Illinois Journal of Mathematics
Volume 50, Number 1, Spring 2006, Pages 1–32
S 0019-2082

HOT–SPOTS FOR CONDITIONED BROWNIAN MOTION

RODRIGO BAÑUELOS AND PEDRO J. MÉNDEZ-HERNÁNDEZ

In memory of Joseph Leo Doob,
with the greatest respect and admiration

ABSTRACT. Let D be a bounded domain in the plane which is symmetric and convex with respect to both coordinate axes. We prove that the Brownian motion conditioned to remain forever in D, the Doob h-process where h is the ground state Dirichlet eigenfunction in D, has the "hot-spots" property. That is, the first non-constant eigenfunction corresponding to the semigroup of this process with its nodal line on one of the coordinate axes attains its maximum and minimum on the boundary and only on the boundary of the domain. This is the exact analogue for conditioned Brownian motion of the result in [14] for Neumann eigenfunctions.

1. Introduction

The *hot-spots* conjecture, often also referred to as the *hot-spots* property, was formulated by J. Rauch in 1974. It asserts that the maximum and the minimum of the first non-constant Neumann eigenfunction for bounded domains in $\mathbb{R}^n$ are attained on the boundary and only on the boundary of the domain. The first general results on this conjecture, outside of domains where the eigenfunctions can be explicitly written down (such as balls, rectangles, etc., and products of these with other bounded domains, see [15]), were obtained in [4]. That paper also contains various precise formulations of the conjecture. The conjecture has generated a lot of interest for many years, especially since the appearance of [4]. Counterexamples for arbitrary planar domains exist. The conjecture is widely believed to be true for all bounded planar convex domains but it is open even for arbitrary triangles. We refer the reader to [5] and [7] for some of this literature and for a reformulation of the conjecture in terms of mixed Dirichlet–Neumann eigenfunctions, and to [10] for an example of a planar domain with one hole where the conjecture

Received July 5, 2005; received in final form September 12, 2005.
2000 *Mathematics Subject Classification.* Primary 47A75, 60J45, 35J10.
The first author is supported in part by NSF grant #9700585-DMS..

fails. Further references to the recent literature on the *hot–spots* problem are contained in these papers.

Motivated by this conjecture and the intuition that for smooth bounded convex domains the Brownian motion conditioned to remain forever in the domain, the Doob h-process corresponding to the ground state eigenfunction for the Dirichlet problem, behaves similar to reflected Brownian motion near the boundary (see [6]), it was conjectured in [3] that the conditioned Brownian motion on any convex domains also has the *hot-spots* property. More precisely, if $\tilde{T}_t^D$ is the semigroup of Brownian motion conditioned to remain forever in the domain D, and Ψ is its first non-constant eigenfunction, then Ψ takes its maximum and minimum on, and only on, the boundary of D. The purpose of this paper is to prove this conjecture for planar domains which are strictly convex in each variable and symmetric with respect to both coordinate axes. This is done by proving inequalities for ratios of survival time probabilities along the lines of those obtained in [8] and [22]. Our results parallel those proved by Jerison and Nadirashvili in [14] for the Rauch conjecture.

Before stating our results, we make the above statements more precise. Let D be a bounded domain in $\mathbb{R}^2$. For any positive superharmonic function h in D the *Doob h-Brownian motion in D* is the strong Markov process with transition densities given by

$$p_t^{h,D}(z,w) = p_t^D(z,w)\frac{h(w)}{h(z)},$$

where z is the starting point and w is the ending point and $p_t^D(z,w)$ is the transition density of the Brownian motion killed upon leaving the domain D. Locally, this process behaves like "ordinary" Brownian motion but its long term behavior is influenced by the superharmonic function h. These processes were introduced by Doob in his seminal paper [13], where he used them to study the boundary behaviour of harmonic functions. The behavior of these processes and their many connections and applications to various areas of probability, analysis, geometry and PDE, have been the subject of countless papers for many years. We refer the reader to [3] and [12] where many connections related to the survival time probabilities of these processes are discussed. In particular, [3] discusses connections to "intrinsic ultraconctractivity," the boundary Harnack principle, hyperbolic and quasi–hyperbolic geometry, logarithmic Sobolev inequalities, and it contains many references to these topics. In this paper we are interested in the case when the positive superharmonic function h is φ_1, the ground state Dirichlet eigenfunction of $-\frac{1}{2}\Delta$ in the domain D corresponding to the smallest eigenvalue λ_1. Since φ_1 vanishes on the boundary, the process, while it approaches the boundary, never really reaches it. For this reason this Doob h-process is frequently called "Brownian motion conditioned to remain forever in D." The semigroup $\tilde{T}_t^D$ of this process on $L^2(D, d\mu)$, where $d\mu = \varphi_1^2 dz$ (we normalize φ_1 by $\|\varphi_1\|_2 = 1$), is given by

