+1}^2 + (2b_{m+1}^0 + 2\gamma_{m+1}^0) x_{m+1} t + (b_{m+1}^0 + \gamma_{m+1}^0) b_{m+1}^0 t^2, \\ &E_{x_{m+1}}((x_t^{(m+1)} - x_{m+1})^2) = 2\gamma_{m+1}^0 x_{m+1} t + b_{m+1}^0 (b_{m+1}^0 + \gamma_{m+1}^0) t^2, \\ &E_{x_{m+1}}\left(\int_0^t x_s^{(m+1)} ds\right) = x_{m+1} t + \frac{b_{m+1}^0}{2} t^2. \end{aligned}$$

(b)

$$E_{x_{m+1}}\left(\left(\int_0^t x_s^{(m+1)} ds\right)^{-p}\right) \le c_7(p) t^{-p} (t + x_{m+1})^{-p} \ \ \forall p > 0.$$

(c)

$$E_{x_{m+1}}((x_t^{(m+1)} + s)^{-2}) \le c_7 (x_{m+1} + s)^{-2} \textit{ for all } s \ge \gamma_{m+1}^0 t.$$

*Proof.* (a) These identities are standard.

(b) To simplify the notation we set $b = b^0_{m+1}$, $\gamma = \gamma^0_{m+1}$ and $x = x_{m+1}$ in the following calculation and also in (c).

$$\begin{aligned}
E_x\left(\left(\int_0^t x_s^{(m+1)} ds\right)^{-p}\right) &= c_p\, E_x\left(\int_{I_t}^\infty u^{-p-1} du\right)\\
&= c_p \int_0^\infty P(I_t \le u) u^{-p-1} du\\
&\le c_p e \int_0^\infty E(e^{-u^{-1} I_t}) u^{-p-1} du\\
&\le c_p e \int_0^\infty \left[\frac{e^{\sqrt{\frac{\gamma}{u}}\, t} + e^{-\sqrt{\frac{\gamma}{u}}\, t}}{2}\right]^{-b/\gamma}\\
&\quad \cdot \exp\left\{-x\sqrt{\frac{\gamma}{u}}\left[\frac{e^{\sqrt{\frac{\gamma}{u}}\, t} - e^{-\sqrt{\frac{\gamma}{u}}\, t}}{e^{\sqrt{\frac{\gamma}{u}}\, t} + e^{-\sqrt{\frac{\gamma}{u}}\, t}}\right]\right\} u^{-p-1} du\\
&\le c_{p,2} \int_0^\infty e^{-\sqrt{\frac{\gamma}{u}} \frac{tb}{\gamma}} \exp\left\{-x\sqrt{\frac{\gamma}{u}} c\left(\sqrt{\frac{\gamma}{u}}\, t \wedge 1\right)\right\} u^{-p-1} du,
\end{aligned}$$

where we used $\frac{e^x - e^{-x}}{e^x + e^{-x}} \ge c(x \wedge 1)$. Set $v = \sqrt{\frac{\gamma}{u}} t$, so $u = \frac{\gamma}{v^2} t^2$ and $du = -2\gamma t^2 dv v^{-3}$, to see

$$\begin{aligned}
E_x(I_t^{-p}) &\le 2c_{p,2} \int_0^\infty \gamma^{-p} e^{-\frac{bv}{\gamma}} e^{-\frac{x}{t} vc(v\wedge 1)} v^{2p+2-3} dv t^2 t^{-2p-2}\\
&\le 2c_{p,2}\gamma^{-p} t^{-2p} \int_0^\infty v^{2p-1} e^{-c\frac{x}{t} v(v\wedge 1)} e^{\frac{-bv}{\gamma}} dv\\
&\equiv 2c_{p,2}\gamma^{-p} t^{-2p}\ J.
\end{aligned}$$

Now $J \le \int_0^\infty v^{2p-1} e^{\frac{-bv}{\gamma}} dv < \infty$ $(p > 0)$ and so we can choose $c_{p,3}$ so that

$$E_x(I_t^{-p}) \le c_{p,3} t^{-2p}. \tag{21}$$

Assume $x > t$ now. Then

$$\begin{aligned}
J &\le \int_0^1 v^{2p-1} e^{-c(\frac{x}{t}) v^2} dv + \int_1^\infty v^{2p-1} e^{-c\frac{x}{t} v} dv\\
&= \int_0^{\sqrt{\frac{cx}{t}}} w^{2p-1} e^{-w^2} dw \left(\frac{cx}{t}\right)^{-p} + \int_{x/t}^\infty w^{2p-1} e^{-cw} dw \left(\frac{x}{t}\right)^{-2p}\\
&\le c_p\left[\left(\frac{t}{x}\right)^p + \left(\frac{t}{x}\right)^{2p}\right]
\end{aligned}$$

and so

$$E_x(I_t^{-p}) \le c_{p,4} t^{-2p} \left[ \left(\frac{t}{x}\right)^p + \frac{t^{2p}}{x^{2p}} \right] \le c_{p,4} \left[ \frac{1}{x^p t^p} + \frac{1}{x^{2p}} \right] \le \frac{c_{p,4}}{x^p t^p} \quad \text{as } x > t. \tag{22}$$

(21) and (22) together give

$$E_x(I_t^{-p}) \le \frac{c(p)}{t^p(x^p + t^p)} \le \frac{c(p)}{t^p(x+t)^p}.$$

(c) It follows from Proposition 6 that

$$E_x(e^{-\lambda x_t^{(m+1)}}) = (1 + \lambda\gamma t)^{-b/\gamma} \exp\left\{ \frac{-x\lambda}{1 + \lambda t \gamma} \right\}. \tag{23}$$

If $s \ge \gamma t$, this gives

$$\begin{aligned}
E_x((x_t^{(m+1)} + s)^{-2}) &= \int_s^\infty 2u^{-3} P_x(x_t^{(m+1)} + s \le u) du \\
&\le \int_s^\infty 2u^{-3} e E_x(e^{-x_t^{(m+1)}(u-s)^{-1}}) du \\
&\le 2e \int_s^\infty u^{-3} \left[ 1 + \frac{\gamma t}{u - s} \right]^{-b/\gamma} \exp\left\{ \frac{-x}{u - s + \gamma t} \right\} du \\
&\le 2e \int_s^\infty u^{-3} \exp(-x/u) du \\
&\le 2e \int_0^{x/s} w x^{-2} \exp(-w) dw \\
&\le c(x+s)^{-2}.
\end{aligned}$$

□

Now let $\{P_x^0 : x \ge 0\}$ denote the laws of the Feller branching process $X$ with no immigration (equivalently, the 0-dimensional squared Bessel process) with generator $\mathcal{L}^0 f(x) = \gamma x f''(x)$. If $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ let $\zeta(\omega) = \inf\{t > 0 : \omega(t) = 0\}$. There is a unique $\sigma$-finite measure $\mathbb{N}_0$ on

$$C_{ex} = \{\omega \in C(\mathbb{R}_+, \mathbb{R}_+) : \omega(0) = 0,\ \zeta(\omega) > 0,\ \omega(t) = 0\ \forall t \ge \zeta(\omega)\}$$

such that for each $h > 0$, if $\Xi^h$ is a Poisson point process on $C_{ex}$ with intensity $h\mathbb{N}_0$, then

$$X = \int_{C_{ex}} \nu \Xi^h(d\nu) \text{ has law } P_h^0; \tag{24}$$

see, e.g., Theorem II.7.3 of [P] which can be projected down to the above by considering the total mass function. Moreover we also have

$$\mathbb{N}_0(\nu_\delta > 0) = (\gamma\delta)^{-1} \tag{25}$$

and by [P], Thm. II.7.2(iii), for $t > 0$,

$$\int_{C_{ex}} \nu_t d\mathbb{N}_0(\nu) = 1. \tag{26}$$

For $t > 0$ let $P_t^*$ denote the probability on $C_{ex}$ defined by

$$P_t^*(A) = \frac{\mathbb{N}_0(A \cap \{\nu_t > 0\})}{\mathbb{N}_0(\nu_t > 0)}. \tag{27}$$

LEMMA 8. *For all $h > 0$*

$$P_h^0(\zeta > t) = P_h^0(X_t > 0) = 1 - e^{-h/(t\gamma)} \leq \frac{h}{t\gamma}. \tag{28}$$

*Proof.* The first equality is immediate from the fact that $X$ is a non-negative martingale. The second equality follows from (24) and (25). □

The following result is easy to prove, for example, by modifying the arguments in the proof of Theorem II.7.3 of [P].

PROPOSITION 9. *Let $f : C(\mathbb{R}_+, \mathbb{R}_+) \to \mathbb{R}$ be bounded and continuous. Then for any $\delta > 0$,*

$$\lim_{h \downarrow 0} h^{-1} E_h^0(f(X) 1(X_\delta > 0)) = \int_{C_{ex}} f(\nu) 1(\nu_\delta > 0) d\mathbb{N}_0(\nu).$$

The representation (24) leads to the following decompositions of the process $x_t^{(m+1)}$ that will be used below. Recall $x_t^{(m+1)}$ is the Feller branching immigration process with coefficients $b_{m+1}^0 > 0, \gamma_{m+1}^0 > 0$ ($b_{m+1}^0 \geq 0$ suffices for this result) starting at $x_{m+1}$ and with law $P_{x_{m+1}}$.

LEMMA 10. *Let $0 \leq \rho \leq 1$.* (a) *We may assume*

$$x^{(m+1)} = X_0' + X_1, \tag{29}$$

*where $X_0'$ is a diffusion with generator $\mathcal{A}_0' f(x) = \gamma_{m+1}^0 x f''(x) + b_{m+1}^0 f'(x)$ starting at $\rho x_{m+1}$, $X_1$ is diffusion with generator $\gamma_{m+1}^0 x f''(x)$ starting at $(1-\rho)x_{m+1} \geq 0$, and $X_0', X_1$ are independent. In addition, we may assume*

$$X_1(t) = \int_{C_{ex}} \nu_t \Xi(d\nu) = \sum_{j=1}^{N_t} e_j(t), \tag{30}$$

*where $\Xi$ is a Poisson point process on $C_{ex}$ with intensity $(1-\rho)x_{m+1}\mathbb{N}_0$, $\{e_j, j \in \mathbb{N}\}$ is an iid sequence with common law $P_t^*$, and $N_t$ is a Poisson random variable (independent of the $\{e_j\}$) with mean $\frac{(1-\rho)x_{m+1}}{t\gamma_{m+1}^0}$.*

(b) *We also have*

$$\int_0^t X_1(s)ds = \int_{C_{ex}} \int_0^t \nu_s ds 1(\nu_t \neq 0)\Xi(d\nu) \tag{31}$$
$$+ \int_{C_{ex}} \int_0^t \nu_s ds 1(\nu_t = 0)\Xi(d\nu)$$
$$\equiv \sum_{j=1}^{N_t} r_j(t) + I_1(t),$$

$$\int_0^t x_s^{(m+1)} ds = \sum_{j=1}^{N_t} r_j(t) + I_2(t), \tag{32}$$

*where* $r_j(t) = \int_0^t e_j(s)ds$, $I_2(t) = I_1(t) + \int_0^t X_0'(s)ds$.

(c) *Let* $\Xi^h$ *be a Poisson point process on* $C_{ex}$ *with intensity* $h_{m+1}\mathbb{N}_0$ *(*$h_{m+1} > 0$*), independent of the above processes. Set* $\Xi^{x+h} = \Xi + \Xi^h$ *and* $X_t^h = \int \nu_t \Xi^h(d\nu)$*. Then*

$$X_t^{x+h} \equiv x_t^{(m+1)} + X^h(t) = \int_{C_{ex}} \nu_t \Xi^{x+h}(d\nu) + X_0'(t) \tag{33}$$

*is a diffusion with generator* $\mathcal{A}_0'$ *starting at* $x_{m+1} + h_{m+1}$*. In addition*

$$\int_{C_{ex}} \nu_t \, \Xi^{x+h}(d\nu) = \sum_{j=1}^{N_t'} e_j(t), \tag{34}$$

*where* $N_t'$ *is a Poisson random variable with mean* $((1-\rho)x_{m+1} + h_{m+1})(\gamma_{m+1}^0 t)^{-1}$*, such that* $\{e_j\}$ *and* $(N_t, N_t')$ *are independent.*

*Also*

$$\int_0^t X_s^{x+h} ds = \sum_{j=1}^{N_t'} r_j(t) + I_2(t) + I_3^h(t), \tag{35}$$

*where* $I_3^h(t) = \int_{C_{ex}} \int_0^t \nu_s ds 1(\nu_t = 0)\Xi^h(d\nu)$.

*Proof.* (a) (29) follows from Theorem XI.1.2 of [RY]. (30) follows from (24) and (25). The other parts, (b), (c), follow in a similar way. □

We next give a first application of the representation of the catalytic semigroup to obtain some preliminary results that will be needed below.

LEMMA 11. *Let* $G_{t,x}$ *be as in (19). Then:*

(a) *For* $i = 1, \dots, m$,

$$\left|\frac{\partial G_{t,x}}{\partial x_i}(y_1, y_2)\right| \le \|f\|_\infty (\gamma_i^0 y_1)^{-1/2}, \tag{36}$$

*and more generally for any* $j \in \mathbb{N}$, *there is a* $c_j$ *such that*

$$\left|\frac{\partial^j G_{t,x}}{\partial x_i^j}(y_1, y_2)\right| \le c_j \|f\|_\infty y_1^{-j/2}. \tag{37}$$

(b) *We have*

$$\left|\frac{\partial G_{t,x}}{\partial y_1}(y_1, y_2)\right| \le c_1 \|f\|_\infty / y_1, \tag{38}$$

*and more generally there is a sequence* $\{c_j\}$ *such that for* $i_1, i_2 \in \{1, \dots, m\}$,

$$\left|\frac{\partial^{j+k_1+k_2}}{\partial x_{i_1}^{k_1} \partial x_{i_2}^{k_2} \partial y_1^j} G_{t,x}(y_1, y_2)\right| \le c_{j+k_1+k_2} \|f\|_\infty y_1^{-j-(k_1+k_2)/2} \tag{39}$$

*for all* $j, k_1, k_2 \in \mathbb{N}$.

(c) $\forall y_2 \ge 0,\ (x, y_1) \to G_{t,x}(y_1, y_2)$ *is* $C^3$ *on* $\mathbb{R}^m \times (0, \infty)$.

*Proof.* (a) We have

$$G_{t,x}(y_1, y_2) = \int f(w_1 + b_1^0 t, \dots, w_m + b_m^0 t, y_2) \prod_{j=1}^m p_{2\gamma_j^0 y_1}(w_j - x_j)\, dw_j$$

and so

$$\begin{aligned} &\left|\frac{\partial G_{t,x}}{\partial x_i}(y_1, y_2)\right| \\ &= \left|\int f(w_1 + b_1^0 t, \dots, w_m + b_m^0 t, y_2) \prod_{j=1}^m p_{2\gamma_j^0 y_1}(w_j - x_j) \frac{(w_i - x_i)}{2\gamma_i^0 y_1}\, dw_j\right| \\ &\le \|f\|_\infty \int_{-\infty}^{\infty} \frac{|w|}{2\gamma_i^0 y_1} p_{2\gamma_i^0 y_1}(w) dw \le \|f\|_\infty (\gamma_i^0 y_1)^{-1/2}. \end{aligned} \tag{40}$$

The general case can be proved by an induction.

(b) If $j = 0$, this follows by arguing as in (a) and using the product form of the density. In fact one can handle any number of $x_{i_j}$ variables. Recall that

$$\frac{\partial G_{t,x}}{\partial y_1}(y_1, y_2) = \sum_{j=1}^m \gamma_j^0 \frac{\partial^2 G_{t,x}}{\partial x_j^2}(y_1, y_2)$$

by the heat equation. Hence the general case follows from the $j = 0$ case, as extended above.

(c) This is an exercise in Dominated Convergence. □

Lemma 12. *If $f : \mathbb{R}^m \times \mathbb{R}_+$ is a bounded Borel function, then, for each $t > 0$, $P_t f \in C_b(\mathbb{R}^m \times \mathbb{R}_+)$; in fact,*

$$|P_t f(x) - P_t f(x')| \leq c_{12} \|f\|_\infty t^{-1} |x - x'|.$$

*Proof.* Recalling (20), we have for $x, x' \in \mathbb{R}^m$,

$$\begin{aligned} &|P_t f(x_1, \ldots, x_m, x_{m+1}) - P_t f(x'_1, \ldots, x'_m, x_{m+1})| \\ &\qquad \leq |E_{x_{m+1}}(G_{t,x}(I_t, x_t^{(m+1)}) - G_{t,x'}(I_t, x_t^{(m+1)}))| \\ &\qquad \leq \|f\|_\infty \sum_{i=1}^m \frac{|x_i - x'_i|}{\sqrt{\gamma_i^0}} E_{x_{m+1}}(I_t^{-1/2}) \quad (\text{by}(36)) \qquad (41) \\ &\qquad \leq c\|f\|_\infty t^{-1} \max_i \{(\gamma_i^0)^{-1/2}\} \sum_{i=1}^m |x_i - x'_i| \qquad (\text{by Lemma 7(b)}). \qquad (42) \end{aligned}$$

For $h > 0$ let $x^h$ denote an independent copy of $x^{(m+1)}$ starting at $h$ but with $b^0_{m+1} = 0$, and let $T_h = \inf\{t \geq 0 : x^h_t = 0\}$. Then $x^{(m+1)} + x^h$ has law $P_{x_{m+1}+h}$ and so if $I_h(t) = \int_0^t x^h_s ds$, then

$$\begin{aligned} &|P_t f(x_1, \ldots, x_{m+1} + h) - P_t f(x_1, \ldots, x_m, x_{m+1})| \\ &\quad = |E(G(I_t + I_h(t), x_t^{(m+1)} + x_t^h) - G(I_t, x_t^{(m+1)}))| \\ &\quad \leq E(c_1 \|f\|_\infty I_h(t) I_t^{-1} + |G(I_t, x_t^{(m+1)} + x_t^h) - G(I_t, x_t^{(m+1)})|) \\ &\qquad\quad (\text{by}(38)) \\ &\quad \leq c_1 \|f\|_\infty E(I_h(t)) E(I_t^{-1}) + 2\|f\|_\infty P(T_h > t) \\ &\quad \leq c\|f\|_\infty htt^{-2} + 2\|f\|_\infty h(t\gamma^0_{m+1})^{-1} \quad (\text{by (28) and Lemma 7(b)}) \\ &\quad = c\|f\|_\infty h(t^{-1}). \end{aligned}$$

This and (42) imply the result. □

Finally, we give an elementary calculus inequality that will be used below.

Lemma 13.

(a) *Let $g : (0, \infty) \to \mathbb{R}$ be $C^2$. Then for all $\Delta, \Delta', \Delta'', y \in (0, \infty)$,*

$$\begin{aligned} &\frac{|g(y + \Delta + \Delta') - g(y + \Delta) - g(y + \Delta') + g(y)|}{(\Delta\Delta')} \\ &\qquad\qquad\qquad\qquad \leq \sup_{\{y_1 \in [y, y+\Delta+\Delta']\}} |g''(y_1)| \end{aligned}$$

*and*

$$|g(y+\Delta+\Delta'+\Delta'')-g(y+\Delta+\Delta')-g(y+\Delta+\Delta'')-g(y+\Delta'+\Delta'') \\ +g(y+\Delta)+g(y+\Delta')+g(y+\Delta'')-g(y)|(\Delta\Delta'\Delta'')^{-1} \\ \leq \sup_{\{y_1\in[y,y+\Delta+\Delta'+\Delta'']\}}|g'''(y_1)|.$$

(b) *Let* $f:\mathbb{R}\times(0,\infty)\to\mathbb{R}$ *be* $C^3$. *Then for all* $\Delta_1\in\mathbb{R}$ *non-zero, and all* $\Delta_2,\Delta_2'>0$,

$$\frac{|f(y_1+\Delta_1,y_2,+\Delta_2)-f(y_1+\Delta_1,y_2)-f(y_1,y_2+\Delta_2)+f(y_1,y_2)|}{(|\Delta_1|\Delta_2)} \\ \leq \sup_{y_1'\in I_1,y_2'\in I_2}\left|\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_1}f(y_1',y_2')\right|,$$

*and*

$$|f(y_1+\Delta_1,y_2+\Delta_2+\Delta_2')-f(y_1+\Delta_1,y_2+\Delta_2)-f(y_1+\Delta_1,y_2+\Delta_2') \\ +f(y_1+\Delta_1,y_2)-f(y_1,y_2+\Delta_2+\Delta_2')+f(y_1,y_2+\Delta_2) \\ +f(y_1,y_2+\Delta_2')-f(y_1,y_2)|\,(|\Delta_1|\Delta_2\Delta_2')^{-1} \\ \leq \sup_{y_1'\in I_1,y_2'\in I_2'}\left|\frac{\partial^2}{\partial y_2^2}\frac{\partial}{\partial y_1}f(y_1',y_2')\right|,$$

*where* $I_j$ *is the closed interval between* $y_j$ *and* $y_j+\Delta_j$, *and* $I_2'$ *is the interval between* $y_2$ *and* $y_2+\Delta_2+\Delta_2'$.