$$\tilde{T}_t^D f(z) = \int_D \tilde{p}_t^D(z,w) f(w) d\mu(w),$$

where

$$\tilde{p}_t^D(z,w) = \frac{e^{\lambda_1 t} p_t^D(z,w)}{\varphi_1(z)\varphi_1(w)},$$

and its generator is

$$L = \frac{1}{2}\Delta + \frac{\nabla \varphi_1}{\varphi_1} \cdot \nabla. \tag{1.1}$$

If $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$ and $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ are the eigenvalues and eigenfunctions for $-\frac{1}{2}\Delta$ with Dirichlet boundary conditions, then $\{0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \dots\}$ and $\{1, \varphi_2/\varphi_1, \varphi_3/\varphi_1, \dots\}$ are the eigenvalues and eigenfunctions for $-L$ in D. That is, for any bounded domain $D \subset \mathbb{R}^2$ [21],

$$L\left(\frac{\varphi_n}{\varphi_1}\right) = -(\lambda_n - \lambda_1)\left(\frac{\varphi_n}{\varphi_1}\right). \tag{1.2}$$

If, in addition, the domain has some boundary smoothness, these eigenfunctions satisfy the Neumann boundary condition. For instance, this is the case if the domain has piecewise smooth boundary. That is, it follows from [20] that

$$\frac{\partial}{\partial \eta}\left(\frac{\varphi_2}{\varphi_1}\right) = 0, \tag{1.3}$$

for all points on the boundary of D where D is smooth. Here, and for the rest of the paper, $\frac{\partial}{\partial \eta}$ is the outer normal derivative on the boundary of D. We will say that ∂D is piecewise smooth if, with the exception of perhaps a finite number of points, every boundary point has a neighborhood where the boundary is at least C^2. For example, a polygon has piecewise smooth boundary. Of course, (1.3) may hold under much weaker conditions than piecewise smooth boundary but in this paper we are not really interested in smoothness "per se" but rather in the convexity properties of D. Indeed, smoothness is only used in Corollary 1.1 to give the exact location of the max and min. The following conjecture was first formulated in [3]; it appears there as *Problem 10*.

Conjecture 1.1. Suppose D is a bounded convex domain in $\mathbb{R}^2$. Then D has the *hot–spots* property for Brownian motion conditioned to remain forever in D. More precisely, $\Psi(z) = \varphi_2(z)/\varphi_1(z)$ attains its maximum (and minimum) on, and only on, the boundary of D.

As explained in [5] and [7], the *hot–spots* conjecture of Rauch can be related to the long term behavior of the survival time probabilities for Brownian motion with killing and reflection. In our setting the *hot–spots* conjecture can be related to the long term behavior of the survival time probabilities of killed Brownian motion. It is well known that for a large class of domains D, including all bounded convex domains in any dimension,

$$\lim_{t\to\infty} e^{\lambda_1 t} p_t^D(z,w) = \varphi_1(z)\varphi_1(w), \tag{1.4}$$

uniformly for $z, w \in D$; see [3] and references therein. In addition, if we let τ_D be the first exit time of Brownian motion from the domain D, we have

$$\lim_{t\to\infty} e^{\lambda_1} P_z\{\tau_D > t\} = \varphi_1(z)\int_D \varphi_1(w)dw, \tag{1.5}$$

uniformly for $z \in D$. From this we see that the functions $P_z\{\tau_D > t\}$ and $p_t^D(z,w)$ both give information on the function φ_1. We can also relate φ_2 to the transition densities of Brownian motion. To do this, we recall some basic properties of nodal lines for convex domains. The set $\gamma = \overline{\{x \in D : \varphi_2(x) = 0\}}$, the closure of the set of zeros of φ_2, is called the *nodal line (also nodal curve)* of φ_2. It is known that planar convex domains (see [1] and [16]) do not have closed nodal lines. That is, for convex planar domains γ is a smooth simple curve intersecting the boundary at exactly two points. The curve divides D into two simply connected domains, D_1 and D_2, called *nodal domains*. We may take $\varphi_2 > 0$ on D_1, and $\varphi_2 < 0$ on D_2. When restricted to D_1, φ_2 is the ground state eigenfunction corresponding to the smallest eigenvalue for the Dirichlet Laplacian on D_1. Of course, in this case the lowest eigenvalue for D_1 is just λ_2. Thus we can relate φ_2, as before, to the transition densities and exit time of killed Brownian motion in D_1. This gives rise to the following conjecture which parallels the problems formulated in [7] for Brownian motion with killing and reflection.

Conjecture 1.2. Suppose D is a bounded convex domain in $\mathbb{R}^2$. Let D_1, D_2 and γ be the nodal domains and nodal line as above. For $z \in \overline{D}_1$ and $t > 0$, set

$$\Psi(z,t) = \frac{P_z\{\tau_{D_1} > t\}}{P_z\{\tau_D > t\}}. \tag{1.6}$$

Then, for each $t > 0$ arbitrarily fixed, the function Ψ attains its maximum on, and only on, $\partial D_1 \backslash \gamma$.

As in the case of the Brownian motion with killing and reflection, a major obstacle with this conjecture is our lack of understanding (for arbitrary convex domains) of the geometry of, particularly the "location" of, the nodal line. In this paper we prove *Conjecture 1.2* when the domain is symmetric relative to both coordinate axes. This leads to a proof of *Conjecture 1.1* under the same assumptions on the domain. In fact, our results only require symmetry and convexity of the domain in both coordinate axes. Our result is the exact analogue of the *hot–spots* result for Neumann eigenfunctions proved by Jerison and Nadirashvili in [14].

Let D be a bounded domain in $\mathbb{R}^2$. We say that D is convex with respect to both axes if all the vertical and horizontal cross sections of D are intervals. Note that a domain can be convex with respect to both axes without being convex. The following result should be compared to those in [7] for Brownian motion with killing and reflection. Set

$$D^+ = D \cap \{(x, y) \in \mathbb{R}^2 : y > 0\},$$
$$D^- = D \cap \{(x, y) \in \mathbb{R}^2 : y < 0\}.$$

THEOREM 1.1. *Let D be a bounded domain in $\mathbb{R}^2$ which is symmetric and convex with respect to both axes.*

(i) *If $z_1 = (x, y_1) \in D^+$, $z_2 = (x, y_2) \in D^+$ and $y_1 < y_2$, then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} < \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}}, \tag{1.7}$$

for any $t > 0$. In particular, the function

$$\Psi(z, t) = \frac{P_z\{\tau_{D^+} > t\}}{P_z\{\tau_D > t\}},$$

for each $t > 0$ arbitrarily fixed, cannot have a maximum at an interior point of D^+.

(ii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| \le |x_1|$, then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} \le \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}}, \tag{1.8}$$

for any $t > 0$.

In order to relate the above result to eigenfunctions, we need information on the location of the nodal line. The study of the geometry of nodal curves has been of interest for many years. We refer the reader to [1] and [16] for some of these results. In the case of planar domains which are symmetric and convex in both axes, as those in Theorem 1.1, L. Payne [19] proved that there are no closed nodal curves and that there is a second eigenfunction whose nodal line is the intersection of the domain with one of the coordinate axes. As we will show below, Theorem 1.1 will give the following result.

THEOREM 1.2. *Let $D \subset \mathbb{R}^2$ be a bounded domain which is symmetric and convex with respect to both coordinate axes. Let φ_2 be such that its nodal line is the intersection of the x-axis with the domain. Assume without loss of generality that $\varphi_2 > 0$ in D^+ and $\varphi_2 < 0$ in D^-. Set $\Psi = \varphi_2/\varphi_1$.*

(i) *If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then*

$$\Psi(z_1) < \Psi(z_2). \tag{1.9}$$

(ii) *If* $z_1 = (x, y_1) \in D^-$ *and* $z_2 = (x, y_2) \in D^-$ *with* $y_2 < y_1$, *then*

$$\Psi(z_1) < \Psi(z_2). \tag{1.10}$$

In particular, Ψ *cannot attain a maximum nor a minimum in the interior of* D.