*Proof.* (a) Fix $\Delta'>0$ and let $h(z)=(g(z+\Delta')-g(z))/\Delta'$. By the mean value theorem,

$$\begin{aligned}|g(y+\Delta+\Delta')&-g(y+\Delta)-g(y+\Delta')+g(y)|(\Delta\Delta')^{-1}\\ &=|h(y+\Delta)-h(y)|\Delta^{-1}\\ &=|h'(\Delta''+y)|\quad \exists\Delta''\in(0,\Delta)\\ &=|g'(\Delta''+y+\Delta')-g'(\Delta''+y))|(\Delta')^{-1}\\ &=|g''(y+\Delta''+\Delta''')|\quad \exists\Delta'''\in(0,\Delta').\end{aligned}$$

Now consider the second bound. If $h(x)=\frac{g(x+\Delta'')-g(x)}{\Delta''}$, the left-hand side is

$$\frac{|h(y+\Delta+\Delta')-h(y+\Delta)-h(y+\Delta')+h(y)|}{(\Delta\Delta')}$$

and so we may apply the first bound to $h$ and then the mean value theorem to get the second bound.

(b) We only prove the slightly more involved second bound. If $g = \frac{\partial f}{\partial y_1}$, then the left-hand side is

$$\left|\int_{y_1}^{y_1+\Delta_1} g(y_1', y_2+\Delta_2+\Delta_2') - g(y_1', y_2+\Delta_2) - g(y_1', y_2+\Delta_2') + g(y_1', y_2) dy_1'\right|.$$

Now apply (a) to $y_2 \to g(y_1', y_2)$ to obtain the required bound. □

**2.2. Existence of derivatives of the catalytic semigroup.** In this subsection $\mathcal{A}^1$ and $P_t$ are as in the previous subsection. We will show that this semigroup takes bounded Borel functions to $C^2$ functions and describe the derivatives in terms of the canonical measure $\mathbb{N}_0$ introduced in the previous subsection. In order to simplify the notation, in this and the next two subsections, we will work with the special case $m = 1$ and then indicate how the general case $m \geq 1$ will follow by some simple modifications. This means we have $N_2 = \emptyset$, $I = Z \cap C = \{2\}$, $N_1 = R_2 = \{1\}$, $S_0 = \mathbb{R} \times \mathbb{R}_+$ and so for $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$,

$$|f|_{C_w^\alpha} = |f|_{\alpha,2} = \sup\{|f(x+h) - f(x)|[|h|^{-\alpha} x_2^{\alpha/2} \vee |h|^{\alpha/2}] : h_2 > 0, x \in \mathbb{R} \times \mathbb{R}_+\}. \tag{43}$$

Let $G$ be given as in (19) with $m = 1$. Then

$$P_t f(x_1, x_2) = E_{x_2}(G_{t,x_1}(I_t, x_t^{(2)})),$$

where now

$$I_t = \int_0^t x_s^{(2)} ds. \tag{44}$$

If $X \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\nu, \nu' \in C_{ex}$, let

$$\begin{aligned}\Delta G_{t,x_1}(X, \nu', \nu) = {} & G_{t,x_1}\left(\int_0^t X_s + \nu_s' + \nu_s ds, X_t + \nu_t' + \nu_t\right) \\ & - G_{t,x_1}\left(\int_0^t X_s + \nu_s' ds, X_t + \nu_t'\right) \\ & - G_{t,x_1}\left(\int_0^t X_s + \nu_s ds, X_t + \nu_t\right) + G_{t,x_1}\left(\int_0^t X_s ds, X_t\right).\end{aligned} \tag{45}$$

Proposition 14. *If $f$ is a bounded Borel function on $\mathbb{R} \times \mathbb{R}_+$ and $t > 0$, then $P_t f \in C_b^2(\mathbb{R} \times \mathbb{R}_+)$ and for $i, j \in \{1, 2\}$*

$$\|(P_t f)_{ij}\|_\infty \leq c_{14} \frac{\|f\|_\infty}{t^2}. \tag{46}$$

*Moreover if $f$ is bounded and continuous on $\mathbb{R}\times\mathbb{R}_+$, then*

$$(47)\qquad (P_tf)_2(x)=E_{x_2}\biggl(\int G_{t,x_1}\biggl(\int_0^t x_s^{(2)}+\nu_s ds, x_t^{(2)}+\nu_t\biggr) - G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds, x_t^{(2)}\biggr)\mathbb{N}_0(d\nu)\biggr),$$

$$(48)\qquad (P_tf)_{22}(x)=E_{x_2}\biggl(\iint \Delta G_{t,x_1}(x^{(2)},\nu',\nu)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\biggr).$$

*Proof.* In view of Lemma 12, and the fact that $P_tf=P_{t/2}(P_{t/2}f)$, it suffices to consider bounded continuous $f$. Let us assume $(P_tf)_2$ exists and is given by (47). We will use this to prove the existence of, and corresponding formula for, $(P_tf)_{22}(x)$. It should be then clear how to derive (47). Let $0<\delta\le t$. If $\nu'_\delta=\nu_t=0$, use Lemmas 13 and 11(b) to see that

$$\begin{aligned}(49)\quad &|\Delta G_{t,x_1}(x^{(2)},\nu',\nu)|\\ &=\biggl|G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds+\int_0^t \nu_s ds+\int_0^\delta \nu'_s ds, x_t^{(2)}\biggr)\\ &\quad - G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds+\int_0^\delta \nu'_s ds, x_t^{(2)}\biggr)\\ &\quad - G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds+\int_0^t \nu_s ds, x_t^{(2)}\biggr)+G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds, x_t^{(2)}\biggr)\biggr|\\ &\le c\|f\|_\infty\biggl(\int_0^t x_s^{(2)}ds\biggr)^{-2}\int_0^\delta \nu'_s ds\int_0^t \nu_s ds.\end{aligned}$$

If $\nu'_\delta=0$ and $\nu_t>0$, then by Lemma 11(b)

$$\begin{aligned}(50)\quad &|\Delta G_{t,x_1}(x^{(2)},\nu',\nu)|\\ &\le\biggl|G_{t,x_1}\biggl(\int_0^t x_s^{(2)}+\nu_s ds+\int_0^\delta \nu'_s ds, x_t^{(2)}+\nu_t\biggr)\\ &\quad - G_{t,x_1}\biggl(\int_0^t x_s^{(2)}+\nu_s ds, x_t^{(2)}+\nu_t\biggr)\biggr|\\ &\quad +\biggl|G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds+\int_0^\delta \nu'_s ds, x_t^{(2)}\biggr)-G_{t,x_1}\biggl(\int_0^t x_s^{(2)}ds, x_t^{(2)}\biggr)\biggr|\\ &\le 2c_1\|f\|_\infty\biggl(\int_0^t x_s^{(2)}ds\biggr)^{-1}\int_0^\delta \nu'_s ds.\end{aligned}$$

A similar argument shows if $\nu'_\delta>0$ and $\nu_t=0$, then

$$(51)\qquad |\Delta G_{t,x_1}(x^{(2)},\nu',\nu)|\le 2c_1\|f\|_\infty\biggl(\int_0^t x_s^{(2)}ds\biggr)^{-1}\int_0^t \nu_s ds.$$

Finally, if $\nu_\delta > 0,\ \nu_t > 0$, then we have the trivial bound

$$|\Delta G_{t,x_1}(x^{(2)},\nu',\nu)| \le 4\|f\|_\infty. \tag{52}$$

Combine (49)–(52) to conclude

$$\begin{aligned}
|\Delta G_{t,x_1}&(x^{(2)},\nu',\nu)| \\
&\le \Big[1(\nu'_\delta = 0, \nu_t = 0)\Big(\int_0^t x_s^{(2)}ds\Big)^{-2}\int_0^\delta \nu'_s ds \int_0^t \nu_s ds) \\
&\qquad + 1(\nu'_\delta = 0, \nu_t > 0)\Big(\int_0^t x_s^{(2)}ds\Big)^{-1}\int_0^\delta \nu'_s ds) \\
&\qquad + 1(\nu'_\delta > 0, \nu_t = 0)\Big(\int_0^t x_s^{(2)}ds\Big)^{-1}\int_0^t \nu_s ds) \\
&\qquad + 1(\nu'_\delta > 0, \nu_t > 0)\Big]\cdot c\|f\|_\infty \\
&\equiv \bar g_{t,\delta}(x^{(2)},\nu',\nu).
\end{aligned} \tag{53}$$

Let $X^h_\cdot$ be independent of $x^{(2)}$ satisfying

$$X_t^h = h + \int_0^t \sqrt{2\gamma_2^0 X_s^h}\,dB'_s,\ (h > 0)$$

(i.e., $X^h$ has law $P_h^0$) so that $x^{(2)} + X^h$ has law $P_{x_2+h}$. Therefore (47) implies

$$\begin{aligned}
&\frac{1}{h}[(P_t f)_2(x + he_2) - (P_t f)_2(x)] \\
&\qquad = \frac{1}{h}\iiint \Delta G(x^{(2)}, X^h, \nu)\,d\mathbb{N}_0(\nu)\,dP_{x_2}\,dP_h^0.
\end{aligned} \tag{54}$$

In addition (53) implies (use also (25) and (26) and Lemma 7(b) with $p = 1$ or 2)

$$\begin{aligned}
&\frac{1}{h}\iiint |\Delta G(x^{(2)}, X^h, \nu)|1(X_\delta^h = 0)\,d\mathbb{N}_0\,dP_{x_2}\,dP_h^0 \\
&\le c\|f\|_\infty\Big\{E_{x_2}\Big(\int_0^t x_s^{(2)}ds\Big)^{-2}\frac{1}{h}E_h^0\Big(\int_0^\delta X_s^h ds\Big)\int\int_0^t \nu_s\,ds\,d\mathbb{N}_0 \\
&\quad + E_{x_2}\Big(\Big(\int_0^t x_s^{(2)}ds\Big)^{-1}\Big)\frac{1}{h}E_h^0\Big(\int_0^\delta X_s^h ds\Big)\mathbb{N}_0(\nu_t > 0)\Big\} \\
&\le c\|f\|_\infty(t^{-2}(x_2+t)^{-2}\delta t + t^{-1}(x_2+t)^{-1}\delta t^{-1}) \\
&\le c\|f\|_\infty(t^{-3})\delta.
\end{aligned} \tag{55}$$

As $G$ is bounded and continuous on $\mathbb{R}_+ \times (0,\infty)$, Proposition 9 implies

$$\lim_{h\downarrow 0} h^{-1} E_h^0(\Delta G(x^{(2)}, X^h, \nu) 1(X_\delta^h > 0)) = \int \Delta G(x^{(2)}, \nu', \nu) 1(\nu'_\delta > 0) d\mathbb{N}_0(\nu') \tag{56}$$

for all $\delta > 0$, pointwise in $(x,\nu) \in C(\mathbb{R}_+, \mathbb{R}_+) \times C_{ex}$. Use (53) to see that

$$\begin{aligned} &h^{-1} E_h^0(|\Delta G(x^{(2)}, X^h, \nu)| 1(X_\delta^h > 0)) \\ &\quad\le c\|f\|_\infty \left[\frac{P_h^0(X_\delta^h > 0)}{h}\right] \left[\left(\int_0^t x_s^{(2)} ds\right)^{-1} \int_0^t \nu_s ds + 1(\nu_t > 0)\right] \\ &\quad\le c\|f\|_\infty \delta^{-1} \left[\left(\int_0^t x_s^{(2)} ds\right)^{-1} \int_0^t \nu_s ds + 1(\nu_t > 0)\right], \end{aligned}$$

the last by (28). The final expression is integrable with respect to $P_{x_2} \times \mathbb{N}_0$ and so by dominated convergence we conclude from (56) that

$$\begin{aligned} &\lim_{h\downarrow 0} h^{-1} \iiint \Delta G(x^{(2)}, X^h, \nu) 1(X_\delta^h > 0) d\mathbb{N}_0(\nu) dP_{x_2} dP_h^0 \\ &\quad= E_{x_2}\left(\iint \Delta G(x^{(2)}, \nu', \nu) 1(\nu'_\delta > 0) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right) \ \forall\, \delta > 0. \end{aligned} \tag{57}$$

Use (53) as in the derivation of (55) to see

$$E_{x_2}\left(\int \sup_{x_1} |\Delta G_{t,x_1}(x^{(2)}, \nu', \nu)| 1(\nu'_\delta = 0) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right) \le c\|f\|_\infty (t^{-3})\delta. \tag{58}$$

Use (54), (55), (57) and (58) and take $\delta \downarrow 0$ to conclude

$$\frac{\partial^+}{\partial x_2^+}(P_t f)_2(x) = E_{x_2}\left(\iint \Delta G_{t,x_1}(x^{(2)}, \nu', \nu) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right). \tag{59}$$

Recall from Lemma 11(a) that

$$\left|\frac{\partial G_{t,x_1}(y_1, y_2)}{\partial x_1}\right| \le (\gamma_1^0)^{-1/2} \|f\|_\infty y_1^{-1/2}.$$

This, together with (58) and Lemma 7(b), implies for $0 < \delta \le t$

$$\begin{aligned} &\left| E_{x_2}\left(\iint (\Delta G_{t,x_1}(x^{(2)}, \nu', \nu) - \Delta G_{t,x'_1}(x^{(2)}, \nu', \nu)) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right)\right| \\ &\le c\|f\|_\infty (t^{-3})\delta \\ &\quad + E_{x_2}\left(\iint 1(\nu_\delta > 0, \nu'_\delta > 0) \frac{c\|f\|_\infty}{(\int_0^t x_s^{(2)} ds)^{1/2}} d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right) |x_1 - x'_1| \\ &\le c\|f\|_\infty [(t^{-3})\delta + \delta^{-2} t^{-1} |x_1 - x'_1|]. \end{aligned}$$

(28) was used in the last line. By first choosing $\delta$ small and then $|x_1 - x_1'|$ small, one sees the right hand side of (59) is continuous in $x_1$, uniformly in $x_2 \geq 0$. If $x_2^n \uparrow x_2$, then we may construct $\{x^n\}$ such that $x^n \uparrow x^{(2)}$ in $C(\mathbb{R}_+, \mathbb{R}_+)$, $x^n$ with law $P_{x_2^n}$ and $x^{(2)}$ with law $P_{x_2}$ (e.g., $x^{n-1} - x^n$ has law $P^0_{x_2^{n-1}-x_2^n}$ and are independent). Then $\Delta G_{t,x_1}(x^n, \nu', \nu) \to \Delta G_{t,x_1}(x^{(2)}, \nu', \nu)$ pointwise (by an elementary argument using (45) and the continuity of $f$) and (by (53))

$$|\Delta G_{t,x_1}(x^n, \nu', \nu)| \leq \bar{g}_{t,\delta}(x^1, \nu', \nu),$$

which is integrable with respect to $P_{x_2} \times \mathbb{N}_0 \times \mathbb{N}_0$ by Lemma 7(b). Dominated convergence now shows that

$$\lim_{n\to\infty} E_{x_2^n}\left(\iint \Delta G_{t,x_1}(x^{(2)}, \nu', \nu) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right)$$
$$= E_{x_2}\left(\iint \Delta G_{t,x_1}(x^{(2)}, \nu', \nu) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right).$$

A similar argument holds if $x_2^n \downarrow x_2$, so the right-hand side of (59) is also continuous in $x_2$ for each $x_1$. Combined with the above this shows $\frac{\partial^+}{\partial x_2^+}(P_t f)_2(x)$ is continuous in $x \in \mathbb{R} \times \mathbb{R}_+$. An elementary calculus exercise using the continuity in $x_2$ shows this in fact equals $(P_t f)_{22}(x)$ and so

$$(P_t f)_{22}(x) = E_{x_2}\left(\iint \Delta G_{t,x_1}(x^{(2)}, \nu', \nu) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\right).$$

This together with (53), Lemma 7(b), (25) and (26) give the upper bound

$$\|(P_t f)_{22}\| \leq c\frac{\|f\|_\infty}{t^2}.$$

Turning to derivatives with respect to $x_1$, let us assume $2\gamma_1^0 = 1$ to ease the notation.

Lemma 7(b) and dominated convergence allows us to differentiate through the integral sign and conclude (by Lemmas 7(b) and 11(a)) that

$$\frac{\partial}{\partial x_1} P_t f(x) = E_{x_2}\left(G'_{t,x_1}\left(\int_0^t x_s^{(2)} ds, x_t^{(2)}\right)\right) \leq c\frac{\|f\|_\infty}{t}, \tag{60}$$

$$\frac{\partial^2}{\partial x_1^2} P_t f(x) = E_{x_2}\left(G''_{t,x_1}\left(\int_0^t x_s^{(2)} ds, x_t^{(2)}\right)\right) \leq c\frac{\|f\|_\infty}{t^2}. \tag{61}$$

Now use (61), Lemma 11(b) with $j = 3$, and Lemma 7(b) to see that $\frac{\partial^2}{\partial x_1^2} P_t f(x_1, x_2)$ is continuous in $x_1$ uniformly in $x_2$. The weak continuity of $E_{x_2}$ in $x_2$ (e.g., by our usual coupling argument), the continuity of $\frac{\partial^2}{\partial x_1^2} G_{t,x_1}(y_1, y_2)$ in $y_1 \in (0, \infty)$ (see Lemma 11(b)), the bound (37) with $j = 2$, and Lemma 7(b) imply $\frac{\partial^2}{\partial x_1^2} P_t f(x_1, x_2)$ is continuous in $x_2$ for each $x_1$. Therefore $(P_t f)_{11}$ is jointly continuous.

For the mixed partial, first note that by Lemma 11(b)

$$|\frac{\partial}{\partial y_1} G'_{t,x_1}(y)| \le c\|f\|_\infty y_1^{-3/2}. \tag{62}$$

Let

$$\Delta_1 G'_{t,x_1}(x^{(2)},\nu) = G'_{t,x_1}\left(\int_0^t x_s^{(2)} + \nu_s ds, x_t^{(2)} + \nu_t\right) - G'_{t,x_1}\left(\int_0^t x_s^{(2)} ds, x_t^{(2)}\right)$$

and argue as for $(P_t f)_{22}$, using (62) in place of Lemma 11(a) (in fact the argument is simpler now), to see that

$$(P_t f)_{12}(x_1,x_2) = E_{x_2}\left(\int \Delta_1 G'_{t,x_1}(x^{(2)},\nu)\mathbb{N}_0(d\nu)\right).$$

From (62) we have for $0 < \delta \le t$,

$$\begin{aligned} &E_{x_2}\left(\int \sup_{x_1} |\Delta_1 G'_{t,x_1}(x^{(2)},\nu)| 1(\nu_\delta = 0) d\mathbb{N}_0(\nu)\right) \\ &\le c\|f\|_\infty E_{x_2}\left(\left(\int_0^t x_s^{(2)} ds\right)^{-3/2}\right) \int \int_0^\delta \nu_s ds d\mathbb{N}_0(\nu) \\ &\le c\|f\|_\infty t^{-3}\delta, \end{aligned} \tag{63}$$

the last from Lemma 7(b) and (26). Just as for $(P_t f)_{22}$, we may use this with (37) (for $j = 2$) and dominated convergence to conclude that $(P_t f)_{12}$ is continuous in $x_1$, uniformly in $x_2$. Continuity in $x_2$ for each $x_1$ is obtained by an easy modification of the argument for $(P_t f)_{22}$, using the bound (63). This completes the proof that $P_t f$ is $C^2$.

Finally to get a (crude) upper bound on $|(P_t f)_{12}|$ use (63) with $\delta = t$ and (37) with $j = 1$ to see

$$\begin{aligned} &|(P_t f)_{12}(x)| \\ &\le E_{x_2}\left(\int |\Delta_1 G'_{t,x_1}(x^{(2)},\nu)| 1(\nu_t = 0) d\mathbb{N}_0(\nu)\right) \\ &\quad + E_{x_2}\left(\int 1(\nu_t > 0)\left[\left|G'_{t,x_1}\left(\int_0^t x_s^{(2)} + \nu_s\, ds, x_t^{(2)} + \nu_t\right)\right|\right.\right. \\ &\qquad \left.\left. + \left|G'_{t,x_1}\left(\int_0^t x_s^{(2)} ds, x_t^{(2)}\right)\right| d\mathbb{N}_0(\nu)\right]\right) \\ &\le c\|f\|_\infty t^{-2} + c\|f\|_\infty E_{x_2}\left(\left(\int_0^t x_s^{(2)} ds\right)^{-1/2}\right)\mathbb{N}_0(\nu_t > 0) \\ &\le c\|f\|_\infty t^{-2}, \end{aligned}$$

by Lemma 7(b) and (25). □

Remark 15. It is straightforward to extend Proposition 14 to general $m \geq 1$. One need only replace the one-dimensional Gaussian distribution by an $m$-dimensional one and make minor changes. One can then apply this to the semigroup $P_t$ in (12) from Section 1.3 via a Fubini argument and conclude that if $i \in C \cap Z$, $\bar{x}_i = (\{x_j, j \in R_i\}, x_i)$, $\hat{x}_i = \{x_j : j \notin R_i \cup \{i\}\}$ and $j_1, j_2 \in R_i$, then $(P_t f)_{j_1 j_2}$ is continuous in $\bar{x}_i$ for each $\hat{x}_i$ and

$$\|(P_t f)_{j_1 j_2}\|_\infty \leq c \frac{\|f\|_\infty}{t^2}.$$

**2.3.** $L^\infty$ **bounds on the catalytic semigroup and equivalence of norms.** We continue to work with the semigroup $P_t$ on the state space $\mathbb{R} \times \mathbb{R}_+$ from Section 2.2 associated with $\mathcal{A}^1$ and $m = 1$. The main objective of this section is to establish $L^\infty$ bounds on the first order partial derivatives of the semigroup and use these results to establish the equivalence of the weighted Hölder norm and semigroup norm from Section 1.4. The derivatives with respect to $x_1$ are considerably easier.

Proposition 16. *If $f$ is a bounded Borel function on $\mathbb{R} \times \mathbb{R}_+$, then*

$$\left\| \frac{\partial}{\partial x_1} P_t f \right\|_\infty \leq \frac{c_{16} \|f\|_\infty}{\sqrt{t}\sqrt{x_2 + t}}, \tag{64}$$

*and*

$$\left\| x_2 \frac{\partial^2}{\partial x_1^2} P_t f \right\|_\infty \leq \frac{c_{16} \|f\|_\infty}{t}. \tag{65}$$

*If $f \in S^\alpha$, then*

$$\left\| \frac{\partial}{\partial x_1} P_t f \right\|_\infty \leq \frac{c_{16} |f|_\alpha t^{\frac{\alpha}{2} - \frac{1}{2}}}{\sqrt{x_2 + t}} \leq c_{16} |f|_\alpha t^{\frac{\alpha}{2} - 1}, \tag{66}$$

*and*

$$\left\| x_2 \frac{\partial^2}{\partial x_1^2} P_t f \right\|_\infty \leq c_{16} |f|_\alpha t^{\frac{\alpha}{2} - 1}. \tag{67}$$

*Proof.* We begin with the first derivative for $f$ bounded Borel measurable. Use (41) (here $m = 1$), Proposition 14 (for the existence of $(P_t f)_1$) and Lemma 7(b) to see that

$$\begin{aligned} \left| \frac{\partial}{\partial x_1} P_t f(x) \right| &\leq c \parallel f \parallel_\infty E_{x_2}\left( \frac{1}{I_t^{1/2}} \right) \\ &\leq \frac{c \parallel f \parallel_\infty}{\sqrt{t}\sqrt{x_2 + t}}. \end{aligned}$$

We now turn to the second derivative. Note that $\mathcal{A}^1$ and $\frac{\partial}{\partial x_1}$ commute and therefore the semigroup $P_t$ and $\frac{\partial}{\partial x_1}$ commute. Therefore a double application of (64) gives

$$
\left\|\frac{\partial^2}{\partial x_1^2}P_t f\right\|_\infty = \left\|\frac{\partial}{\partial x_1}P_{\frac{t}{2}}\frac{\partial}{\partial x_1}P_{\frac{t}{2}}f\right\|_\infty = \frac{c}{\sqrt{t}\sqrt{x_2+t}}\left\|\frac{\partial}{\partial x_1}P_{t/2}f\right\|_\infty \leq \frac{c\|f\|_\infty}{t(t+x_2)}. \tag{68}
$$

This proves the first two inequalities and also shows that

$$
\lim_{t\to\infty}\left\|\frac{\partial}{\partial x_1}P_t f(x)\right\|_\infty = 0. \tag{69}
$$

If $f \in S^\alpha$, we proceed as in [ABP] and write

$$
\left|\frac{\partial}{\partial x_1}P_{2t}f - \frac{\partial}{\partial x_1}P_t f\right| = \left|\frac{\partial}{\partial x_1}P_t(P_t f - f)\right|.
$$

Applying the previous estimate to $g = P_t f - f$ and using the definition of $|f|_\alpha$ we get

$$
\left|\frac{\partial}{\partial x_1}P_{2t}f - \frac{\partial}{\partial x_1}P_t f\right| \leq \frac{c \parallel g \parallel_\infty}{\sqrt{t}\sqrt{x_2+t}} \leq |f|_\alpha t^{\alpha/2}\frac{c}{\sqrt{t}\sqrt{x_2+t}}\ .
$$

This together with (69) implies that

$$
\left|\frac{\partial}{\partial x_1}P_t f(x)\right| \leq \sum_{k=0}^{\infty}\left|\frac{\partial}{\partial x_1}(P_{2^k t}f - P_{2^{(k+1)}t}f)(x)\right| \leq |f|_\alpha \sum_{k=0}^{\infty}(2^k t)^{\frac{\alpha}{2}-\frac{1}{2}}\frac{1}{\sqrt{x_2+t2^k}} \leq c|f|_\alpha t^{\frac{\alpha}{2}-\frac{1}{2}}\frac{1}{\sqrt{x_2+t}}\ .
$$

This then immediately yields (66). Use the above in (68) to derive (67). □

Notation. If $w > 0$, set $p_j(w) = \frac{w^j}{j!}e^{-w}$. If $\{r_j(t)\}$ and $\{e_j(t)\}$ are as in Lemma 10, let $R_k = R_k(t) = \sum_{j=1}^k r_j(t)$ and $S_k = S_k(t) = \sum_{j=1}^k e_j(t)$.

Proposition 17. *If $f$ is a bounded Borel function on $\mathbb{R}\times\mathbb{R}_+$, then*

$$
\left\|\frac{\partial}{\partial x_2}P_t f\right\|_\infty \leq \frac{c_{17}\|f\|_\infty}{\sqrt{t}\sqrt{x_2+t}}, \tag{70}
$$

*and*

$$\left\| x_2 \frac{\partial^2}{\partial x_2^2} P_t f \right\|_\infty \leq \frac{c_{17} x_2 \|f\|_\infty}{t(t+x_2)} \leq \frac{c_{17}\|f\|_\infty}{t}. \tag{71}$$

*If* $f \in S^\alpha$, *then*

$$\left\| \frac{\partial}{\partial x_2} P_t f \right\|_\infty \leq \frac{c_{17}|f|_\alpha t^{\frac{\alpha}{2}-\frac{1}{2}}}{\sqrt{x_2+t}} \leq c_{17}|f|_\alpha t^{\frac{\alpha}{2}-1}, \tag{72}$$

*and*

$$\left\| x_2 \frac{\partial^2}{\partial x_2^2} P_t f \right\|_\infty \leq c_{17}|f|_\alpha t^{\frac{\alpha}{2}-1}. \tag{73}$$

*Proof.* As in the proof of Proposition 14 we may assume without loss of generality that $f$ is bounded and continuous. From Proposition 14 we have

$$\begin{aligned}(P_t f)_2(x) = E_{x_2}\bigg(\int \bigg[&G_{t,x_1}\bigg(\int_0^t x_s^{(2)} + \nu_s ds, x_t^{(2)}\bigg) \\ &- G_{t,x_1}\bigg(\int_0^t x_s^{(2)} ds, x_t^{(2)}\bigg)\bigg] 1(\nu_t = 0) d\mathbb{N}_0(\nu)\bigg) \\ &+ E_{x_2}\bigg(\int \bigg[G_{t,x_1}\bigg(\int_0^t x_s^{(2)} + \nu_s ds, x_t^{(2)} + \nu_t\bigg) \\ &- G_{t,x_1}\bigg(\int_0^t x_s^{(2)} ds, x_t^{(2)}\bigg)\bigg] 1(\nu_t > 0) d\mathbb{N}_0(\nu)\bigg) \\ \equiv\ & E_1 + E_2.\end{aligned}$$

By Lemmas 11 and 7, and (26),

$$\begin{aligned}|E_1| &\leq c\|f\|_\infty \int \int_0^t \nu_s ds d\mathbb{N}_0(\nu) E_{x_2}\bigg(\frac{1}{\int_0^t x_s^{(2)} ds}\bigg) \\ &\leq c\|f\|_\infty \frac{t}{t(t+x_2)} = \frac{c\|f\|_\infty}{t+x_2}.\end{aligned} \tag{74}$$

We now use the decomposition of Lemma 10 with $\rho = 0$. Use (30) and (31) to conclude that if $G = G_{t,x_1}$, then

$$\begin{aligned}E_2 = ct^{-1} E\bigg(&G\bigg(\sum_{j=1}^{N_t+1} r_j(t) + I_2(t), \sum_{j=1}^{N_t+1} e_j(t) + X_0'(t)\bigg) \\ &- G\bigg(\sum_{j=1}^{N_t} r_j(t) + I_2(t), \sum_{j=1}^{N_t} e_j(t) + X_0'(t)\bigg)\bigg).\end{aligned}$$

Let $w = \frac{x_2}{\gamma_2^0 t}$ and recall that $\|G\|_\infty \le \|f\|_\infty$. We may sum by parts and use the independence of $N_t$ from $(I_t, X_0'(t), \{e_j\})$ to see that

$$\begin{aligned}
|E_2| &= ct^{-1}\bigg|\sum_{k=0}^{\infty} p_k(w)E(G(R_{k+1}+I_2(t), S_{k+1}+X_0'(t)) \\
&\qquad\qquad - G(R_k + I_2(t), S_k + X_0'(t)))\bigg| \\
&\le ct^{-1}\|f\|_\infty \sum_{k=1}^{\infty}|p_{k-1}(w)-p_k(w)| + ct^{-1}e^{-w}\|f\|_\infty \\
&\le ct^{-1}\|f\|_\infty \sum_{k=1}^{\infty} p_k(w)\frac{|k-w|}{w} + ct^{-1}e^{-w}\|f\|_\infty \\
&\le ct^{-1}\|f\|_\infty \sum_{k=0}^{\infty} p_k(w)\frac{|k-w|}{w} \\
&\le c\|f\|_\infty t^{-1}w^{-1}[(E((N_t-w)^2)^{1/2} \wedge E(N_t+w)] \\
&\le \frac{c\|f\|_\infty}{\sqrt{t}\sqrt{t+x_2}}.
\end{aligned}$$

This and (74) give (70).

Next consider the second derivative for continuous $f$ and use the notation and conventions in the above argument. Proposition 14 and symmetry allows us to write

$$\begin{aligned}
(P_t f)_{22}(x) =& E_{x_2}\left(\iint \Delta G(x^{(2)},\nu,\nu')1(\nu_t=0,\nu_t'=0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\right) \\
&+ 2E_{x_2}\left(\iint \Delta G(x^{(2)},\nu,\nu')1(\nu_t=0,\nu_t'>0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\right) \\
&+ E_{x_2}\left(\iint \Delta G(x^{(2)},\nu,\nu')1(\nu_t>0,\nu_t'>0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\right) \\
\equiv& E_1 + 2E_2 + E_3.
\end{aligned}$$

Use Lemma 11(b), (c) and Lemma 13(a) with $g(y_1) = G(y_1, x_t^{(2)})$ to show that if $\chi_t = 1(\nu_t = \nu_t' = 0)$, then

$$\begin{aligned}
|E_1| &= \bigg|E_{x_2}\bigg(\iiint\bigg[G\bigg(\int_0^t (x_s^{(2)}+\nu_s+\nu_s')\,ds, x_t^{(2)}\bigg) - G\bigg(\int_0^t (x_s^{(2)}+\nu_s')\,ds, x_t^{(2)}\bigg) \\
&\quad - G\bigg(\int_0^t (x_s^{(2)}+\nu_s)\,ds, x_t^{(2)}\bigg) + G\bigg(\int_0^t x_s^{(2)}\,ds, x_t^{(2)}\bigg)\bigg]\chi_t d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\bigg)\bigg| \\
&\le c\|f\|_\infty E_{x_2}\bigg(\iint \int_0^t \nu_s\,ds \int_0^t \nu_s'\,ds d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\bigg(\int_0^t x_s^{(2)}\,ds\bigg)^{-2}\bigg) \\
&\le c\|f\|_\infty \frac{t^2}{t^2(t+x_2)^2} = \frac{c\|f\|_\infty}{(t+x_2)^2},
\end{aligned}$$

where in the last line we have used (26) and Lemma 7(b).

For $E_2$ we may drop $\nu_t$ from the expression for $G(X,\nu,\nu')$, regroup terms, and use Lemma 11(b) and (25) to write

$$\begin{aligned}|E_2| &\le E_{x_2}\Bigg(\iint\Bigg\{\Bigg|G\Bigg(\int_0^t x_s^{(2)}+\nu_s+\nu'_s\,ds, x_t^{(2)}+\nu'_t\Bigg)\\ &\qquad - G\Bigg(\int_0^t x_s^{(2)}+\nu'_s\,ds, x_t^{(2)}+\nu'_t\Bigg)\Bigg|1(\nu'_t>0)\\ &\quad+\Bigg|G\Bigg(\int_0^t x_s^{(2)}+\nu_s\,ds, x_t^{(2)}\Bigg)-G\Bigg(\int_0^t x_s^{(2)}\,ds, x_t^{(2)}\Bigg)\Bigg|1(\nu'_t>0)\Bigg\}\\ &\qquad\qquad d\mathbb{N}_0(d\nu)d\mathbb{N}_0(\nu')\Bigg)\\ &\le \frac{c}{t}\Bigg[\|f\|_\infty\int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)E_{x_2}\Bigg(\Bigg(\int_0^t x_s^{(2)}\,ds\Bigg)^{-1}\Bigg)\Bigg]\\ &\le \frac{c\|f\|_\infty}{t(x_2+t)},\end{aligned}$$

where in the last line we have again used (26) and Lemma 7(b).

Again using the decomposition and notation of Lemma 10 with $\rho=0$ and (25), we have

(75)

$$\begin{aligned}|E_3| &= (\gamma_2^0)^{-2}t^{-2}\Bigg|E_{x_2}\Bigg(\iint G\Bigg(\int_0^t x_s^{(2)}+\nu_s+\nu'_s\,ds, x_t^{(2)}+\nu'_t+\nu_t\Bigg)\\ &\quad - G\Bigg(\int_0^t x_s^{(2)}+\nu'_s\,ds, x_t^{(2)}+\nu'_t\Bigg)-G\Bigg(\int_0^t x_s^{(2)}+\nu_s\,ds, x_t^{(2)}+\nu_t\Bigg)\\ &\quad + G\Bigg(\int_0^t x_s^{(2)}\,ds, x_t^{(2)}\Bigg)dP_t^*(\nu)dP_t^*(\nu')\Bigg)\Bigg|\\ &= \frac{c}{t^2}\Bigg|E\Bigg(G\Bigg(\sum_{j=1}^{N_t+2} r_j(t)+I_2(t), \sum_{j=1}^{N_t+2} e_j(t)+X'_0(t)\Bigg)\\ &\quad - 2G\Bigg(\sum_{j=1}^{N_t+1} r_j(t)+I_2(t), \sum_{j=1}^{N_t+1} e_j(t)+X'_0(t)\Bigg)\\ &\quad + G\Bigg(\sum_{j=1}^{N_t} r_j(t)+I_2(t), \sum_{j=1}^{N_t} e_j(t)+X'_0(t)\Bigg)\Bigg)\Bigg|\\ &= c(x_2t)^{-1}\Bigg|\sum_{k=0}^\infty wp_k(w)[G(R_{k+2}+I_2(t),S_{k+2}+X'_0(t))\\ &\quad - 2G(R_{k+1}+I_2(t),S_{k+1}+X'_0(t))+G(R_k+I_2(t),S_k+X'_0(t))]\Bigg|.\end{aligned}$$

Recall here that $w = \frac{x_2}{\gamma_2^0 t}$. Now sum by parts twice and use $|G| \le \|f\|_\infty$ to bound the above by

$$\begin{aligned}
&c\|f\|_\infty (x_2 t)^{-1} \left| w[-p_0(w) + p_1(w)] + \sum_{k=2}^{\infty} w[p_{k-2}(w) - 2p_{k-1}(w) + p_k(w)] \right| \\
&\quad \le c\|f\|_\infty (x_2 t)^{-1} \left[ \sum_{k=2}^{\infty} p_k(w) \frac{|[w-k]^2 - k|}{w} + wp_0(w) + wp_1(w) \right] \\
&\quad \le c\|f\|_\infty (x_2 t)^{-1} \left[ \sum_{k=0}^{\infty} p_k(w) \left[ \frac{(w-k)^2 + k}{w} \right] + 2p_1(w) \right] \\
&\quad \le c\|f\|_\infty (x_2 t)^{-1}.
\end{aligned}$$

On the other hand if we use the trivial bound $|G| \le \|f\|_\infty$ in (75) we get $|E_3| \le c\|f\|_\infty t^{-2}$ and so

$$|E_3| \le \frac{c\|f\|_\infty}{t(t + x_2)}.$$

Combine the above bounds on $E_1$, $E_2$ and $E_3$ to obtain (71).

The bounds for $f \in S^\alpha$ are then obtained from the above just as in the proof of Proposition 16. □

Set $J_t = 2\gamma_1^0 I_t$, where $I_t$ is given by (44) and recall Convention 1, as adapted in (17). Recall that in our current setting, $|f|_{C_w^\alpha}$ is as in (43).