(iii) *If* $z_1 = (x_1, y) \in D^+$ *and* $z_2 = (x_2, y) \in D^+$ *with* $|x_2| < |x_1|$, *then*

$$\Psi(z_1) \leq \Psi(z_2). \tag{1.11}$$

Using Theorem 1.2 and the boundary condition (1.3) we will proof the following Corollary in §5.

Corollary 1.1. *Suppose* $D \subset \mathbb{R}^2$ *is a bounded domain with piecewise smooth boundary which is symmetric and convex with respect to both coordinate axes and that* φ_2 *is as in Theorem 1.2. Then strict inequality holds in* (1.11) *unless* D *is a rectangle. The maximum and minimum of* Ψ *on* $\overline{D}$ *are achieved at the points where the* y*-axis meets* ∂D *and, except for the rectangle, at no other points.*

In [11], an example of a planar domain is given where the Neumann eigenfunction attains a maximum in the interior of the domain. It was proved in [17] that there exists a domain D such that the nodal line of the second Dirichlet eigenfunction φ_2 is closed. Thus the *hot-spots* property for the conditioned Brownian motion, as stated in Conjecture (1.1), also fails in general domains. It would be interesting to know, as in the case of the Neumann eigenfunctions ([9], [10]), if in this setting there is also a domain for which both the maximum and the minimum are attained in the interior of the domain and if such a domain can be constructed with only one hole as was done in [10] for the Neumann problem. We would be very surprised if this were not the case. Also of interest would be extensions of the above results to other domains for which the Rauch conjecture is now known, such as convex domains with only one line of symmetry ([7], [18]), and the lip1 domains studied in [2]. We note here that the multiple integral techniques used in this paper are completely different from the coupling techniques used in the above mentioned papers dealing with the Neumann problem. However, we believe that coupling techniques (applied to the conditioned Brownian motion) should also be explored for this problem and that a better understanding of the "conditioned" *hot–spots* property will lead to a better understanding of the "classical" *hot–spots* property.

The paper is organized as follows. In §2, we state the general multiple integral inequalities needed for the proof of Theorem 1.1 and set some notation. The proofs of the multiple integral inequalities are by induction on the number of integrals as the arguments in, for example, [8] and [22]. However, in all the previous works we were interested in inequalities where the starting points were fixed and the "polarization" arguments, as complicated as they appear to be, were, more or less, "straightforward." The situation here is more complicated and great care has to be given to the many reflections that arise. In §3, we prove the case $n = 1$; the general case is proved in §4. The proofs of Theorems 1.1, 1.2 and Corollary 1.1 are given in §5. We end §5 with some results and question on *"hot-spots"* for Schrödinger operators of the form $L_V = -\Delta + V$.

2. Multiple integrals

For the rest of this section we assume that D is a bounded domain which is symmetric and convex in both axes. We may assume that D is of the form

$$D = \left\{ \ (x, y) \in \mathbb{R}^2 : x \in (-a, a), -f(x) < y < f(x) \ \right\}, \tag{2.1}$$

where $f : [-a, a] \to \mathbb{R}^+$ is an even, non–increasing, function on $[0, a]$ with $f(-a) = f(a) = 0$, and maximum $b = f(0)$. With this notation we also have

$$D^+ = \left\{ (x, y) \in \mathbb{R}^2 : x \in (-a, a), 0 < y < f(x) \right\}.$$

For $(x, y) \in \mathbb{R}^2$, $\{t_k\}_{k=1}^{\infty} \subset (0, \infty)$, and $n \geq 2$, we define the following functions in $\mathbb{R}^2$:

$$\Phi_1(x, y) = \int_D p_{t_1}(x_1 - x, x_2 - y)\, dx_1 dx_2,$$

$$\Phi_1^+(x, y) = \int_{D^+} p_{t_1}(x_1 - x, x_2 - y)\, dx_1 dx_2,$$

$$\Phi_n(x, y) = \int_D p_{t_n}(x_1 - x, x_2 - y)\, \Phi_{n-1}(x_1, x_2)\, dx_1 dx_2,$$