LEMMA 18. *For each $M \ge 1$ and $\alpha \in (0,1)$ there is a $c_{18} = c_{18}(M, \alpha) > 0$ such that if $M^0 \le M$, then*

$$|fg|_\alpha \le c_{18} |f|_{C_w^\alpha} \|g\|_\infty + \|f\|_\infty |g|_\alpha \tag{76}$$

*and*

$$\|fg\|_\alpha \le c_{18} [\|f\|_{C_w^\alpha} \|g\|_\infty + \|f\|_\infty |g|_\alpha] \tag{77}$$

*Proof.* Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+$ and define $\tilde{f}(y) = f(y) - f(x)$. Then (18) gives

$$\begin{aligned}
|P_t(fg)(x) - fg(x)| &\le |P_t(\tilde{f}g)(x)| + |f(x)||P_t g(x) - g(x)| \\
&\le \|g\|_\infty E_{x_2} \left( \int |\tilde{f}(z, x_t^{(2)})| p_{J_t}(z - x_1 - b_1^0 t)\, dz \right) \\
&\qquad + \|f\|_\infty |g|_\alpha t^{\alpha/2}.
\end{aligned} \tag{78}$$

Write

$$E_{x_2}\left(\int |\tilde{f}(z,x_t^{(2)})|p_{J_t}(z-x_1-b_1^0t)dz\right) \tag{79}$$
$$\le E_{x_2}\left(\int |\tilde{f}(z,x_t^{(2)})-\tilde{f}(z,x_2)|p_{J_t}(z-x_1-b_1^0t)dz\right)$$
$$+ E_{x_2}\left(\int |f(z,x_2)-f(x_1+b_1^0t,x_2)|p_{J_t}(z-x_1-b_1^0t)dz\right)$$
$$+ |f(x_1+b_1^0t,x_2)-f(x)|$$
$$\equiv E_1+E_2+E_3.$$

The definition of $|f|_{\alpha,i}$ gives

$$E_1 \le |f|_{\alpha,2}E_{x_2}(|x_t^{(2)}-x_2|^\alpha x_2^{-\alpha/2} \wedge |x_t^{(2)}-x_2|^{\alpha/2})$$
$$\le |f|_{\alpha,2}\left[E_{x_2}(|x_t^{(2)}-x_2|^2)^{\alpha/2}x_2^{-\alpha/2} \wedge E_{x_2}(|x_t^{(2)}-x_2|^2)^{\alpha/4}\right].$$

Lemma 7 (a) gives

$$E_{x_2}(|x_t^{(2)}-x_2|^2) \le cM^2(t^2+x_2t)$$

for some universal constant $c$. Therefore

$$E_1 \le |f|_{\alpha,2}c(M,\alpha)[((t^2+x_2t)^{\alpha/2}x_2^{-\alpha/2}) \wedge (t^2+x_2t)^{\alpha/4}]$$
$$\le c(M,\alpha)|f|_{\alpha,2}t^{\alpha/2}[(t/x_2+1)^{\alpha/2} \wedge (1+x_2/t)^{\alpha/4}]$$
$$\le c(M,\alpha)2^{\alpha/2}|f|_{\alpha,2}t^{\alpha/2}.$$

Similarly we have

$$E_2 \le |f|_{\alpha,2}E_{x_2}\left(\int [|z-(b_1^0t+x_1)|^\alpha x_2^{-\alpha/2}] \wedge [|z-(b_1^0t+x_1)|^{\alpha/2}]\right.$$
$$\left. p_{J_t}(z-x_1-b_1^0t)dz\right)$$
$$\le c|f|_{\alpha,2}\left(E_{x_2}(J_t^{\alpha/2}x_2^{-\alpha/2}) \wedge E_{x_2}(J_t^{\alpha/4})\right)$$
$$\le c|f|_{\alpha,2}\left(\left(E_{x_2}(J_t)^{\alpha/2}x_2^{-\alpha/2}\right) \wedge E_{x_2}(J_t)^{\alpha/4}\right).$$

Lemma 7(a) shows that $E_{x_2}(J_t) \le 2M(t^2+x_2t)$. Put this in the above bound on $E_2$ and argue as above to see

$$E_2 \le c(M,\alpha)|f|_{\alpha,2}t^{\alpha/2}.$$

Put the above bounds on $E_1$ and $E_2$ into (79) and then in (78) to see that

$$|P_t(fg)(x)-fg(x)|$$
$$\le \|g\|_\infty[c(M,\alpha)(|f|_{\alpha,2}+|f|_{\alpha,2}+|b_1^0|^{\alpha/2}|f|_{\alpha,2})]t^{\alpha/2}+\|f\|_\infty|g|_\alpha t^{\alpha/2}$$

and so

$$|fg|_\alpha \le c(M,\alpha)|f|_{C^\alpha_w}\|g\|_\infty + \|f\|_\infty |g|_\alpha.$$

This gives (76) and (77) is then immediate. ☐

THEOREM 19.

$$(2+c_{16}+c_{17})^{-1}|f|_{C^\alpha_w} \le |f|_\alpha \le c_{18}|f|_{C^\alpha_w}. \tag{80}$$

*In particular* $C^\alpha_w = S^\alpha$ *and so* $S^\alpha$ *contains* $C^1$ *functions with compact support in* $\mathbb{R}\times\mathbb{R}_+$.

*Proof.* Set $g=1$ in Lemma 18 to obtain the second inequality in (80).

Let $x,h\in\mathbb{R}\times\mathbb{R}_+$, $t>0$ and use Propositions 16 and 17 to see that

$$\begin{aligned}|f(x+h)-f(x)| &\le 2|f|_\alpha t^{\alpha/2} + |P_tf(x+h)-P_tf(x)| \\ &\le 2|f|_\alpha t^{\alpha/2} + (c_{16}+c_{17})|f|_\alpha t^{\alpha/2-1/2}(x_2+t)^{-1/2}|h|.\end{aligned} \tag{81}$$

First set $t=|h|$ and bound $(x_2+t)^{-1/2}$ by $t^{-1/2}$ to see that (81) is at most

$$(2+c_{16}+c_{17})|f|_\alpha |h|^{\alpha/2}. \tag{82}$$

Next for $x_2>0$, set $t=|h|^2/x_2$ and bound $(x_2+t)^{-1/2}$ by $x_2^{-1/2}$ to bound (81) by

$$(2+c_{16}+c_{17})|f|_\alpha x_2^{-\alpha/2}|h|^\alpha. \tag{83}$$

The first inequality in (80) is now immediate from (82) and (83) and the proof is complete. ☐

We next state versions of Propositions 16 and 17 for general $m\ge 1$, i.e., for the semigroup $P_t$ on $\mathbb{R}^m\times\mathbb{R}_+$ given by (18). The proofs are minor modifications of those already given for $m=1$ as one replaces a one-dimensional Gaussian density by an $m$-dimensional one and then makes some obvious changes. We have only stated the extensions we will actually need below.

PROPOSITION 20.

(a) *If* $f$ *is a bounded Borel function on* $\mathbb{R}^m\times\mathbb{R}_+$, *then*

$$\sum_{j=1}^{m+1}\left\|\frac{\partial}{\partial x_j}P_tf\right\|_\infty \le \frac{c_{20}\|f\|_\infty}{\sqrt{t}\sqrt{x_{m+1}+t}}. \tag{84}$$

(b) *If* $f\in S^\alpha$, *then*

$$\sum_{j=1}^{m+1}\left\|\frac{\partial}{\partial x_j}P_tf\right\|_\infty \le \frac{c_{20}|f|_\alpha t^{\frac{\alpha}{2}-\frac12}}{\sqrt{x_{m+1}+t}} \le c_{20}|f|_\alpha t^{\frac{\alpha}{2}-1}, \tag{85}$$

*and*

$$\sum_{j=1}^{m+1}\left\|x_{m+1}\frac{\partial^2}{\partial x_j^2}P_tf\right\|_\infty \le c_{20}|f|_\alpha t^{\frac{\alpha}{2}-1}. \tag{86}$$

REMARK 21. The proof of Theorem 19 now goes through with only minor changes to prove (80) in this $m+1$ dimensional setting. For $m=0$, i.e., for the semigroup of $\mathcal{A}^1 f(x)=b_1^0 f'(x)+\gamma_1^0 x f''(x)$ $(x\ge 0)$, this result was proved in Section 7 of [ABP] assuming only $b_1^0\ge 0$ (as opposed to the $b_1^0>0$ assumed here). This strengthening will be used in Section 3 below.

**2.4. Weighted Hölder bounds on the catalytic semigroup.** In this section we will obtain bounds on the weighted Hölder norms of $P_tf$. We continue to work in the same setting as Sections 2.2 and 2.3 with $m=1$. As usual, the $x_1$ derivatives are easier.

PROPOSITION 22. *If $f$ is a bounded Borel function on $\mathbb{R}\times\mathbb{R}_+$, then for all $x,h\in\mathbb{R}\times\mathbb{R}_+$,*

$$\left|\frac{\partial P_tf}{\partial x_1}(x+h)-\frac{\partial P_tf}{\partial x_1}(x)\right|\le \frac{c_{22}\|f\|_\infty}{t^{3/2}(x_2+t)^{1/2}}|h|, \tag{87}$$

*and*

$$\left|(x+h)_2\frac{\partial^2 P_tf}{\partial x_1^2}(x+h)-x_2\frac{\partial^2 P_tf}{\partial x_1^2}(x)\right|\le \frac{c_{22}\|f\|_\infty}{t^{3/2}(x_2+t)^{1/2}}|h|. \tag{88}$$

*If $f\in S^\alpha$, then for all $x,h\in\mathbb{R}\times\mathbb{R}_+$,*

$$\left|\frac{\partial P_tf}{\partial x_1}(x+h)-\frac{\partial P_tf}{\partial x_1}(x)\right|\le c_{22}|f|_\alpha t^{\frac{\alpha}{2}-\frac{3}{2}}(x_2+t)^{-1/2}|h|, \tag{89}$$

*and*

$$\left|(x+h)_2\frac{\partial^2 P_tf}{\partial x_1^2}(x+h)-x_2\frac{\partial^2 P_tf}{\partial x_1^2}(x)\right|\le c_{22}|f|_\alpha t^{\frac{\alpha}{2}-\frac{3}{2}}(x_2+t)^{-1/2}|h|. \tag{90}$$

*Proof.* As in the proof of Proposition 14 it suffices to consider $f$ bounded and continuous. As (87) is simpler, we only give a proof of (88). Recall $\check{e}_i$ is the $i$th unit basis vector and the definition (44) of $I_t$. From (61) and Lemma 11(a) we have (recall $G''_{t,x_1}$ denotes the second derivative with respect to $x_1$),

$$\begin{aligned} &|x_2(P_tf)_{11}(x+h_1\check{e}_1)-x_2(P_tf)_{11}(x)| \\ &\quad = x_2|E_{x_2}((G''_{t,x_1+h_1}-G''_{t,x_1})(I_t,x_t^{(2)}))| \\ &\quad \le x_2c_3\|f\|_\infty E_{x_2}(I_t^{-3/2})|h_1| \\ &\quad \le x_2c_3\|f\|_\infty c_7t^{-3/2}(x_2+t)^{-3/2}|h_1| \text{ (by Lemma 7(b))} \\ &\quad \le c\|f\|_\infty t^{-3/2}(x_2+t)^{-1/2}|h_1|. \end{aligned} \tag{91}$$

Turning now to increments in $x_2$, (61) implies that for $h_2 \ge 0$,

$$\begin{aligned}
&(x_2+h_2)(P_tf)_{11}(x+h_2\check{e}_2) - x_2(P_tf)_{11}(x)\\
&\qquad = [(x_2+h_2)E_{x_2+h_2} - x_2E_{x_2}](G''_{t,x_1}(I_t, x_t^{(2)}))\\
&\qquad = h_2E_{x_2+h_2}(G''_{t,x_1}(I_t, x_t^{(2)})) + x_2[E_{x_2+h_2} - E_{x_2}](G''_{t,x_1}(I_t, x_t^{(2)}))\\
&\qquad = E_1 + E_2.
\end{aligned}$$

By Lemmas 11(a) and 7(b),

$$|E_1| \le h_2 c_{11}\|f\|_\infty E_{x_2+h_2}(I_t^{-1}) \le c\|f\|_\infty t^{-1}(x_2+t)^{-1}h_2.$$

For $E_2$ we use the decompositions (33), (34), (35) and notation from Lemma 10 with $\rho = \frac{1}{2}$. Then

$$\begin{aligned}
|E_2| = x_2\Bigg|E\Bigg(&G''_{t,x_1}\Bigg(\sum_{j=1}^{N'_t} r_j(t) + I_2(t) + I_3^h(t), \sum_{j=1}^{N'_t} e_j(t) + X'_0(t)\Bigg)\\
&- G''_{t,x_1}\Bigg(\sum_{j=1}^{N_t} r_j(t) + I_2(t), \sum_{j=1}^{N_t} e_j(t) + X'_0(t)\Bigg)\Bigg)\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\le x_2\Bigg|E\Bigg(G''_{t,x_1}\Bigg(\sum_{j=1}^{N'_t} r_j(t) + I_2(t) + I_3^h(t), \sum_{j=1}^{N'_t} e_j(t) + X'_0(t)\Bigg)\\
&\qquad - G''_{t,x_1}\Bigg(\sum_{j=1}^{N'_t} r_j(t) + I_2(t), \sum_{j=1}^{N'_t} e_j(t) + X'_0(t)\Bigg)\Bigg)\Bigg|\\
&\quad + x_2\Bigg|E\Bigg(G''_{t,x_1}\Bigg(\sum_{j=1}^{N'_t} r_j(t) + I_2(t), \sum_{j=1}^{N'_t} e_j(t) + X'_0(t)\Bigg)\\
&\qquad - G''_{t,x_1}\Bigg(\sum_{j=1}^{N_t} r_j(t) + I_2(t), \sum_{j=1}^{N_t} e_j(t) + X'_0(t)\Bigg)\Bigg)\Bigg|\\
&\equiv E_{2a} + E_{2b}.
\end{aligned}$$

By Lemmas 11(b) and 7(b), and the independence of $x^{(2)}$ and $\Xi^h$,

$$\begin{aligned}
|E_{2a}| &\le x_2\, c\|f\|_\infty E_{x_2}\Bigg(\Bigg(\int_0^t x_s^{(2)}ds\Bigg)^{-2}\Bigg)E(I_3^h(t))\\
&\le x_2\, c\|f\|_\infty c_7 t^{-2}(x_2+t)^{-2}h_2 \int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)\\
&\le c\|f\|_\infty t^{-1}(x_2+t)^{-1}h_2,
\end{aligned}$$

where (26) is used in the last line. Recall that $R_k = \sum_{j=1}^k r_j(t)$ and $p_k(w) = e^{-w}w^k/k!$. The independence of $N_t'$ from the other random variables appearing in the first term in $E_{2b}$ allows us to condition on its value. The same is true for $N_t$ in the second term in $E_{2b}$. Therefore, if $w = x_2/2\gamma_2^0 t$ and $w' = w + (h_2/\gamma_2^0 t)$, then by Lemma 11(a),

$$\begin{aligned}
|E_{2b}| &= x_2\left|\sum_{k=0}(p_k(w') - p_k(w))E(G''_{t,x_1}(R_k + I_2(t), S_k + X_0'(t)))\right| \\
&\le ctw\sum_{k=0}^{\infty}\left|\int_w^{w'} p_k'(u)\,du\right|\|f\|_\infty E\left(\left(\int_0^t X_0'(s)\,ds\right)^{-1}\right) \\
&\le c\|f\|_\infty tw\int_w^{w'}\sum_{k=0}^{\infty} p_k(u)\frac{|k-u|}{u}\,du t^{-1}\left(\frac{x_2}{2}+t\right)^{-1} \text{ (by Lemma 7(b))} \\
&\le c\|f\|_\infty tw\int_w^{w'} u^{-1/2} du t^{-1}(x_2+t)^{-1} \\
&\le c\|f\|_\infty w^{1/2}h_2 t^{-1}(x_2+t)^{-1} \\
&\le c\|f\|_\infty h_2 t^{-3/2}(x_2+t)^{-1/2}.
\end{aligned}$$

The above bounds on $|E_1|$, $|E_{2a}|$ and $|E_{2b}|$, and (91) give (88).

Let $f \in S^\alpha$. We only prove (90) as (89) is then proved by the same argument. If $g = P_t f - f$, then $\|g\|_\infty \le |f|_\alpha t^{\alpha/2}$, and so by (88),

$$\begin{aligned}
&|(x+h)_2(P_{2t}f - P_t f)_{11}(x+h) - x_2(P_{2t}f - P_t f)_{11}(x)| \\
&\quad = |(x+h)_2(P_t g)_{11}(x+h) - x_2(P_t g)_{11}(x)| \\
&\quad \le c|f|_\alpha t^{\frac{\alpha}{2}-\frac{3}{2}}(x_2+t)^{-1/2}|h| \to 0 \text{ as } t \to \infty.
\end{aligned}$$

Therefore we may write the left-hand side of (90) as a telescoping sum and use the above bound to show

$$\begin{aligned}
&|(x+h)_2(P_t f)_{11}(x+h) - x_2(P_t f)_{11}(x)| \\
&\quad = \left|\sum_{k=1}^{\infty}(x+h)_2(P_{2^k t}f - P_{2^{k-1}t}f)_{11}(x+h) - x_2(P_{2^k t}f - P_{2^{k-1}t}f)_{11}(x)\right| \\
&\quad \le \sum_{k=1}^{\infty} c|f|_\alpha (2^{k-1}t)^{\frac{\alpha}{2}-\frac{3}{2}}(x_2 + 2^{k-1}t)^{-1/2}|h| \\
&\quad \le c|f|_\alpha t^{\frac{\alpha}{2}-\frac{3}{2}}(x_2+t)^{-1/2}|h|. \qquad \square
\end{aligned}$$

PROPOSITION 23. *If $f$ is a bounded Borel function on $\mathbb{R}\times\mathbb{R}_+$, then for all $x,h\in\mathbb{R}\times\mathbb{R}_+$,*

$$\left|\frac{\partial P_t f}{\partial x_2}(x+h)-\frac{\partial P_t f}{\partial x_2}(x)\right|\le\frac{c_{23}\|f\|_\infty}{t(x_2+t)}|h|, \tag{92}$$

*and*

$$\left|(x+h)_2\frac{\partial^2 P_t f}{\partial x_2^2}(x+h)-x_2\frac{\partial^2 P_t f}{\partial x_2^2}(x)\right|\le\frac{c_{23}\|f\|_\infty}{t^{3/2}(x_2+t)^{1/2}}|h|. \tag{93}$$

*If $f\in S^\alpha$, then for all $x,h\in\mathbb{R}\times\mathbb{R}_+$,*

$$\left|\frac{\partial P_t f}{\partial x_2}(x+h)-\frac{\partial P_t f}{\partial x_2}(x)\right|\le c_{23}|f|_\alpha t^{\frac{\alpha}{2}-1}(x_2+t)^{-1}|h|, \tag{94}$$

*and*

$$\left|(x+h)_2\frac{\partial^2 P_t f}{\partial x_2^2}(x+h)-x_2\frac{\partial^2 P_t f}{\partial x_2^2}(x)\right|\le c_{23}|f|_\alpha t^{\frac{\alpha}{2}-\frac{3}{2}}(x_2+t)^{-1/2}|h|. \tag{95}$$

*Proof.* The last two inequalities follow from the first two just as in the proof of Proposition 22. As the proof of (92) is similar to, but much easier than, that of (93), we only prove the latter. As usual we may assume $f$ is bounded and continuous.

To simplify the write-up we assume $\gamma_1^0=1$ but note that our constants, as usual, will be uniform in $\epsilon\le\gamma_1^0\le\epsilon^{-1}$. Recall the notation $\Delta G_{t,x_1}(X,\nu,\nu')$ from (45) just before Proposition 14. That result shows

$$\begin{aligned}(P_tf)_{22}(x)&=E_{x_2}\left(\iint\Delta G_{t,x_1}(x^{(2)},\nu,\nu')d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\right)\\&=\sum_{j=1}^4 E_{x_2}(\Delta G^j_{t,x_1}(x^{(2)})),\end{aligned} \tag{96}$$

where

$$\Delta G^1_{t,x_1}(X)=\iint\Delta G_{t,x_1}(X,\nu,\nu')1(\nu_t=\nu'_t=0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu'),$$

$$\Delta G^2_{t,x_1}(X)=\iint\Delta G_{t,x_1}(X,\nu,\nu')1(\nu_t>0,\nu'_t=0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu'),$$

$$\Delta G^3_{t,x_1}(X)=\iint\Delta G_{t,x_1}(X,\nu,\nu')1(\nu_t=0,\nu'_t>0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu'),$$

and

$$\begin{aligned}\Delta G^4_{t,x_1}(X)&=\iint\Delta G_{t,x_1}(X,\nu,\nu')1(\nu_t>0,\nu'_t>0)d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu'),\\&=(\gamma_2^0t)^{-2}\iint\Delta G_{t,x_1}(X,\nu,\nu')1(\nu_t>0,\nu'_t>0)dP^*_t(\nu)dP^*_t(\nu').\end{aligned}$$

In the last line $P_t^*$ is the probability defined by (27) and we have used (25). $\Delta G^j_{t,x_1}(X)$ depends on $X$ only through $X_t$ and $\int_0^t X_s ds$, and so we will abuse the notation and write $\Delta G^j_{t,x_1}(X_t, \int_0^t X_s ds)$ when it is convenient.

Consider first the increments in $x_2$. Let $h_2 \geq 0$ and use (96) to write

$$\begin{aligned}(97)\qquad & |(x_2+h_2)(P_t f)_{22}(x_1,x_2+h_2) - x_2(P_t f)_{22}(x_1,x_2)| \\ &\quad \leq h_2|(P_t f)_{22}(x_1,x_2+h_2)| \\ &\qquad + \sum_{j=1}^4 |((x_2+2h_2)E_{x_2+h_2} - x_2 E_{x_2})(\Delta G^j_{t,x_1}(X))|.\end{aligned}$$

In the following lemmas we again use the decompositions of $x_t^{(2)}$ and $X_t^{x+h}$ from Lemma 10 with $\rho = \frac{1}{2}$.

Lemma 24. *We have*

$$|((x_2+2h_2)E_{x_2+h_2} - x_2 E_{x_2})(\Delta G^1_{t,x_1}(X))| \leq \frac{c_{24}\|f\|_\infty}{t(x_2+t)} h_2.$$

*Proof.* Write $\Delta G^j$ for $\Delta G^j_{t,x_1}$ and $G$ for $G_{t,x_1}$. The left-hand side is bounded by

$$\begin{aligned}& 2h_2 E_{x_2+h_2}(|\Delta G^1(X)|) \\ &+ x_2\left|E\left(\Delta G^1\left(\int_0^t x_s^{(2)} + X_s^h ds, X_t^{x+h}\right) - \Delta G^1\left(\int_0^t x_s^{(2)} ds, X_t^{x+h}\right)\right)\right| \\ &+ x_2\left|E\left(\Delta G^1\left(\int_0^t x_s^{(2)} ds, x_t^{(2)} + X_t^h\right) - \Delta G^1\left(\int_0^t x_s^{(2)}, x_t^{(2)}\right)\right)\right| \\ &= E_1 + E_2 + E_3.\end{aligned}$$

Use Lemmas 13(a) and 11(b), and (26) to get

$$\begin{aligned}E_1 &\leq 2h_2\bigg|E_{x_2+h_2}\bigg(\iiint\bigg(G\left(\int_0^t X_s ds + \int_0^t \nu_s ds + \int_0^t \nu'_s ds, X_t\right) \\ &\qquad - G\left(\int_0^t X_s ds + \int_0^t \nu_s ds, X_t\right) - G\left(\int_0^t X_s ds + \int_0^t \nu'_s ds, X_t\right) \\ &\qquad + G\left(\int_0^t X_s ds, X_t\right)\bigg) 1(\nu_t = \nu'_t = 0)\bigg) d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu')\bigg)\bigg| \\ &\leq 2h_2 c_2\|f\|_\infty E_{x_2+h_2}\left(\left(\int_0^t X_s ds\right)^{-2}\right)\left[\int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)\right]^2 \\ &\leq ch_2\|f\|_\infty t^{-2}(x_2+t)^{-2} t^2 = ch_2\|f\|_\infty (x_2+t)^{-2}.\end{aligned}$$

The integrand in $E_2$ is a third order difference in $G(\cdot, X_t^{x+h})$ to which we may apply Lemma 13(b) and the bound on $\frac{\partial^3 G}{\partial y_1^3}(y_1,y_2)$ from Lemma 11, and the independence of $x^{(2)}$ and $X^h$ to conclude that

$$\begin{aligned}E_2 &\le x_2 c_3 \|f\|_\infty E_{x_2}\Big(\Big(\Big(\int_0^t x_s^{(2)} ds\Big)^{-3}\Big) E\Big(\int_0^t X_s^h ds)\Big)\Big(\int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)\Big)^2 \\ &\le c x_2 \|f\|_\infty t^{-3}(x_2+t)^{-3} h_2 t^3 \\ &\le c\|f\|_\infty (x_2+t)^{-2} h_2.\end{aligned}$$

Finally bound $E_3$ by

$$\begin{aligned}x_2 E\Big(1(X_t^h>0)\Big|\Delta G^1\Big(\int_0^t x_s^{(2)} ds, x_t^{(2)} + X_t^h\Big)\Big|\Big) \\ + x_2 E\Big(1(X_t^h>0)\Big|\Delta G^1\Big(\int_0^t x_s^{(2)} ds, x_t^{(2)}\Big)\Big|\Big).\end{aligned}$$

Each term can be handled in the same way, so consider the first. The integrand defining $\Delta G^1(\int_0^t x_s^{(2)} ds, x_t^{(2)} + X_t^h)$ is a second order difference in $G(\cdot, x_t^{(2)} + X_t^h)$ to which we may apply Lemma 13(a) and the bound on $\frac{\partial^2 G}{\partial y_1^2}(y_1, y_2)$ from Lemma 11. This allows us to bound $E_3$ by

$$\begin{aligned}c_2\|f\|_\infty x_2 E_{x_2}\Big(\Big(\Big(\int_0^t x_s^{(2)} ds\Big)^{-2}\Big) P(X_t^h>0)\Big(\int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)\Big)^2 \\ \le c\|f\|_\infty x_2 t^{-2}(t+x_2)^{-2} h_2 t^{-1} t^2 \\ \le c\|f\|_\infty t^{-1}(x_2+t)^{-1} h_2.\end{aligned}$$

We have also used (28) in the second line to bound $P(X_t^h > 0)$. The above bounds on $E_1 - E_3$ give the required result. □

LEMMA 25. *For $j = 2, 3$ we have*

$$|((x_2 + 2h_2)E_{x_2+h_2} - x_2 E_{x_2})(\Delta G_{t,x_1}^j(X))| \le \frac{c_{25}\|f\|_\infty}{t(x_2+t)} h_2.$$

*Proof.* By symmetry we only need consider $j = 2$. As before we let $w = \frac{x_2}{2\gamma_2^0 t}$, $w_h = (\frac{x_2}{2} + h_2)(\gamma_2^0 t)^{-1}$, $S_k = \sum_{j=1}^k e_j(t)$, $R_k = \sum_{j=1}^k r_j(t)$, and $\mathbb{N}_0(\cdot \cap \{\nu_t > 0\}) = (\gamma_2^0 t)^{-1} P_t^*(\cdot)$ by (27). We also write $G$ for $G_{t,x_1}$. Let $Q_h$ be the law of $I_3^h(t)$. As this last random variable is independent of the others appearing below by an elementary property of Poisson point processes, we may condition on it and use (33), (34) and (35) to conclude

$$
\begin{aligned}
&(x_2+2h_2)E_{x_2+h_2}\Big(\Delta G^2\Big(\int_0^t X_s ds, X_t\Big)\Big)\\
&= 2w_h E\Big(\iiint\\
&\Big[G\Big(I_2(t)+z+R_{N'_t}+\int_0^t \nu_s ds+\int_0^t \nu'_s ds, X'_0(t)+S_{N'_t}+\nu_t\Big)\\
&\quad - G\Big(I_2(t)+z+R_{N'_t}+\int_0^t \nu'_s ds, X'_0(t)+S_{N'_t}\Big)\\
&\quad - G\Big(I_2(t)+z+R_{N'_t}+\int_0^t \nu_s ds, X'_0(t)+S_{N'_t}+\nu_t\Big)\\
&\quad + G(I_2(t)+z+R_{N'_t}, X'_0(t)+S_{N'_t})\Big]1(\nu'_t=0)dP^*_t(\nu)d\mathbb{N}_0(\nu')dQ_h(z)\Big).
\end{aligned}
$$

Recall that $p_k(u)=e^{-u}u^k/k!$. $N'_t$ is independent of $(I_2(t),\{R_k\},\{S_k\},X'_0(t))$ and so we may condition on its value to see that the above equals

$$
\begin{aligned}
(98)\qquad & 2\sum_{k=0}^{\infty} w_h p_k(w_h)\cdot E\Big(\iint 1(\nu'_t=0)\Big[G\Big(I_2(t)+z\\
&\qquad + R_{k+1}+\int_0^t \nu'_s ds, X'_0(t)+S_{k+1}\Big)\\
&\qquad - G\Big(I_2(t)+z+R_k+\int_0^t \nu'_s ds, X'_0(t)+S_k\Big)\\
&\qquad - G(I_2(t)+z+R_{k+1}, X'_0(t)+S_{k+1})\\
&\qquad + G(I_2(t)+z+R_k, X'_0(t)+S_k)\Big]d\mathbb{N}_0(\nu')dQ_h(z)\Big)\\
&= 2\sum_{k=0}^{\infty} q_k(w_h)\int (G_{k+1}(z)-G_k(z))dQ_h(z),
\end{aligned}
$$

where $q_k(y)=yp_k(y)=(k+1)p_{k+1}(y)$ and

$$
\begin{aligned}
G_k(z) = E\Big(\int 1(\nu'_t=0)\Big[G\Big(I_2(t)+z+R_k+\int_0^t \nu'_s ds, X'_0(t)+S_k\Big)\\
- G(I_2(t)+z+R_k, X'_0(t)+S_k)\Big]d\mathbb{N}_0(\nu')\Big).
\end{aligned}
$$

When working under $E_{x_2}$ there is no $I_3^h(t)$ term and so similar reasoning leads to

$$
(99)\qquad x_2 E_{x_2}\Big(\Delta G^2\Big(\int_0^t x_s^{(2)}\,ds, x_t^{(2)}\Big)\Big) = 2\sum_{k=0}^{\infty} q_k(w)(G_{k+1}(0)-G_k(0)).
$$

If $H_k = \int G_k(z) - G_k(0) dQ_h(z)$, and $d_k = q_k(w_h) - q_k(w)$, where $d_{-1} = q_{-1} = 0$, we may take differences between (98) and (99) and then sum by parts to get

(100)

$$\begin{aligned}
&\left|\Big[(x_2+h_2)E_{x_2+2h_2} - x_2 E_{x_2}\Big]\left(\Delta G^2\left(\int_0^t X_s ds, X_t\right)\right)\right| \\
&\qquad = 2\left|\sum_{k=0}^{\infty} q_k(w_h)\left[\int G_{k+1}(z) - G_k(z) dQ_h(z)\right] - q_k(w)(G_{k+1}(0) - G_k(0))\right| \\
&\qquad = 2\left|\sum_{k=0}^{\infty}\left[d_k \int G_{k+1}(z) - G_k(z) dQ_h(z) + \sum_{k=0}^{\infty} q_k(w)(H_{k+1} - H_k)\right]\right| \\
&\qquad = 2\left|\sum_{k=0}^{\infty}(d_{k-1} - d_k)\int G_k(z) dQ_h(z) + \sum_{k=0}^{\infty}(q_{k-1}(w) - q_k(w))H_k\right|.
\end{aligned}$$

Now

$$\begin{aligned}
H_k = E\bigg(\iint 1(\nu'_t = 0)\bigg[&G\left(I_2(t) + z + R_k + \int_0^t \nu'_s ds, X'_0(t) + S_k\right) \\
&- G(I_2(t) + z + R_k, X'_0(t) + S_k) \\
&- G\left(I_2(t) + R_k + \int_0^t \nu'_s ds, X'_0(t) + S_k\right) \\
&+ G(I_2(t) + R_k, X'_0(t) + S_k)\bigg] d\mathbb{N}_0(\nu') dQ_h(z)\bigg).
\end{aligned}$$

The expression inside the square brackets is a second order difference in the first variable of $G$ to which we may apply Lemmas 13(a) and 11(b), and conclude

(101)

$$\begin{aligned}
|H_k| &\le c\|f\|_\infty \int 1(\nu'_t = 0)\int_0^t \nu'_s ds d\mathbb{N}_0(\nu') \int z dQ_h(z) E\left(\left(\int_0^t X'_0(s) ds\right)^{-2}\right) \\
&\le c\|f\|_\infty t h_2 \int\int_0^t \nu'_s ds d\mathbb{N}_0(\nu')\left(\frac{x_2}{2} + t\right)^{-2} t^{-2} \quad \text{(by Lemma 7(b))} \\
&\le c\|f\|_\infty h_2 (x_2 + t)^{-2}.
\end{aligned}$$

For all $k \ge 0$, $q_{k-1}(w) - q_k(w) = p_k(w)(k - w)$ and we therefore may conclude that

$$\sum_{k=0}^{\infty} |q_{k-1}(w) - q_k(w)| = \sum_{k=0}^{\infty} p_k(w)|k - w| \le w^{1/2}. \tag{102}$$

The bound on $|\partial G/\partial y_1|$ from Lemma 11(b) implies that

$$
\begin{aligned}
(103)\qquad |G_k(z)| &\le \|f\|_\infty E\left(\left(\int_0^t X_0'(s)ds\right)^{-1}\right)\int\int_0^t \nu_s' ds d\mathbb{N}_0(\nu') \\
&\le c\|f\|_\infty (x_2+t)^{-1} \text{ (by (26) and Lemma 7(b)).}
\end{aligned}
$$

Use the identity $kp_k'(u) = k(p_{k-1}(u) - p_k(u))$, for all $k \ge 0$, and some algebra to see that

$$
\begin{aligned}
(104)\qquad \sum_{k=0}^\infty |d_{k-1} - d_k| &= \sum_{k=0}^\infty \left|\int_w^{w_h} q_{k-1}'(u) - q_k'(u)du\right| \\
&= \sum_{k=0}^\infty \left|\int_w^{w_h} kp_k'(u) - (k+1)p_{k+1}'(u)du\right| \\
&= \sum_{k=0}^\infty \left|\int_w^{w_h} p_k(u)\left[\frac{(k-u)^2}{u} - 1\right]du\right| \\
&\le \int_w^{w_h}\left[\sum_{k=0}^\infty p_k(u)(k-u)^2u^{-1}\right] + 1du \\
&= 2h_2(\gamma_2^0 t)^{-1}.
\end{aligned}
$$

Use (101)–(104) in (100) to see that

$$
\begin{aligned}
&\left|((x_2+2h_2)E_{x_2+h_2} - x_2E_{x_2})\left(\Delta G^2\left(\int_0^t X_s ds, X_t\right)\right)\right| \\
&\quad \le \frac{c\|f\|_\infty h_2}{\gamma_2^0 t(x_2+t)} + \frac{\sqrt{x_2}}{\sqrt{\gamma_2^0 t}}\frac{c\|f\|_\infty h_2}{(x_2+t)^2} \\
&\quad \le \frac{c\|f\|_\infty h_2}{t(x_2+t)},
\end{aligned}
$$

as required. □

LEMMA 26. *We have*

$$
|((x_2+2h_2)E_{x_2+h_2} - x_2E_{x_2})(\Delta G_{t,x_1}^4(X))| \le \frac{c_{26}\|f\|_\infty}{t^{3/2}(x_2+t)^{1/2}}h_2.
$$

*Proof.* We use the same setting and notation as in Lemma 25. Also introduce

$$
\hat{G}_k(z) = E(G(I_2(t) + R_k + z, X_0'(t) + S_k)),
$$

and

$$
\hat{H}_k = \int (\hat{G}_k(z) - \hat{G}_k(0))\, dQ_h(z).
$$

As in the derivation of (98) we have

(105)
$$
\begin{aligned}
&(x_2+2h_2)E_{x_2+h_2}\Big(\Delta G^4\Big(\int_0^t X_s ds, X_t\Big)\Big)\\
&\quad=\frac{(x_2+2h_2)}{(\gamma_2^0 t)^2}E\Big(\iiint\Big[G\Big(I_2(t)+R_{N'_t}+z\\
&\qquad+\int_0^t \nu_s ds+\int_0^t \nu'_s ds, X'_0(t)+S_{N'_t}+\nu_t+\nu'_t\Big)\\
&\qquad-G\Big(I_2(t)+R_{N'_t}+z+\int_0^t \nu_s ds, X'_0(t)+S_{N'_t}+\nu_t\Big)\\
&\qquad-G\Big(I_2(t)+R_{N'_t}+z+\int_0^t \nu'_s ds, X'_0(t)+S_{N'_t}+\nu'_t\Big)\\
&\qquad+G(I_2(t)+R_{N'_t}+z, X'_0(t)+S_{N'_t})\Big]dP_t^*(\nu)dP_t^*(\nu')dQ_h(z)\Big)\\
&\quad=2(\gamma_2^0 t)^{-1}\sum_{k=0}^\infty w_h p_k(w_h)\int(\hat G_{k+2}(z)-2\hat G_{k+1}(z)+\hat G_k(z))dQ_h(z)\\
&\quad=2(\gamma_2^0 t)^{-1}\sum_{k=0}^\infty q_k(w_h)\int(\hat G_{k+2}(z)-2\hat G_{k+1}(z)+\hat G_k(z))dQ_h(z).
\end{aligned}
$$

A similar argument holds under $P_{x_2}$ but now there is no $I_3^h(t)$ term and hence no $Q_h$ integration. This leads to

(106)
$$
\begin{aligned}
&x_2Ex_2\Big(\Delta G^4\Big(\int_0^t X_s ds, X_t\Big)\Big)\\
&\qquad=2(\gamma_2^0 t)^{-1}\sum_{k=0}^\infty q_k(w)(\hat G_{k+2}(0)-2\hat G_{k+1}(0)+\hat G_k(0)).
\end{aligned}
$$

Recall that $d_k=q_k(w_h)-q_k(w)$ and $p_k=d_k=q_k=0$ if $k<0$. Take differences between (105) and (106) and then sum by parts twice to see that

(107)
$$
\begin{aligned}
&\Big|((x_2+2h_2)E_{x_2+h_2}-x_2E_{x_2})\Big(\Delta G^4\Big(\int_0^t X_s ds, X_t\Big)\Big)\Big|\\
&\quad=2(\gamma_2^0 t)^{-1}\Big|\sum_{k=0}^\infty q_k(w_h)\int(\hat G_{k+2}(z)-2\hat G_{k+1}(z)+\hat G_k(z))dQ_h(z)\\
&\qquad\qquad-q_k(w)(\hat G_{k+2}(0)-2\hat G_{k+1}(0)+\hat G_k(0))\Big|
\end{aligned}
$$

$$= 2(\gamma_2^0 t)^{-1}\Big|\sum_{k=0}^{\infty} d_k \int (\hat{G}_{k+2} - 2\hat{G}_{k+1} + \hat{G}_k)(z)dQ_h(z)$$
$$+ q_k(w)[\hat{H}_{k+2} - 2\hat{H}_{k+1} + \hat{H}_k]\Big|$$
$$= 2(\gamma_2^0 t)^{-1}\Big|\sum_{k=0}^{\infty}(d_{k-2} - 2d_{k-1} + d_k)\int \hat{G}_k(z)dQ_h(z)$$
$$+ \sum_{k=0}^{\infty}(q_{k-2}(w) - 2q_{k-1}(w) + q_k(w))\hat{H}_k\Big|.$$

Lemmas 11(a) and 7(b) imply

$$\begin{aligned} |\hat{H}_k| &= |E(G(I_2(t) + R_k + I_3^h(t), X_0'(t) + S_k) \\ &\quad - G(I_2'(t) + R_k, X_0'(t) + S_k)| \\ &\le c\|f\|_\infty E\left(\left(\int_0^t X_0'(s)ds\right)^{-1}\right)E(I_3^h(t)) \\ &\le c\|f\|_\infty t^{-1}\left(\frac{x_2}{2} + t\right)^{-1}\int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)h_2 \\ &\le c\|f\|_\infty (x_2 + t)^{-1}h_2. \end{aligned} \tag{108}$$

We also have

$$\begin{aligned} &\sum_{k=0}^{\infty}|q_{k-2}(w) - 2q_{k-1}(w) + q_k(w)| \\ &\qquad = \sum_{k=0}^{\infty} e^{-w}\frac{w^k}{k!}\left|\frac{k(k-1) - 2kw + w^2}{w}\right| \\ &\qquad \le \sum_{k=0}^{\infty} p_k(w)\frac{[(w-k)^2 + k]}{w} = 2. \end{aligned} \tag{109}$$

Finally $q_k(u) = (k+1)p_{k+1}(u)$ implies for $k \ge -2$,

$$q_k'(u) = (k+1)p_{k+1}'(u) = (k+1)(p_k(u) - p_{k+1}(u)).$$

Use this and a bit of algebra to see that if

$$h(k,u) = (k-u)^3 - 2(k-u)^2 + 3u(u-k) + k,$$

then

$$\sum_{k=0}^{\infty}|d_{k-2}-2d_{k-1}+d_k| = \sum_{k=0}^{\infty}\left|\int_w^{w_h} q'_{k-2}(u)-2q'_{k-1}(u)+q'_k(u)\,du\right| = \sum_{k=0}^{\infty}\left|\int_w^{w_h}\frac{p_k(u)}{u^2}h(k,u),du\right|. \tag{110}$$

If $N=N^{(u)}$ is Poisson with mean $u$, then for $u\geq 1$,

$$u^{-2}E(|h(N,u)|)\leq cu^{-2}(u^{3/2}+u+3u^{3/2}+u)\leq cu^{-1/2}. \tag{111}$$

For $0<u\leq 1$,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{p_k(u)}{u^2}|h(k,u)| &= e^{-u}u^{-2}|-u^3+u^2|+e^{-u}u^{-1}|u(-2+4u-u^2)| \\ &\quad+\sum_{k=2}^{\infty}\frac{e^{-u}u^{k-2}}{k!}|h(k,u)| \\ &\leq e^{-u}(u+1)+e^{-u}|-2+4u-u^2|+\sum_{k=2}^{\infty}\frac{e^{-u}u^{k-2}}{k!}ck^2(k-1) \\ &\leq c+c\sum_{k=2}^{\infty}\frac{e^{-u}u^{k-2}}{(k-2)!}k=c+c(u+2)\leq c.\end{aligned} \tag{112}$$

Use (111) and (112) in (110) to show that

$$\begin{aligned}\sum_{k=0}^{\infty}|d_{k-2}-2d_{k-1}+d_k| &\leq \int_w^{w_h}\sum_{k=0}^{\infty}p_k(u)u^{-2}|h(k,u)|du \\ &\leq \int_w^{w_h}c(u+1)^{-1/2}du\leq c(w+1)^{-1/2}h_2(\gamma_2^0t)^{-1} \\ &\leq \frac{ch_2}{\sqrt{t}\sqrt{x_2+t}}.\end{aligned} \tag{113}$$

Substitute (108), (109), (113) and the trivial bound $|\hat{G}_k(z)|\leq\|f\|_\infty$ into (107) to conclude

$$\left|((x_2+2h_2)E_{x_2+h_2}-x_2E_{x_2})\left(\Delta G^4\left(\int_0^t X_s ds, X_t\right)\right)\right| \leq \frac{c\|f\|_\infty h_2}{t^{3/2}(x_2+t)^{1/2}}+\frac{c\|f\|_\infty h_2}{t(x_2+t)},$$

as required. □

Finally we consider the increments in $x_1$.