and

$$\Phi_n^+(x, y) = \int_{D^+} p_{t_n}(x_1 - x, x_2 - y)\, \Phi_{n-1}^+(x_1, x_2)\, dx_1 dx_2,$$

where

$$p_t(x, y) = \frac{1}{2\pi t} e^{-(x^2+y^2)/2t}.$$

Notice that for all $x, y \in \mathbb{R}$ and all $n \geq 1$,

$$\Phi_n(x, y) = \Phi_n(-x, y) = \Phi_n(x, -y), \tag{2.2}$$

and

$$\Phi_n^+(x, y) = \Phi_n^+(-x, y). \tag{2.3}$$

The main result of this section, which is used to derive the inequalities in §1, is the following theorem.

Theorem 2.1. *Suppose $0 \leq x_1 \leq x_2$ and $0 \leq y_1 \leq y_2$. Then for $n \geq 1$*

$$\Phi_n^+(x_1, y_2)\Phi_n(x_2, y_1) \geq \Phi_n^+(x_1, y_1)\Phi_n(x_2, y_2), \tag{2.4}$$

$$\Phi_n^+(x_1, y_2)\Phi_n(x_2, y_1) \geq \Phi_n^+(x_2, y_2)\Phi_n(x_1, y_1), \tag{2.5}$$

$$\begin{aligned}\Phi_n^+(x_1,y_2)\Phi_n(x_2,y_1)+\Phi_n^+(x_2,y_2)\Phi_n(x_1,y_1)\geq\\ \Phi_n^+(x_1,y_1)\Phi_n(x_2,y_2)+\Phi_n^+(x_2,y_1)\Phi_n(x_1,y_2),\end{aligned}\tag{2.6}$$

and

$$\begin{aligned}\Phi_n^+(x_1,y_2)\Phi_n(x_2,y_1)+\Phi_n^+(x_1,y_1)\Phi_n(x_2,y_2)\geq\\ \Phi_n^+(x_2,y_2)\Phi_n(x_1,y_1)+\Phi_n^+(x_2,y_1)\Phi_n(x_1,y_2).\end{aligned}\tag{2.7}$$

From (2.4) and (2.5) we have the following corollary.

Corollary 2.1. *Suppose $0 \leq y_1 \leq y_2$. Then for all $n \geq 1$*

$$\frac{\Phi_n^+(x,y_1)}{\Phi_n(x,y_1)}\leq\frac{\Phi_n^+(x,y_2)}{\Phi_n(x,y_2)},\tag{2.8}$$

for any $x \in \mathbb{R}$. If $|x_1| \leq |x_2|$, then

$$\frac{\Phi_n^+(x_2,y)}{\Phi_n(x_2,y)}\leq\frac{\Phi_n^+(x_1,y)}{\Phi_n(x_1,y)},\tag{2.9}$$

for any $y > 0$.

The proof of Theorem 2.1 is by induction on n. We prove the case $n = 1$ in §3 and the general case in §4. Here we present two general lemmas and introduce some more notation that will be used throughout the paper.

Lemma 2.1. *Suppose $x_1 \leq x_2$, $0 \leq u_1 \leq u_2$, $v_2 \leq 0$, and $0 \leq v_1$. Then for all $t > 0$*

$$p_t(x_1-u_1,x_2-u_2)\geq p_t(x_2-u_1,x_1-u_2),\tag{2.10}$$

and

$$p_t(x_1-v_2,x_2-v_1)\geq p_t(x_2-v_2,x_1-v_1).\tag{2.11}$$

In addition, if $0 \leq |v_2| \leq v_1$, and $0 \leq x_1$, then

$$p_t(x_1-v_2,x_2-v_1)\geq p_t(x_1+v_1,x_2+v_2).\tag{2.12}$$

Proof. A simple computation shows that

$$\begin{aligned}(x_1-u_1)^2+(x_2-u_2)^2\leq(x_1-u_2)^2+(x_2-u_1)^2\Leftrightarrow\\ -x_1u_1-x_2u_2\leq-x_1u_2-x_2u_1\Leftrightarrow\\ x_1(u_2-u_1)\leq x_2(u_2-u_1),\end{aligned}$$

and

$$\begin{aligned}(x_1-v_2)^2+(x_2-v_1)^2\leq(x_2-v_2)^2+(x_1-v_1)^2\Leftrightarrow\\ -x_1v_2-x_2v_1\leq-x_2v_2-x_1v_1\Leftrightarrow\\ x_1(v_1-v_2)\leq x_2(v_1-v_2).\end{aligned}$$