LEMMA 27. *If $f$ is a bounded Borel function on $\mathbb{R}\times\mathbb{R}_+$, then for all $x\in\mathbb{R}\times\mathbb{R}_+$ and all $h_1\in\mathbb{R}$,*

$$\left|x_2\frac{\partial^2 P_t f}{\partial x_2^2}(x_1+h_1,x_2)-x_2\frac{\partial^2 P_t f}{\partial x_2^2}(x_1,x_2)\right| \leq c_{27}\|f\|_\infty t^{-3/2}(x_2+t)^{-1/2}|h_1|. \tag{114}$$

*Proof.* Let

$$\Delta^2 G^j_{t,x_1}(X)=\Delta G^j_{t,x_1+h_1}(X)-\Delta G^j_{t,x_1}(X),\ j=1,\dots,4,$$

so that by (96),

$$x_2(P_tf)_{22}(x_1+h_1,x_2)-x_2(P_tf)_{22}(x_1,x_2)=\sum_{j=1}^4 x_2E_{x_2}(\Delta^2 G^j_{t,x_1}(x^{(2)})). \tag{115}$$

Considering the $j=1$ contribution in the above sum first, write

$$\begin{aligned}
&x_2E_{x_2}(\Delta^2G^1_{t,x_1}(x^{(2)}))\\
&=x_2E_{x_2}\Bigg(\iint 1(\nu_t=\nu'_t=0)\bigg[G_{t,x_1+h_1}\Big(\int x^{(2)}_s+\nu_s+\nu'_s ds, x^{(2)}_t\Big)\\
&\quad -G_{t,x_1+h_1}\Big(\int x^{(2)}_s+\nu_s ds, x^{(2)}_t\Big)-G_{t,x_1+h_1}\Big(\int x^{(2)}_s+\nu'_s ds, x^{(2)}_t\Big)\\
&\quad +G_{t,x_1+h_1}\Big(\int x^{(2)}_s ds, x^{(2)}_t\Big)-G_{t,x_1}\Big(\int x^{(2)}_s+\nu_s+\nu'_s ds, x^{(2)}_t\Big)\\
&\quad +G_{t,x_1}\Big(\int x^{(2)}_s+\nu_s ds, x^{(2)}_t\Big)+G_{t,x_1}\Big(\int x^{(2)}_s+\nu'_s ds, x^{(2)}_t\Big)\\
&\quad -G_{t,x_1}\Big(\int x^{(2)}_s ds, x^{(2)}_t\Big)\bigg]d\mathbb{N}_0(\nu)d\mathbb{N}_0(\nu')\Bigg).
\end{aligned}$$

The expression in square brackets on the righthand side is a third order difference of $G_{t,x_1}(y_1,X_t)$ (first order in $x_1$ and second order in $y_1$) to which we may apply Lemma 13(b) and the bound on $\frac{\partial^3 G_{t,x_1}}{\partial x_1\partial y_1^2}(y_1,y_2)$ from Lemma 11, and conclude

$$\begin{aligned}
&|x_2E_{x_2}(\Delta^2G^1_{t,x_1}(x^{(2)}))|\\
&\leq cx_2\|f\|_\infty E_{x_2}\bigg(\Big(\int_0^t x^{(2)}_s ds\Big)^{-5/2}\bigg)|h_1|\Big(\int\int_0^t \nu_s ds d\mathbb{N}_0(\nu)\Big)^2\\
&\leq cx_2\|f\|_\infty t^{-5/2}(x_2+t)^{-5/2}|h_1|t^2\\
&\leq c\|f\|_\infty|h_1|t^{-1/2}(x_2+t)^{-3/2}.
\end{aligned} \tag{116}$$

Turning to $\Delta^2 G^2_{t,x_1}$, set

$$H_{t,x_1}(I_t, X_t, \nu, h_1) = G_{t,x_1+h_1}\left(I_t + \int_0^t \nu_s ds, X_t\right) - G_{t,x_1+h_1}(I_t, X_t) - G_{t,x_1}\left(I_t + \int_0^t \nu_s ds, X_t\right) + G_{t,x_1}(I_t, X_t).$$

Then

(117) $$|x_2 E_{x_2}(\Delta^2 G^2_{t,x_1}(x^{(2)}))| = \left| x_2 E_{x_2}\left( \iint 1(\nu_t = 0, \nu'_t > 0)\left[ H_{t,x_1}\left( \int_0^t x_s^{(2)} + \nu'_s ds, x_t^{(2)} + \nu'_t, \nu, h_1 \right) - H_{t,x_1}\left( \int_0^t x_s^{(2)} ds, x_t^{(2)}, \nu, h_1 \right) \right] d\mathbb{N}_0(\nu) d\mathbb{N}_0(\nu') \right)\right|.$$

We may apply Lemmas 13(b) and 11(b) to see that

$$|H_{t,x_1}(I_t, X_t, \nu, h_1)| \le c\|f\|_\infty I_t^{-3/2} |h_1| \int_0^t \nu_s ds.$$

Use this to bound each term in the integrand on the righthand side of (117) and so conclude

(118)

$$\begin{aligned} &|x_2 E_{x_2}(\Delta^2 G^2_{t,x_1}(x^{(2)}))| \\ &\le c\|f\|_\infty x_2 E_{x_2}\left( \left( \int_0^t x_s^{(2)} ds \right)^{-3/2} \right) |h_1| \int \int_0^t \nu_s ds d\mathbb{N}_0(\nu) \mathbb{N}_0(\{\nu' : \nu'_t > 0\}) \\ &\le c\|f\|_\infty x_2 t^{-3/2}(x_2+t)^{-3/2} |h_1| t t^{-1} \\ &\le c\|f\|_\infty t^{-3/2}(x_2+t)^{-1/2}|h_1|. \end{aligned}$$

By symmetry the same bound holds for $|x_2 E_{x_2}(\Delta^2 G^3_{t,x_1}(x^{(2)}))|$.

To bound the $j = 4$ term in (115), we use the notation and setting introduced for Lemma 10 (with $\rho = 1/2$) and in the proof of Lemma 25. For $h_1$ fixed, let $DG_{t,x_1}(y_1, y_2) = G_{t,x_1+h_1}(y_1, y_2) - G_{t,x_1}(y_1, y_2)$ and define

$$\Delta G_k = E(DG_{t,x_1}(R_k + I_2(t), S_k + X'_0(t))).$$

The Mean Value Theorem and the bound on $|\partial G_{t,x_1}/\partial x_1(y)|$ in Lemma 11 implies

(119) $$|\Delta G_k| \le c\|f\|_\infty |h_1| E_{x_2}\left( \left( \int_0^t X'_0(s) ds \right)^{-1/2} \right) \le c\|f\|_\infty |h_1| t^{-1/2}(x_2+t)^{-1/2}.$$

Recall that $w = x_2(2\gamma_2^0 t)^{-1}$ and $q_k(w) = w p_k(w)$.

Then

$$\begin{aligned}
&|x_2 E_{x_2}(\Delta^2 G^4_{t,x_1}(x^{(2)}))| \\
&= x_2(\gamma_2^0 t)^{-2}\bigg|E_{x_2}\bigg\{\iint DG_{t,x_1}\bigg(\int_0^t x_s^{(2)} + \nu_s + \nu'_s ds, x_t^{(2)} + \nu_t + \nu'_t\bigg) \\
&\quad - DG_{t,x_1}\bigg(\int_0^t x_s^{(2)} + \nu_s ds, x_t^{(2)} + \nu_t\bigg) \\
&\quad - DG_{t,x_1}\bigg(\int_0^t x_s^{(2)} + \nu'_s ds, x_t^{(2)} + \nu'_t\bigg) \\
&\quad + DG_{t,x_1}\bigg(\int_0^t x_s^{(2)} ds, x_t^{(2)}\bigg) dP_t^*(\nu) dP_t^*(\nu')\bigg\}\bigg| \\
&= \frac{cw}{t}|E(DG_{t,x_1}(I_2(t) + R_{N_t+2}, X'_0(t) + S_{N_t+2}) \\
&\quad - 2DG_{t,x_1}(I_2(t) + R_{N_t+1}, X'_0(t) + S_{N_t+1}) \\
&\quad + DG_{t,x_1}(I_2(t) + R_{N_t}, X'_0(t) + S_{N_t}))| \\
&= ct^{-1}\bigg|\sum_{k=0}^{\infty} wp_k(w)[\Delta G_{k+2} - 2\Delta G_{k+1} + \Delta G_k]\bigg| \\
&= ct^{-1}\bigg|\sum_{k=0}^{\infty}(q_{k-2}(w) - 2q_{k-1}(w) + q_k(w))\Delta G_k\bigg|,
\end{aligned}$$

where we sum by parts twice in the last line and use $q_k = 0$ for $k < 0$. Use (109) and (119) in the above to see that

$$\begin{aligned}
|x_2 E_{x_2}(\Delta^2 G^4_{t,x_1}(x^{(2)}))| &\le ct^{-1}\|f\|_\infty |h_1| t^{-1/2}(x_2+t)^{-1/2} \\
&\le c\|f\|_\infty |h_1| t^{-3/2}(x_2+t)^{-1/2}.
\end{aligned} \tag{120}$$

Putting (116), (118) and (120) into (115), we complete the proof of Lemma 27. □

We now may finish the proof of Proposition 23. Use Lemmas 24, 25, and 26, and (71) from Proposition 17 in (97) to obtain

$$\begin{aligned}
&|(x_2+h_2)(P_t f)_{22}(x_1, x_2+h_2) - x_2(P_t f)_{22}(x_1,x_2)| \\
&\qquad \le \frac{c_{17}\|f\|_\infty h_2}{t(t+x_2)} + \frac{c\|f\|_\infty h_2}{t^{3/2}(t+x_2)^{1/2}} \le \frac{c\|f\|_\infty h_2}{t^{3/2}(t+x_2)^{1/2}}.
\end{aligned}$$

Lemma 27 gives the corresponding bound for increments in $x_1$ and so the proof of (93) is complete, as required. □

Finally we state the required extensions of Propositions 22 and 23 to general $m \ge 1$, i.e., for the semigroup $P_t$ on $\mathbb{R}^m \times \mathbb{R}_+$ given by (18). The proofs are again minor modifications of those already given for $m = 1$.

PROPOSITION 28.

(a) *If $f$ is a bounded Borel function on $\mathbb{R}^m \times \mathbb{R}_+$, then for all $x, h \in \mathbb{R}^m \times \mathbb{R}_+$,*

$$\sum_{j=1}^{m+1} \left| \frac{\partial P_t f}{\partial x_j}(x+h) - \frac{\partial P_t f}{\partial x_j}(x) \right| \le c_{28} \|f\|_\infty t^{-\frac{3}{2}} (x_{m+1}+t)^{-1/2} |h|.$$

(b) *If $f \in S^\alpha$, then for all $x, h \in \mathbb{R}^m \times \mathbb{R}_+$,*

$$\sum_{j=1}^{m+1} \left| \frac{\partial P_t f}{\partial x_j}(x+h) - \frac{\partial P_t f}{\partial x_j}(x) \right| + \left| (x+h)_{m+1} \frac{\partial^2 P_t f}{\partial x_j^2}(x+h) - x_{m+1} \frac{\partial^2 P_t f}{\partial x_j^2}(x) \right| \le c_{28} |f|_\alpha t^{\frac{\alpha}{2}-\frac{3}{2}} (x_{m+1}+t)^{-1/2} |h|.$$

*Proof.* The only step which is slightly different is the derivation of the bound (91) for

$$|x_{m+1}(P_t f)_{ii}(x+h\check{e}_j) - x_{m+1}(P_t f)_{ii}(x)| \tag{121}$$

for $i \neq j$. We again omit the analogous bound for the first derivative. As in (91), (121) equals

$$x_{m+1} \left| E_{x_{m+1}} \left( \left( \frac{\partial^2 G_{t,x+h\check{e}_j}}{\partial x_i^2} - \frac{\partial^2 G_{t,x}}{\partial x_i^2} \right) (I_t, x_t^{(m+1)}) \right) \right|.$$

As in Lemma 11 one easily checks that

$$\left| \frac{\partial^2}{\partial x_i^2} \frac{\partial}{\partial x_j} G_{t,x}(y_1, y_2) \right| \le c \|f\|_\infty y_1^{-3/2}$$

and the upper bound (91) for (121) now follows just as before. The other steps in the proof are trivial modifications of those used for $m = 1$. □

REMARK 29. If $m = 0$ so that $P_t$ is the semigroup of

$$\mathcal{A}^1 f(x) = b_1^0 f'(x) + \gamma_1^0 x f''(x),$$

Propositions 20 and 28 continue to hold as we may consider functions which depend only on $x_{m+1}$. Here we assume $b_1^0 > 0$ but in fact these results (with $m = 0$) were proved in [BP1] (see Section 4 of that reference) for $b_1^0 \ge 0$. There the weighted Hölder norm $|\ |_{C_w^\alpha}$ was used but these are equivalent by Remark 21. This extension will be used in the next section to handle the coordinates $i \in N_2$.

## 3. Multi-dimensional bounds

We return to the setting of Section 1.3 and recall the notation introduced there. In particular we will work with the multi-dimensional semigroup $P_t = \prod_{i\in I} P_t^i$ on the state space $S_0 = \{(x_1,\dots,x_d) : x_i \geq 0\ \forall i \notin N_1\}$, where $I = (Z\cap C)\cup N_2$. It is now relatively straightforward to use the results from the previous sections on the semigroups $P_t^i$, $i\in I$, to prove the required bounds on the resolvent $R_\lambda$ of $P_t$. The arguments are easy modifications of those in Section 5 of [BP1], although the setting is a bit more complex as in [BP1] the $P_t^i$ are all one-dimensional. Recall that if $i \in Z\cap C$, then $b_i^0 > 0$.

For $k\in I$ let

$$\bar y_k = (\{y_j\}_{j\in R_k}, y_k),\ \text{if } k\in Z\cap C,$$
$$\bar y_k = y_k \text{ if } k\in N_2,$$

$$\hat y_k = (\{y_j\}_{j\notin R_k\cup\{k\}}) \text{ if } k\in Z\cap C,$$
$$\hat y_k = (y_1,\dots,y_{k-1},y_{k+1},\dots,y_d) \text{ if } k\in N_2,$$

and $\bar{\mathbb{R}}_k = \mathbb{R}^{|R_k|}\times\mathbb{R}_+$ if $k\in Z\cap C$ and $\bar{\mathbb{R}}_k = \mathbb{R}$ if $k\in N_2$. If $\bar h_k \in \bar{\mathbb{R}}_k$ and $j\in R_k$, set $\bar h_k \bar e_k = h_k \check e_k + \sum_{j\in R_k} h_j \check e_j$, respectively $h_k\check e_k$, according as $k\in Z\cap C$ (and $\bar h_k = (\{h_j\}_{j\in R_k}, h_k)$) or $k\in N_2$ (and $\bar h_k = h_k$).

Let $f : S_0\to\mathbb{R}$ be a bounded Borel function. For $k\in I$ define

$$F_k(\bar y_k; t, \hat x_k) = \int f(y) \prod_{i\in I, i\neq k} P_t^i(\bar x_i, d\bar y_i).$$

Then

$$P_t f(x) = P_t^k(F_k(\cdot; t,\hat x_k))(\bar x_k) \text{ for all } k\in I. \tag{122}$$

If $\bar h\in\bar{\mathbb{R}}_k$, and $k\in Z\cap C$, then

$$\begin{aligned}|F_k(\bar y_k + \bar h; t,\hat x_k) &- F_k(\bar y_k; t,\hat x_k)|\\ &= \left|\int [f(y+\bar h\bar e_k) - f(y)] \prod_{i\in I,i\neq k} P_t^i(\bar x_i, d\bar y_i)\right|\\ &\leq |f|_{\alpha,k}[|\bar h|^\alpha y_k^{-\alpha/2}\wedge |\bar h|^{\alpha/2}].\end{aligned}$$

A similar argument works if $k\in N_2$ with $h^\alpha y_k^{-\alpha/2}$ in place of $[|\bar h|^\alpha y_k^{-\alpha/2}\wedge |\bar h|^{\alpha/2}]$.

Therefore

$$|F_k(\cdot; t,\hat x_k)|_{C_w^\alpha} \leq |f|_{\alpha,k} \text{ for all } k\in I. \tag{123}$$

Here $|F_k(\cdot; t,\hat x_k)|_{C_w^\alpha}$ is defined as in Section 1.4 with $\{k\}\cap(Z\cap C)$, $\{k\}\cap N_2$ playing the roles of $Z\cap C$, and $N_2$, respectively.

If $F : S_0 \to \mathbb{R}$, let

$$\frac{\partial F}{\partial \bar{x}_k} = \left( \left\{ \frac{\partial F}{\partial x_j} \right\}_{j \in R_k}, \frac{\partial F}{\partial x_k} \right), \tag{124}$$

$$\left| \frac{\partial F}{\partial \bar{x}_k} \right| = \sum_{j \in R_k} \left| \frac{\partial F}{\partial x_j} \right| + \left| \frac{\partial F}{\partial x_k} \right| \quad \text{if } k \in Z \cap C,$$

$$\frac{\partial F}{\partial \bar{x}_k} = \frac{\partial F}{\partial x_k}, \quad \text{if } k \in N_2,$$

and for $k \in I$, set

$$\left\| \frac{\partial F}{\partial \bar{x}_k} \right\|_\infty = \sup \left\{ \left| \frac{\partial F}{\partial \bar{x}_k}(x) \right| : x \in S_0 \right\}. \tag{125}$$

Similarly introduce

$$\Delta_k F = \left( \left\{ x_k \frac{\partial^2 F}{\partial x_j^2} \right\}_{j \in R_k}, x_k \frac{\partial^2 F}{\partial x_k^2} \right), \quad \text{if } k \in Z \cap C,$$

$$\Delta_k F = x_k \frac{\partial^2 F}{\partial x_k^2}, \quad \text{if } k \in N_2,$$

and define $|\Delta_k F|$ and $\|\Delta_k F\|_\infty$ by the obvious modifications of (124) and (125).