On the other hand, if $-v_2 \le v_1$, then

$$\begin{aligned} (x_1-v_2)^2+(x_2-v_1)^2 \le (x_1+v_1)^2+(x_2+v_2)^2 \Leftrightarrow \\ -x_1v_2-x_2v_1 \le x_1v_1+x_2v_2 \Leftrightarrow \\ x_1(-v_1-v_2) \le x_2(v_1+v_2). \end{aligned}$$

Thus (2.10), (2.11) and (2.12) follow from the fact that the function $p_t(x,y) = p_t(|x-y|)$ is radial symmetric decreasing. □

Define

$$\begin{aligned} \hat{p}_t(x_1-u_1,x_2-u_2) &= p_t(x_1-u_1,x_2-u_2)+p_t(x_1+u_1,x_2-u_2) \\ &\quad + p_t(x_1-u_1,x_2+u_2)+p_t(x_1+u_1,x_2+u_2). \end{aligned} \tag{2.13}$$

Lemma 2.2. *Suppose $0 \le x_1 \le x_2$, and $0 \le u_1 \le u_2$. Then*

$$\hat{p}_t(x_1-u_1,x_2-u_2) \ge \hat{p}_t(x_2-u_1,x_1-u_2), \tag{2.14}$$

for all $t>0$.

Proof. Without loss of generality we can assume that $t=1/2$. A simple computation shows that (2.14) is equivalent to

$$\begin{aligned} e^{x_1u_1+x_2u_2}+e^{-x_1u_1+x_2u_2}+e^{-(-x_1u_1+x_2u_2)}+e^{-(x_1u_1+x_2u_2)} \ge \\ e^{x_2u_1+x_1u_2}+e^{-x_2u_1+x_1u_2}+e^{-(-x_2u_1+x_1u_2)}+e^{-(x_2u_1+x_1u_2)}. \end{aligned}$$

Consider

$$\begin{aligned} a_1 &= x_1u_1+x_2u_2, \\ a_2 &= -x_1u_1+x_2u_2, \\ b_1 &= x_2u_1+x_1u_2, \\ b_2 &= -x_2u_1+x_1u_2. \end{aligned}$$

Then $a_1 \ge b_1 \ge 0$ and $a_2 \ge |b_2|$. The desired result immediately follows from the fact that e^x+e^{-x} is an even function which is increasing for $x>0$. □

3. Proof of Theorem 2.1, case $n=1$

With f as in (2.1), we introduce some more notation.

$$\begin{aligned} A(u,v) &= [-f(u),f(u)]\times[0,f(v)], \\ A^+(u,v) &= [0,f(u)]\times[0,f(v)], \\ A^-(u,v) &= [-f(u),0]\times[0,f(v)]. \end{aligned}$$

Notice that as functions of u and v, $A(u,v)$, $A^+(u,v)$ and $A^-(u,v)$ are even with respect to both coordinate axes.

Lemma 3.1. *Suppose* $0 \le x_1 \le x_2$ *and* $0 \le y_1 \le y_2$. *Then*

$$\Phi_1^+(x_1, y_2)\Phi_1(x_2, y_1) \ge \Phi_1^+(x_1, y_1)\Phi_1(x_2, y_2). \tag{3.1}$$

Proof. By Fubini's theorem and the symmetry of the domain we see that (3.1) is equivalent to

$$\int_0^a \int_0^a \hat{p}_{t_1}(x_1 - u_1, x_2 - u_2) \int_{A(u_2,u_1)} p_{t_1}(y_1 - v_2, y_2 - v_1) dv_2 dv_1 du_2 du_1 \ge$$
$$\int_0^a \int_0^a \hat{p}_{t_1}(x_1 - u_1, x_2 - u_2) \int_{A(u_2,u_1)} p_{t_1}(y_2 - v_2, y_1 - v_1) dv_2 dv_1 du_2 du_1,$$

where $\hat{p}_{t_1}(x_1 - u_1, x_2 - u_2)$ is given by (2.13). Thus we must prove that