PROPOSITION 30. *There is a constant $c_{30}$ such that*

(a) *for all $f \in C_w^\alpha(S_0)$, $t > 0$, $x \in S_0$, and $k \in I$,*

$$\left| \frac{\partial P_t f}{\partial \bar{x}_k} \right| (x) \le c_{30} |f|_{\alpha,k} t^{\alpha/2 - 1/2} (t + x_k)^{-\frac{1}{2}} \le c_{30} |f|_{\alpha,k} t^{\alpha/2-1},$$

*and*

$$\|\Delta_k P_t f\|_\infty \le c_{30} |f|_{\alpha,k} t^{\alpha/2-1};$$

(b) *for all $f$ bounded and Borel on $S_0$ and all $k \in I$,*

$$\left\| \frac{\partial P_t f}{\partial \bar{x}_k} \right\|_\infty \le c_{30} \|f\|_\infty t^{-1}. \tag{126}$$

*Proof.* By (122) and Proposition 20 (see Remark 29 if $k \in N_2$),

$$\begin{aligned} \|\Delta_k P_t f\|_\infty &\le c \sup_{\hat{x}_k} |F_k(\cdot; t, \hat{x}_k)|_\alpha t^{\alpha/2-1} \\ &\le c \sup_{\hat{x}_k} |F_k(\cdot; t, \hat{x}_k)|_{C_w^\alpha} t^{\alpha/2-1} \text{ (by Theorem 19 and Remark 21)} \\ &\le c |f|_{\alpha,k} t^{\alpha/2-1}, \end{aligned}$$

the last by (123). The inequalities for the first derivatives are similar. Use Proposition 20(a) for part (b). ☐

LEMMA 31. *There is a constant* $c_{31}$ *such that for all bounded measurable* $f : \bar{\mathbb{R}}_k \to \mathbb{R}$, $t > 0$, $k \in I$ *and* $x, \bar{h} \in \bar{\mathbb{R}}_k$,

(a)

$$|P_t^k f(x+\bar{h}) - P_t^k f(x)| \leq c_{31}|f|_{C_w^\alpha} t^{\alpha/2-1/2}(x_k+t)^{-1/2}|\bar{h}|,$$

(b)

$$|P_t^k f(x+\bar{h}) - P_t^k f(x)| \leq c_{31}\|f\|_\infty t^{-1}|\bar{h}|.$$

*Proof.* This is a simple application of the Fundamental Theorem of Calculus and the bounds on $|\partial P_t^k f/\partial \bar{x}_k(x)|$ from Proposition 30. □

Let $I_j = R_j \cup \{j\}$, if $j \in Z \cap C$, and $I_j = \{j\}$, if $j \in N_2$, so that $\bigcup_{j \in I} I_j$ is a partition of $V$. If $j \neq k \in I$, define

$$\hat{x}_{j,k} = (x_i, i \notin I_j \cup I_k) \in \mathbb{R}^{\{1,\ldots,d\}-I_j \cup I_k}.$$

For $f : \mathbb{R}_+^d \to \mathbb{R}$ be a bounded Borel function and $\bar{y}_j \in I_j, \bar{y}_k \in I_k$, let

$$F_{j,k}(\bar{y}_j, \bar{y}_k; t, \hat{x}_{j,k}) = \int f(y) \prod_{i \in I-\{j,k\}} P_t^i(\bar{x}_i, d\bar{y}_i).$$

Define $|F_{j,k}(\cdot; t, \hat{x}_{j,k})|_{\alpha,j}$ and $|F_{j,k}(\cdot; t, \hat{x}_{j,k})|_{\alpha,k}$ as in Section 1.3 with $\{j,k\} \cap Z \cap C$ and $\{j,k\} \cap N_2$ playing the roles of $Z \cap C$ and $N_2$, respectively. Then just as for (122) and (123) we can show

$$P_t f(x) = \iint F_{j,k}(\bar{y}_j, \bar{y}_k; t, \hat{x}_{j,k}) P_t^j(\bar{x}_j, d\bar{y}_j) P_t^k(\bar{x}_k, d\bar{y}_k), \tag{127}$$

and

$$|F_{j,k}(\cdot; t, \hat{x}_{j,k})|_{\alpha,j} \leq |f|_{\alpha,j}, \qquad |F_{j,k}(\cdot; t, \hat{x}_{j,k})|_{\alpha,k} \leq |f|_{\alpha,k}. \tag{128}$$

PROPOSITION 32. *There is a constant* $c_{32}$ *such that for all* $f \in C_w^\alpha(S_0)$, $i, k \in I$ *and* $\bar{h}_i \in \bar{\mathbb{R}}_i$,

(a)

$$\left|\frac{\partial P_t f}{\partial \bar{x}_k}(x+\bar{h}_i\bar{e}_i) - \frac{\partial P_t f}{\partial \bar{x}_k}(x)\right| \leq c_{32}|f|_{\alpha,i} t^{-3/2+\alpha/2}(x_i+t)^{-1/2}|\bar{h}_i|, \tag{129}$$

(b)

$$|\Delta_k(P_t f)(x+\bar{h}_i\bar{e}_i) - \Delta_k(P_t f)(x)| \leq c_{32}|f|_{\alpha,i} t^{-3/2+\alpha/2}(x_i+t)^{-1/2}|\bar{h}_i|. \tag{130}$$

*Proof.* Assume first that $i = k \in I$. By (122) and Proposition 28, we have

$$\begin{aligned} &\left|\frac{\partial(P_t f)}{\partial \bar{x}_i}(x+\bar{h}_i\bar{e}_i) - \frac{\partial(P_t f)}{\partial \bar{x}_i}(x)\right| \\ &\quad \leq \left|\frac{\partial}{\partial \bar{x}_i} P_t^i(F_i(\cdot; t, \hat{x}_i))(\bar{x}_i+\bar{h}_i\bar{e}_i) - \frac{\partial}{\partial \bar{x}_i} P_t^i(F_i(\cdot; t, \hat{x}_i))(\bar{x}_i)\right| \\ &\quad \leq c|F_i(\cdot; t, \hat{x}_i)|_\alpha t^{\alpha/2-3/2}(x_i+t)^{-1/2}|\bar{h}_i|. \end{aligned} \tag{131}$$

Note that if $k \in N_2$ we may still apply Proposition 28 by Remark 29. Theorem 19 (see Remark 21) and (123) imply $|F_i(\cdot; t, \hat{x}_i)|_\alpha \leq c|f|_{\alpha,i}$ and so the proof is complete in this case.

Next consider $i \neq k$. Then (127) implies

$$\frac{\partial(P_t f)}{\partial \bar{x}_k}(x) = \frac{\partial}{\partial \bar{x}_k} \iint F_{i,k}(\bar{y}_i, \bar{y}_k; t, \hat{x}_{i,k}) P_t^k(\bar{x}_k, d\bar{y}_k) P_t^i(\bar{x}_i, d\bar{y}_i) \tag{132}$$
$$= \int \frac{\partial}{\partial \bar{x}_k} P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k}))(\bar{x}_k) P_t^i(\bar{x}_i, d\bar{y}_i),$$

where differentiation through the integral is easily justified using

$$\begin{aligned}\left\| \frac{\partial}{\partial \bar{x}_k} P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k})) \right\|_\infty &\leq c|F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k})|_{\alpha,k} t^{\alpha/2-1} \text{ (by Proposition 30)} \\ &\leq c|f|_{\alpha,k} t^{\alpha/2-1} \text{ (by (128))}.\end{aligned}$$

Let

$$K_{i,k}(\bar{y}_i; t, \hat{x}_i) = \frac{\partial}{\partial \bar{x}_k} P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k}))(\bar{x}_k).$$

For $i \in I$, the above representation and notation together with Lemma 31 give

$$\begin{aligned}&\left| \frac{\partial(P_t f)}{\partial \bar{x}_k}(x + \bar{h}_i \bar{e}_i) - \frac{\partial(P_t f)}{\partial \bar{x}_k}(x) \right| \\ &\quad = \left| \int K_{i,k}(\bar{y}_i; t, \hat{x}_i)[P_t^i(\bar{x}_i + \bar{h}_i \bar{e}_i, d\bar{y}_i) - P_t^i(\bar{x}_i, d\bar{y}_i)] \right| \\ &\quad \leq c|K_{i,k}(\cdot; t, \hat{x}_i)|_{C_w^\alpha} t^{\alpha/2-1/2}(x_i + t)^{-1/2}|\bar{h}_i|.\end{aligned} \tag{133}$$

If $\bar{h}_i \in \bar{\mathbb{R}}_i$ and $k \in Z \cap C$, then

$$\begin{aligned}&|K_{i,k}(\bar{y}_i + \bar{h}_i; t, \hat{x}_i) - K_{i,k}(\bar{y}_i; t, \hat{x}_i))| \\ &\quad = \left| \frac{\partial}{\partial \bar{x}_k} \{P_t^k(F_{i,k}(\bar{y}_i + \bar{h}_i, \cdot; t, \hat{x}_{i,k}))(\bar{x}_k) - P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k}))(\bar{x}_k)\} \right| \\ &\quad \leq \frac{c}{\sqrt{t}\sqrt{t + x_k}} \|F_{i,k}(\bar{y}_i + \bar{h}_i, \cdot; t, \hat{x}_{i,k}) - F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k})\|_\infty,\end{aligned}$$

the last by Proposition 20. The same reasoning also applies to $k \in N_2$ by Remark 29. Now use (128) in the above to conclude that

$$|K_{i,k}(\cdot; t, \hat{x}_i)|_{C_w^\alpha} \leq ct^{-1}|f|_{\alpha,i} \text{ for all } i \in I \text{ and } k \in Z \cap C. \tag{134}$$

Finally, combine (134) with (133) to see that for all $i, k \in I$,

$$\left| \frac{\partial}{\partial \bar{x}_k} P_t f(x + \bar{h}_i \bar{e}_i) - \frac{\partial}{\partial \bar{x}_k} P_t f(x) \right| \leq c|f|_{\alpha,i} t^{\alpha/2-3/2}(x_i + t)^{-1/2}|\bar{h}_i|.$$

This proves (a).

The proof of (b) is similar. Instead of $K_{i,k}$, one works with

$$L_{i,k}(\bar{y}_i; t, \hat{x}_i) = \Delta_k P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k}))(\bar{x}_k). \qquad \square$$

COROLLARY 33. *For $f \in C_w^\alpha$, $P_t f$ is in $C_b^2(S_0)$ and for all $j_1, j_2 \in V$*

$$\|(P_t f)_{j_1 j_2}\|_\infty \leq c_{33} \frac{\|f\|_\infty}{t^2}. \tag{135}$$

*Proof.* We focus on the second order partials. Let $j_1, j_2 \in V$ and choose $k_i \in I$ so that $j_i \in I_{k_i}$. Then

$$(P_t f)_{j_1,j_2}(x) = \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} P_t^k[F_k(\cdot; t, \hat{x}_k)](\bar{x}_k) \quad \text{if } k_1 = k_2 = k, \tag{136}$$

and

(137)

$$(P_t f)_{j_1,j_2}(x) = \frac{\partial}{\partial x_{j_1}} \int \left[ \frac{\partial}{\partial x_{j_2}} P_t^{k_2}(F_{k_1,k_2}(\bar{y}_{k_1}, \cdot; t, \hat{x}_{k_1,k_2}))(\bar{x}_{k_2}) \right] P_t^{k_1}(\bar{x}_{k_1}, d\bar{y}_{k_1}) \quad \text{if } k_1 \neq k_2.$$

Differentiation under the integral sign is easy to justify as in (132). The required bound on $\|(P_t f)_{j_1,j_2}\|_\infty$ now follows from Remark 15 if $k_1 = k_2$, and a double application of Proposition 20(a) if $k_1 \neq k_2$.

Turning now to the continuity of $(P_t f)_{j_1,j_2}$, assume first that $k_1 = k_2 = k$. Then by (136) and Remark 15

$$\frac{\partial^2 P_t f}{\partial x_{j_1} \partial x_{j_2}}(x) \text{ is continuous in } \bar{x}_k \text{ for } \hat{x}_k \text{ fixed.} \tag{138}$$

If $i \in I$ is distinct from $k$, then

$$\begin{aligned}
&|(P_t f)_{j_1,j_2}(x + \bar{h}_i \bar{e}_i) - (P_t f)_{j_1,j_2}(x)| \\
&= \left| \int \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k}))(\bar{x}_k)[P_t^i(\bar{x}_i + \bar{h}_i \bar{e}_i, d\bar{y}_i) - P_t^i(\bar{x}_i, d\bar{y}_i)] \right| \\
&\leq ct^{-1}|\bar{h}_i| \sup_{\bar{y}_i} |(P_t^k(F_{i,k}(\bar{y}_i, \cdot; t, \hat{x}_{i,k}))_{j_1,j_2}(\bar{x}_k)| \quad \text{(by Lemma 31(b))} \\
&\leq ct^{-3}\|f\|_\infty |\bar{h}_i| \qquad \text{(by Remark 15).}
\end{aligned}$$

This and (138) give the continuity of $(P_t f)_{j_1,j_2}$. For $k_1 \neq k_2$ continuity in $\bar{x}_j$ for $j \notin \{k_1, k_2\}$ uniformly in the other variables is proved as above, and continuity in $\bar{x}_{k_1}$ (say) uniformly in the other variables is proved using Proposition 28(a). The details are left for the reader. $\square$

Recall that $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt$ is the resolvent associated with $P_t$.

THEOREM 34. *There is a constant $c_{34}$ such that for all $f \in C^\alpha_w(S_0)$, $\lambda \geq 1$ and $k, i \in I$,*

(a)

$$\left\|\frac{\partial R_\lambda f}{\partial \bar{x}_k}\right\|_\infty + \|\Delta_k R_\lambda f\|_\infty \leq c_{34}\lambda^{-\alpha/2}|f|_{\alpha,k},$$

(b)

$$\left|\frac{\partial R_\lambda f}{\partial \bar{x}_k}\right|_{\alpha,i} + |\Delta_k R_\lambda f|_{\alpha,i} \leq c_{34}[(|f|^{1-\alpha}_{\alpha,k}|f|^{\alpha}_{\alpha,i}) \vee (|f|^{1-\alpha/2}_{\alpha,k}|f|^{\alpha/2}_{\alpha,i})]$$
$$\leq c_{34}|f|_{C^\alpha_w}.$$

*Proof.* (a) This follows by integrating the inequalities in Proposition 30 over time.

(b) If $\bar{t} > 0$ and $\bar{h}_i \in \bar{\mathbb{R}}_i$, then

$$\begin{aligned}(139)\quad & |\Delta_k(R_\lambda f)(x + \bar{h}_i\bar{e}_i) - \Delta_k(R_\lambda f)(x)| \\ &\leq \int_0^{\bar{t}} |\Delta_k(P_t f)(x + \bar{h}_i\bar{e}_i)| + |\Delta_k(P_t f)(x)|dt \\ &\quad + \left|\int_{\bar{t}}^\infty e^{-\lambda t}[\Delta_k(P_t f)(x + \bar{h}_i\bar{e}_i) - \Delta_k(P_t f)(x)]dt\right| \\ &\leq \int_0^{\bar{t}} ct^{\alpha/2-1}|f|_{\alpha,k}\,dt + \int_{\bar{t}}^\infty c|f|_{\alpha,i}t^{\alpha/2-3/2}(x_i + t)^{-1/2}|\bar{h}_i|dt,\end{aligned}$$

where we used Proposition 30 to bound the first term and Proposition 32 to bound the second. The above is at most

$$c|f|_{\alpha,k}\bar{t}^{\alpha/2} + c|f|_{\alpha,i}\bar{t}^{\alpha/2-1/2}x_i^{-1/2}|\bar{h}_i|.$$

Set $\bar{t} = |f|^2_{\alpha,i}|f|^{-2}_{\alpha,k}|\bar{h}_i|^2 x_i^{-1}$, to conclude that

$$(140)\qquad |\Delta_k(R_\lambda f)(x + \bar{h}_i\bar{e}_i) - \Delta_k(R_\lambda f)(x)| \leq c|f|^{1-\alpha}_{\alpha,k}|f|^{\alpha}_{\alpha,i}x_i^{-\alpha/2}|\bar{h}_i|^\alpha.$$

Use $(x_i + t)^{-1/2} \leq t^{-1/2}$ in (139) to conclude that for any $\bar{t} > 0$,

$$|\Delta_k(R_\lambda f)(x + \bar{h}_i\bar{e}_i) - \Delta_k(R_\lambda f)(x)| \leq c|f|_{\alpha,k}\bar{t}^{\alpha/2} + c|f|_{\alpha,i}\bar{t}^{\alpha/2-1}|\bar{h}_i|.$$

Now set $\bar{t} = |f|_{\alpha,i}|f|^{-1}_{\alpha,k}|\bar{h}_i|$ to conclude

$$(141)\qquad |\Delta_k(R_\lambda f)(x + \bar{h}_i\bar{e}_i) - \Delta_k(R_\lambda f)(x)| \leq c|f|^{1-\alpha/2}_{\alpha,k}|f|^{\alpha/2}_{\alpha,i}|\bar{h}_i|^{\alpha/2}.$$

(140) and (141) together imply the required bound on $|\Delta_k R_\lambda f|_{\alpha,i}$ in (b).

The required bound on $|\partial R_\lambda f/\partial \bar{x}_k|_{\alpha,i}$ is proved in the same way. □

## 4. Proof of uniqueness

In this section we complete the proof of uniqueness of solutions to the martingale problem $MP(\mathcal{A},\nu)$, where $\nu$ is a probability on $S$ and

$$\mathcal{A}f(x) = \sum_{j\in R}\gamma_j(x)x_{c_j}x_jf_{jj}(x) + \sum_{j\notin R}\gamma_j(x)x_jf_{jj}(x) + \sum_{j\in V}b_j(x)f_j(x).$$

We first give the proof of Lemma 5 which shows that $S$ is the natural state space for solutions to the martingale problem using the following result.

LEMMA 35. *Let $Z_t \ge 0$ be a continuous adapted process for which $\int_0^t Z_s ds$ is strictly increasing in $t$. Assume $b(s,\omega,x)$ is a $\mathcal{P}\times$Borel real-valued function ($\mathcal{P}$ is the predictable $\sigma$-field) which is continuous in $s$ and satisfies $b(s,\omega,x)\ge \varepsilon_0$ if $x\le\delta$ for some fixed positive $\epsilon_0,\delta$. For some adapted Brownian motion $B$ and stopping time $T\le\infty$ assume that for some $x\ge 0$,*

$$X_t = x + \int_0^t b(s,X_s)ds + \int_0^t \sqrt{Z_sX_s}dB_s \ge 0 \text{ for } t<T.$$

*Then $X>0$ on $\{(s,\omega): 0<s<T, Z_s(\omega)<\varepsilon_0/2\}$ a.s.*

*Proof.* Let $\zeta = \int_0^T Z_s ds \le \infty$ and define $\tau:[0,\zeta)\to[0,T)$ by $\int_0^{\tau_t} Z_s ds = t$. If $\tilde{Z}_u = Z(\tau_u)$ and $\tilde{X}_u = X(\tau_u)$ for $u<\zeta$, then standard arguments allow us to define a time-changed filtration $\tilde{\mathcal{F}}_u$ and an $(\tilde{\mathcal{F}}_u)$-Brownian motion $\tilde{B}$ so that $\tilde{Z}$ and $\tilde{X}$ are $(\tilde{\mathcal{F}}_t)$-predictable and satisfy

$$\tilde{X}_t = x + \int_0^t b(\tau_u,\tilde{X}_u)\tilde{Z}_u^{-1}du + \int_0^t \sqrt{\tilde{X}_u}d\tilde{B}_u, \quad t<\zeta.$$

If $\{[T_i,U_i]: i\in\mathbb{N}\}$ are the stochastic intervals on which $\tilde{Z}$ completes its upcrossings of $[\varepsilon_0/2,\varepsilon_0]$ ($T_i\le\infty$, $U_i\le\zeta$), it suffices to fix $i$ and show

$$\tilde{X}>0 \text{ on } (T_i,U_i]\cap(0,\zeta) \text{ a.s.}$$

Let $V=\inf\{t\ge T_i:\tilde{X}_t\le\delta/2\}\wedge U_i$ and $W=\inf\{t>V:\tilde{X}_t\ge\delta\}\wedge U_i$. Then for $u\in[V,W]$, $X(\tau_u)=\tilde{X}_u\le\delta$ and $\tilde{Z}_u\le\varepsilon_0$ and so

$$\frac{b(\tau_u,\tilde{X}_u)}{\tilde{Z}_u} = \frac{b(\tau_u,X(\tau_u))}{\tilde{Z}_u} \ge 1.$$

A standard comparison theorem (see V.43 of [RW]) shows that $\tilde{X}_t\ge Y(t-V)$ on $[V,W]$, where $Y$ is the pathwise unique solution of

$$Y_t = \tilde{X}_V + t + \int_0^t \sqrt{Y_s}d\tilde{B}_{s+V} > 0 \quad \forall t>0.$$

Here the last inequality holds because $4Y$ is the square of a 4-dimensional Bessel process. This shows $\tilde{X}>0$ on $(V,W]$, and on $[V,W]$ if $\tilde{X}_V>0$. The same reasoning shows $\tilde{X} > 0$ on subsequent upcrossing intervals of $[\delta/2, \delta]$ by $\tilde{X}$ in $(T_i, U_i]$ and so we conclude that $\tilde{X} > 0$ on $(T_i, U_i] \cap (0, \zeta)$ as required. □

*Proof of Lemma 5.* By conditioning we may assume $\nu = \delta_x$. Fix an edge $(i,j) \in \mathcal{E}$. We apply Lemma 35 with $Z_t = 2\gamma_j(x_t)x_t^{(i)}$ and $X_t = x_t^{(j)}$. The fact that $b_i|_{\{x_i=0\}} > 0$ (since $i \in C$) easily shows that $\int_0^t Z_s ds$ is strictly increasing in $t$. If $b_j^M(x) = b_j(x_1 \wedge M, \ldots, x_d \wedge M)$, then $\inf b_j^M|_{\{x_j=0\}} = 2\varepsilon_M > 0$ by Hypothesis 2 (since $j \in R$) and so for some $\delta_M > 0$, $x_j \in [0, \delta_M]$ implies $b_j^M(x) \geq \varepsilon_M$. Applying the previous result with

$$T = T^M = \inf\{t \geq 0 : x_t^{(1)} \vee \cdots \vee x_t^{(d)} \geq M\},$$

we see that $(x_t^{(j)} + x_t^{(i)}) > 0$ $P$-a.s. on $[0, T^M]$ and so letting $M \to \infty$ we are done. □

Let $\mathcal{A}^0$ be given by (7) with coefficients satisfying (8) and (9).