$$\int_0^a \int_{u_1}^a \Bigg[\hat{p}_{t_1}(x_1 - u_1, x_2 - u_2) \int_{A(u_2,u_1)} p_{t_1}(y_1 - v_2, y_2 - v_1) dv_2 dv_1 +$$
$$\hat{p}_{t_1}(x_1 - u_2, x_2 - u_1) \int_{A(u_1,u_2)} p_{t_1}(y_1 - v_2, y_2 - v_1) dv_2 dv_1\Bigg] du_2 du_1 \ge$$
$$\int_0^a \int_{u_1}^a \Bigg[\hat{p}_{t_1}(x_1 - u_1, x_2 - u_2) \int_{A(u_2,u_1)} p_{t_1}(y_2 - v_2, y_1 - v_1) dv_2 dv_1 +$$
$$\hat{p}_{t_1}(x_1 - u_2, x_2 - u_1) \int_{A(u_1,u_2)} p_{t_1}(y_2 - v_2, y_1 - v_1) dv_2 dv_1\Bigg] du_2 du_1.$$

By (2.14) it is enough to prove that for all $0 \le y_1 \le y_2$ and all $0 \le u_1 \le u_2$,

$$\begin{aligned} &\int_{A(u_2,u_1)} p_{t_1}(y_1 - v_2, y_2 - v_1) dv_2 dv_1 \ge \\ &\int_{A(u_2,u_1)} p_{t_1}(y_2 - v_2, y_1 - v_1) dv_2 dv_1, \end{aligned} \tag{3.2}$$

and that

$$\begin{aligned} &\int_{A(u_1,u_2)} p_{t_1}(y_1 - v_2, y_2 - v_1) dv_2 dv_1 + \\ &\int_{A(u_2,u_1)} p_{t_1}(y_1 - v_2, y_2 - v_1) dv_2 dv_1 \ge \\ &\int_{A(u_1,u_2)} p_{t_1}(y_2 - v_2, y_1 - v_1) dv_2 dv_1 + \\ &\int_{A(u_2,u_1)} p_{t_1}(y_2 - v_2, y_1 - v_1) dv_2 dv_1. \end{aligned} \tag{3.3}$$

We begin by proving (3.2). In order to simplify the proof, we break it into several steps. The first two numbers in the "steps" correspond to the inequality we are proving. For example, 3.2.1 corresponds to step 1 in the proof of the inequality (3.2), and so on. This notation will be maintained for the rest of this section.

Step 3.2.1:. Suppose (v_2, v_1) are such that $0 \le v_2 \le f(u_2) \le v_1 \le f(u_1)$. Then by (2.10) we have,

$$p_{t_1}(y_2 - v_2, y_1 - v_1) \le p_{t_1}(y_1 - v_2, y_2 - v_1).$$

Hence

$$\int_{A^+(u_2,u_1)\setminus A^+(u_2,u_2)} p_{t_1}(y_2 - v_2, y_1 - v_1)dv_2dv_1 \le \int_{A^+(u_2,u_1)\setminus A^+(u_2,u_2)} p_{t_1}(y_1 - v_2, y_2 - v_1)dv_2dv_1. \tag{3.4}$$

Step 3.2.2:. Suppose (v_2, v_1) are such that $-f(u_2) \le v_2 \le 0$ and $0 \le v_1 \le f(u_1)$. Then by (2.11),

$$p_{t_1}(y_2 - v_2, y_1 - v_1) \le p_{t_1}(y_1 - v_2, y_2 - v_1).$$

Hence

$$\int_{A^-(u_2,u_1)} p_{t_1}(y_2 - v_2, y_1 - v_1)dv_2dv_1 \le \int_{A^-(u_2,u_1)} p_{t_1}(y_1 - v_2, y_2 - v_1)dv_2dv_1. \tag{3.5}$$

Step 3.2.3:. Suppose (v_2, v_1) are such that $0 \le v_2 \le v_1 \le f(u_2)$. Clearly

$$p_{t_1}(y_2 - v_2, y_1 - v_1) + p_{t_1}(y_1 - v_2, y_2 - v_1) = p_{t_1}(y_1 - v_2, y_2 - v_1) + p_{t_1}(y_2 - v_2, y_1 - v_1).$$

Hence

$$\int_{A^+(u_2,u_2)} p_{t_1}(y_2 - v_2, y_1 - v_1)dv_2dv_1 = \int_{A^+(u_2,u_2)} p_{t_1}(y_1 - v_2, y_2 - v_1)dv_2dv_1. \tag{3.6}$$

But now clearly (3.4), (3.5) and (3.6) imply (3.2).