Recall that $\mathcal{A}^0$ is the generator of a unique diffusion on $S(x^0)$ given by (10) with semigroup $P_t$ and resolvent $R_\lambda$ given by (12) and (11) respectively. Recall also $M^0$ is defined by (15). We next consider perturbations around this diffusion. Let $x^0 \in S$ be fixed.

Proposition 36. *Assume that*

(142)
$$\tilde{\mathcal{A}}f(x) = \sum_{j \in V} \tilde{b}_j(x) f_j(x) + \sum_{j \in N_1} \tilde{\gamma}_j(x) x_{c_j} f_{jj} + \sum_{j \notin N_1} \tilde{\gamma}_j(x) x_j f_{jj}, \quad x \in S(x^0),$$

*where* $\tilde{b}_i : S(x^0) \to \mathbb{R}$ *and* $\tilde{\gamma}_i : S(x^0) \to (0, \infty)$

$$\tilde{\Gamma} = \sum_{i=1}^d \|\tilde{\gamma}_i\|_{C_w^\alpha} + \|\tilde{b}_i\|_{C_w^\alpha} < \infty.$$

*Let*

$$\tilde{\varepsilon}_0 = \sum_{i=1}^d \|\tilde{\gamma}_i - \gamma_i^0\|_\infty + \|\tilde{b}_i - b_i^0\|_\infty,$$

*where* $\{b^0, \gamma^0\}$ *satisfy* (8). *Let* $\mathcal{B}f = (\tilde{\mathcal{A}} - \mathcal{A}^0)f$. *Then there is an* $\varepsilon_1 = \varepsilon_1(M^0) > 0$ *and a* $\lambda_1 = \lambda_1(M^0, \tilde{\Gamma}) \geq 0$ *such that if* $\tilde{\varepsilon}_0 \leq \varepsilon_1$ *and* $\lambda \geq \lambda_1$, *then* $\mathcal{B}R_\lambda : C_w^\alpha \to C_w^\alpha$ *is a bounded operator with* $\|\mathcal{B}R_\lambda\| \leq 1/2$.

*Proof.* Let $f \in C^\alpha_w$ and recall definition (15). Then

$$\begin{aligned}\|\mathcal{B}R_\lambda f\|_{C^\alpha_w} :=\ & \|(\tilde{\mathcal{A}} - \mathcal{A}^0)R_\lambda f\|_{C^\alpha_w} \\ \leq & \sum_{j\in V} \|(\tilde{b}_j(x) - b^0_j)\frac{\partial R_\lambda f}{\partial x_j}\|_{C^\alpha_w} \\ & + \sum_{j\in N_1} \big\|(\tilde{\gamma}_j(x) - \gamma^0_j)x_{c_j}\frac{\partial^2 R_\lambda f}{\partial x_j^2}\big\|_{C^\alpha_w} \\ & + \sum_{j\notin N_1} \|(\tilde{\gamma}_j(x) - \gamma^0_j)x_j\frac{\partial^2 R_\lambda f}{\partial x_j^2}\|_{C^\alpha_w}.\end{aligned}$$

Consider the first term on the right hand side with $j \in V$. Then first using the triangle inequality and then Theorem 34, yields

$$\begin{aligned}&|(\tilde{b}_j(x) - b^0_j)\frac{\partial R_\lambda f}{\partial x_j}(x)|_{\alpha,i} \\ &\quad \leq c[\|\tilde{b}_j(x) - b^0_j\|_{C^\alpha_w}\|\frac{\partial R_\lambda f}{\partial x_j}(x)\|_\infty + \|(\tilde{b}_j(x) - b^0_j)\|_\infty|\frac{\partial R_\lambda f}{\partial x_j}(x)|_{\alpha,i}] \\ &\quad \leq c[(\tilde{\Gamma} + M^0)c_{34}\lambda^{-\alpha/2} + \tilde{\varepsilon}_0 c_{34}]|f|_{C^\alpha_w}\end{aligned}$$

Carrying out similar calculations for the other terms using the appropriate bounds from Theorem 34, we obtain

$$\|\mathcal{B}R_\lambda f\|_{C^\alpha_w} \leq c_0[\tilde{\varepsilon}_0 + (\tilde{\Gamma} + M^0)\lambda^{-\alpha/2}]|f|_{C^\alpha_w} \tag{143}$$

for some $c_0 = c_0(M^0)$ and therefore

$$\|\mathcal{B}R_\lambda f\|_{C^\alpha_w} \leq \frac{1}{2}\|f\|_{C^\alpha_w} \tag{144}$$

provided that $\tilde{\varepsilon}_0 \leq (4c_0)^{-1}$ and $\lambda \geq (4c_0(\tilde{\Gamma} + M^0))^{2/\alpha}$. □

If $\nu$ is a probability on $S(x^0)$, as before we say a probability, $\tilde{P}$, on $C(\mathbb{R}_+, S(x^0))$ solves the martingale problem $MP(\tilde{\mathcal{A}}, \nu)$ if under $\tilde{P}$, the law of $x_0(\omega) = \omega_0$ is $\nu$ and for all $f \in C^2_b(S(x^0))$ $(x_t(\omega) = \omega(t))$,

$$M_f(t) = f(x_t) - f(x_0) - \int_0^t \tilde{\mathcal{A}}f(x_s)\,ds$$

is a local martingale with respect to the canonical right-continuous filtration $(\mathcal{F}_t)$.

THEOREM 37. *Assume that $\tilde{\mathcal{A}}$ is given by (142) with coefficients $\tilde{\gamma}_i : S(x^0) \to (0, \infty)$ and $\tilde{b}_i : S(x^0) \to \mathbb{R}$ which are Hölder continuous of index $\alpha \in (0, 1)$, constant outside a compact set and $\tilde{b}_j|_{x_j=0} \geq 0$ for all $j \in V\backslash N_1$. Assume also that $\tilde{\varepsilon}_0 \leq \varepsilon_1(M^0)$, where $b^0_i, \gamma^0_i$ satisfy (8). Then for each probability $\nu$ on $S(x^0)$ there is a unique solution to $MP(\tilde{\mathcal{A}}, \nu)$.*

*Proof.* Existence of solutions to $MP(\tilde{\mathcal{A}},\nu)$ is standard and the assumptions on the coefficients $\{\tilde{b}_i\}$ ensure solutions remain in $S(x^0)$. Hence, we only need consider uniqueness. By conditioning we may assume $\nu = \delta_x$, $x \in S(x^0)$ (see p. 136 of [B]). By Krylov's Markov selection theorem it suffices to show uniqueness of a strong Markov family $\{P^{x'}, x' \in S(x^0)\}$ of solutions to $MP(\tilde{\mathcal{A}},\delta_x)$ (see the proof of Proposition 2.1 in [ABBP]). Let $(\tilde{R}_\lambda, \lambda > 0)$ be the associated resolvent operators.

LEMMA 38. *For $f \in C_w^\alpha$, $\tilde{R}_\lambda f = R_\lambda f + \tilde{R}_\lambda \mathcal{B} R_\lambda f$.*

*Proof.* An easy application of Fatou's Lemma shows that $\tilde{E}_x(x_j(t)) \le x_j + \|\tilde{b}_j\|_\infty t$ for all $j \in V\backslash N_1$ (recall these coordinates are non-negative). This implies the square functions of the martingale part of each coordinate are integrable. It follows that for $g \in C_b^2(S(x^0))$, $M_g$ is a martingale and so

$$\tilde{E}_x(g(x_t)) = g(x) + \int_0^t \tilde{E}_x(\tilde{\mathcal{A}}g(x_s))\,ds.$$

Multiply by $\lambda e^{-\lambda t}$ and integrate over $t \ge 0$ to see that for $g \in C_b^2(S(x^0))$,

$$\lambda \tilde{R}_\lambda g = g + \tilde{R}_\lambda(\mathcal{B}g) + \tilde{R}_\lambda(\mathcal{A}^0 g). \tag{145}$$

Let $f \in C_w^\alpha$ and for $\delta > 0$, set $g_\delta(x) = \int_\delta^\infty e^{-\lambda t} P_t f(x) dt$. Corollary 33 implies that $g_\delta \in C_b^2(S(x^0))$. Moreover using the bounds in Proposition 30 it is easy to verify that for $i \in V$,

$$(g_\delta)_i(x) \to (R_\lambda f)_i(x) \text{ as } \delta \downarrow 0 \text{ uniformly in } x \in S(x^0), \tag{146}$$

for $i \in C \cap Z$, $j \in R_i$,

$$x_i(g_\delta)_{jj}(x) \to x_i(R_\lambda f)_{jj}(x) \text{ as } \delta \downarrow 0 \text{ uniformly in } x \in S(x^0), \tag{147}$$

and for $i \notin N_1$,

$$x_i(g_\delta)_{ii} \to x_i(R_\lambda f)_{ii} \text{ as } \delta \downarrow 0 \text{ uniformly on } S(x^0). \tag{148}$$

Since $\{\tilde{b}_i\}, \{\tilde{\gamma}_i\}$ are bounded, (146), (147), (148) imply that

$$\mathcal{B}g_\delta \to \mathcal{B}R_\lambda f \text{ as } \delta \downarrow 0 \text{ uniformly on } S(x^0). \tag{149}$$

An easy calculation using $\dot{P}_t g_\delta = P_t \mathcal{A}^0 g_\delta \to \mathcal{A}^0 g_\delta$ as $t \downarrow 0$ shows that

$$\mathcal{A}^0 g_\delta = \lambda g_\delta - e^{-\lambda\delta} P_\delta f \to \lambda R_\lambda f - f \text{ uniformly on } S(x^0) \text{ as } \delta \downarrow 0. \tag{150}$$

Now set $g = g_\delta$ in (145) and use (149), (150) and the obvious uniform convergence of $g_\delta$ to $R_\lambda f$ to see that

$$\lambda \tilde{R}_\lambda(R_\lambda f) = R_\lambda f + \tilde{R}_\lambda(\mathcal{B}R_\lambda f) + \tilde{R}_\lambda(\lambda R_\lambda f - f).$$

Rearranging, we get the required result. $\square$

Continuing with the proof of Theorem 37, note that the Hölder continuity of $\tilde{\gamma}_i$ and $\tilde{b}_i$ and the fact that they are constant outside a compact set imply $\tilde{\Gamma} < \infty$. Therefore we may choose $\lambda_1$ as in Proposition 36 so that for $\lambda \geq \lambda_1$, $\mathcal{B}R_\lambda : C_w^\alpha \to C_w^\alpha$ with norm at most 1/2. If $f \in C_w^\alpha$, we may iterate Lemma 38 to see that

$$\tilde{R}_\lambda f(x) = \sum_{n=0}^{\infty} R_\lambda((\mathcal{B}R_\lambda)^n f)(x),$$

where the series converges uniformly and the error term approaches zero by the bound

$$\|(\mathcal{B}R_\lambda)^n f\|_\infty \leq \|(\mathcal{B}R_\lambda)^n f\|_{C_w^\alpha} \leq 2^{-n}\|f\|_{C_w^\alpha}.$$

This shows that for all $f \in C_w^\alpha$, $\tilde{R}_\lambda f(x)$ is unique for $\lambda \geq \lambda_1$ and hence so is $\tilde{P}_t f(x) = \tilde{E}_x(f(x_t))$ for all $t \geq 0$. As $C_w^\alpha$ is measure determining, uniqueness of $\tilde{P}_t(x, dy)$ and hence $\tilde{P}^x$ follows. □

*Proof of Theorem 4.* Existence of a solution to $MP(\mathcal{A}, \nu)$ is standard because the coefficients are continuous and the $|b_i|$ have linear growth at $\infty$ (see, for example, the proof of Theorem 1.1 in [ABBP]). By Lemma 5 the solutions remain in $S$.

Turning to uniqueness, we may assume by conditioning that $\nu = \delta_z$, $z \in S$. Let $\tau_R = \inf\{t \geq 0 : X_t \notin [0, R]^d\}$. By path continuity of our solutions, $\tau_R \uparrow \infty$ a.s. as $R \uparrow \infty$. Therefore it suffices to prove uniqueness of $X_{t\wedge\tau_R}$ and then let $R \uparrow \infty$. By redefining $b_i, \gamma_i$ outside $[0, R]^d$ we may assume that $\{b_i\}_{i\in V}, \{x_i\gamma_i\}_{i\in V\setminus R}, \{x_{c_i}x_i\gamma_i\}_{i\in R}$ are all bounded and uniformly Hölder continuous (e.g., for $i \in R$ redefine $\gamma_i(x)$ to be $\gamma_i(x_1\wedge R, \ldots, x_d\wedge R)\frac{R^2}{(x_i\vee R)(x_{c_i}\vee R)}$).

By the localization argument of Stroock and Varadhan (see Theorem 6.6.1 of [SV] and also the proof of Theorem 1.1 in [ABBP]) it suffices to show that for all $x^0 \in S$ there is an $r(x^0) > 0$ and continuous $\tilde{b}_i : S(x^0) \to \mathbb{R}$, $\tilde{\gamma}_i : S(x^0) \to (0, \infty)$ such that for $x \in B(x^0, r) \cap \mathbb{R}_+^d$

$$\begin{aligned} \tilde{b}_j(x) &= b_j(x)\ \forall\, j \in V, \\ \tilde{\gamma}_j(x) &= x_j\gamma_j(x) \text{ for } j \in N_1 \\ \tilde{\gamma}_j(x) &= x_{c_j}\gamma_j(x) \text{ for } j \in R\setminus N_1 \\ \tilde{\gamma}_j(x) &= \gamma_j(x) \text{ for } j \notin R, \end{aligned} \tag{151}$$

and

(152) there is a unique solution to $MP(\tilde{\mathcal{A}}, \delta_x)$ for all $x \in S(x^0)$.

Here $\tilde{\mathcal{A}}$ is defined as in (142) and a solution of the martingale problem is defined to be a law on $C(\mathbb{R}_+, S(x^0))$ in the usual way. Some explanation is in order here. First note that it is easy to check that $\mathcal{A}f(x) = \tilde{\mathcal{A}}f(x)$ for $x \in B(x^0, r) \cap S(x^0)$. If $T_R = \inf\{t \geq 0 : x_t^{(i)} + x_t^{(j)} \leq 1/R$ for some $j \in R_i,\ i \in C \cap Z\}$, then by Lemma 5 $T_R \uparrow \infty$ a.s. as $R \to \infty$. It therefore suffices to prove uniqueness of $X(\cdot\wedge\tau_R\wedge T_R)$ and so we may apply the covering argument in Theorem 6.6.1 of [SV] to the compact set $K_R = [0,R]^d - \{x : x_i + x_j < 1/R$ for some $j \in R_i,\ i \in C \cap Z\}$ to arrive at the above reduction. The Borel measurability in the initial point assumed in [SV] follows as in Ex. 6.7.4 of [SV], and the boundedness of $\tilde{b}_i$, $\tilde{\gamma}_i x_i$, etc. (also assumed in [SV]) is not needed (as long as the original coefficients can be assumed to be bounded which is the case by the reduction made above). Also the larger state space $S(x^0)$ is convenient but not really used as the solutions are stopped before they exit $\mathbb{R}^d_+$.

Let $b^0_j = b_j(x^0)$, $\gamma^0_j = \tilde{\gamma}_j(x^0)$ (i.e., the right-hand side of (151) when $x = x^0$), and note by the definition of $S$ (and $x^0 \in S$) $\gamma^0_j > 0$, while Hypothesis 2 implies $b^0_j \geq 0$ if $j \in Z$, and $b^0_j > 0$ if $j \in Z \cap (C \cup R)$. In particular if $M^0 = M^0(x^0)$ is as in (15), then $M^0(x^0) < \infty$ for $x^0 \in S$.

**Case 1.** We first assume that for all $j \in N_2$, $b_j(x^0) \geq 0$. Let $\hat{b}_j(x)$, $\hat{\gamma}_j(x)$ be defined by the right side of (151) for $x \in \mathbb{R}^d_+$ and

$$\tilde{\varepsilon}_0(r) = \sum_{j\in V} \sup_{x\in B(x^0,2r)\cap\mathbb{R}^d_+} |\hat{b}_j(x) - b^0_j| + \sum_{j\in V} \sup_{x\in B(x^0,2r)\cap\mathbb{R}^d_+} |\hat{\gamma}_j(x) - \gamma^0_j|.$$

We use continuity of the coefficients to choose $0 < r < \min_{i\notin Z(x^0)} x^0_i/2$ such that $\tilde{\varepsilon}_0(r) < \varepsilon_1(M^0(x^0))$ ($\varepsilon_1$ is as in Theorem 37). Let $\rho_r : [0,\infty) \to [0,1]$ be the function that is 1 on $[0,r]$, 0 on $[2r,\infty)$ and linear on $[r,2r]$. For $x \in S(x^0)$ define $x^+ = (x^+_1,\ldots,x^+_d)$, and

$$\tilde{b}_i(x) = \rho_r(|x - x^0|)\hat{b}_i(x^+) + (1 - \rho_r(|x - x^0|))b^0_i,$$

and

$$\tilde{\gamma}_i(x) = \rho_r(|x - x^0|)\hat{\gamma}_i(x^+) + (1 - \rho_r(|x - x^0|))\gamma^0_i.$$

Clearly $\tilde{b}_i(x) = \hat{b}_i(x)$ and $\tilde{\gamma}_i(x) = \hat{\gamma}_i(x)$ for $x \in B(x^0,r)\cap\mathbb{R}^d_+$.

We claim that $(\tilde{\gamma}_i, \tilde{b}_i)$ satisfy the hypotheses of Theorem 37. The condition that $b^0_j \geq 0$ for $j \in N_2$ (defining this case) implies $b^0_j \geq 0$ for all $j \notin N_1$ (as $b^0_j > 0$ if $j \in Z \cap C$) and so the required condition (8) on the constants holds. The $\alpha$-Hölder continuity of $\tilde{\gamma}_i$ and $\tilde{b}_i$ follows easily from the $\alpha$-Hölder continuity of $\hat{\gamma}_i$ and $\hat{b}_i = b_i$. Clearly $\tilde{\gamma}_i$ and $\tilde{b}_i$ are constant outside of $B(x^0,2r)$ and hence also bounded. If $i \in (R \cup C) \cap Z(x^0)$, then $b^0_i > 0$ and $b_i|_{x_i=0} > 0$ (by Hypothesis 2) imply $\tilde{b}_i|_{\{x_i=0\}} > 0$. If $i \in Z(x^0)$ the same reasoning shows that $\tilde{b}_i|_{x_i=0} \geq 0$. Note that $r < \min_{i\notin Z(x^0)} x^0_i/2$ implies $\tilde{\gamma}_i(x) > 0$ for all $x$. Finally we have

$$\sum_{j\in V} \sup_{x\in S(x^0)} |\tilde{b}_j(x) - b^0_j| + \sum_{j\in V} \sup_{x\in S(x^0)} |\tilde{\gamma}_j(x) - \gamma^0_j|$$
$$\leq \tilde{\varepsilon}_0(r) < \varepsilon_1(M^0).$$

We may therefore apply Theorem 37 to derive (152) and the proof is complete in this case.

**Case 2.** Finally, we must deal with the case in which $b_j(x^0) < 0$ for some $j \in N_2$, say for $j \in N_2^-$. This implies $x_j^0 > 0$ by Hypothesis 2. In order to prove the required uniqueness (152) we first prove uniqueness for the case in which for $j \in N_2^-$ we replace $b_j(x)$ by $\hat{b}_j(x)$, which satisfies Hypothesis 2, agrees with $b_j(x)$ outside $B(x^0, r)$, and $\hat{b}_j(x^0) = 0$. Here $r > 0$ is chosen so that $x_j \gamma_j(x) \geq \varepsilon_j > 0$ in $B(x^0, 2r)$. The martingale problem with the modified coefficients $\{\hat{b}_j(\cdot)\}$ satisfies the conditions of Case 1 and so the required result is established as above for these modified coefficients. We can then obtain uniqueness for $\mathrm{MP}(\tilde{\mathcal{A}}, \delta_x)$ for all $x \in S(x^0)$, where the coefficients of $\tilde{\mathcal{A}}$ agree with the original coefficients $\{b_j : j \in V\}$ on $B(x^0, r')$ for some $0 < r' \leq r$ using Girsanov's theorem as in Case 2 of the proof of Theorem 1.2 in [BP1]. □

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