The argument for (3.3) is similar and we again break it into several steps.

Step 3.3.1:. Suppose (v_2, v_1) are such that $-f(u_1) \le v_2 \le 0$ and $0 \le v_1 \le f(u_1)$. From (2.11) we obtain

$$p_{t_1}(y_2 - v_2, y_1 - v_1) \le p_{t_1}(y_1 - v_2, y_2 - v_1).$$

Hence

$$\int_{A^-(u_2,u_1)\cup A^-(u_1,u_2)} p_{t_1}(y_1-v_1,y_2-v_2)dv_2dv_1 \leq \int_{A^-(u_2,u_1)\cup A^-(u_1,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1, \tag{3.7}$$

and

$$\int_{A^-(u_2,u_2)} p_{t_1}(y_1-v_1,y_2-v_2)dv_2dv_1 \leq \int_{A^-(u_2,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1. \tag{3.8}$$

Step 3.3.2:. Suppose $(v_2,v_1) \in A^+(u_2,u_1)$ are such that $0 \leq v_2 \leq v_1$. Then

$$p_{t_1}(y_2-v_2,y_1-v_1) + p_{t_1}(y_1-v_2,y_2-v_1) = p_{t_1}(y_1-v_2,y_2-v_1) + p_{t_1}(y_2-v_2,y_1-v_1),$$

and integrating over all such (v_2,v_1) we conclude that

$$\int_{A^+(u_2,u_1)\cup A^+(u_1,u_2)} p_{t_1}(y_2-v_2,y_1-v_1)dv_2dv_1 = \int_{A^+(u_2,u_1)\cup A^+(u_1,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1, \tag{3.9}$$

and that

$$\int_{A^+(u_2,u_2)} p_{t_1}(y_2-v_2,y_1-v_1)dv_2dv_1 = \int_{A^+(u_2,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1. \tag{3.10}$$

Inequality (3.3) follows from (3.7), (3.8), (3.9), and (3.10). This completes the proof of the lemma. □

Lemma 3.2. *Suppose* $0 \leq x_1 \leq x_2$ *and* $0 \leq y_1 \leq y_2$. *Then*

$$\Phi_1^+(x_1,y_2)\Phi_1(x_2,y_1) \geq \Phi_1^+(x_2,y_2)\Phi_1(x_1,y_1). \tag{3.11}$$

Proof. One easily sees that (3.11) is equivalent to

$$\int_0^a \int_{u_1}^a \left[\hat{p}_{t_1}(x_1-u_1,x_2-u_2)\int_{A(u_2,u_1)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1 + \right.$$
$$\left.\hat{p}_{t_1}(x_1-u_2,x_2-u_1)\int_{A(u_1,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1\right] du_2du_1 \geq$$
$$\int_0^a \int_{u_1}^a \left[\hat{p}_{t_1}(x_2-u_1,x_1-u_2)\int_{A(u_2,u_1)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1 + \right.$$
$$\left.\hat{p}_{t_1}(x_2-u_2,x_1-u_1)\int_{A(u_1,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1\right] du_2du_1.$$

Thanks to (2.14)

$$\hat{p}_{t_1}(x_1-u_1,x_2-u_2) \geq \hat{p}_{t_1}(x_1-u_2,x_2-u_1).$$

Thus it is enough to prove that

$$\int_{A(u_2,u_1)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1 \geq \int_{A(u_1,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1, \tag{3.12}$$

if $u_2 \geq u_1 \geq 0$ and $y_2 \geq y_1 \geq 0$.

Step 3.12.1:. Suppose (v_2, v_1) are such that $0 \leq v_2 \leq f(u_2) \leq v_1 \leq f(u_1)$. Then (2.10) implies that

$$p_{t_1}(y_1-v_1,y_2-v_2) \leq p_{t_1}(y_1-v_2,y_2-v_1).$$

Integrating over $A^+(u_2,u_1) \setminus A^+(u_2,u_2)$ we have

$$\int_{A^+(u_1,u_2)\setminus A^+(u_2,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1 \leq \int_{A^+(u_2,u_1)\setminus A^+(u_2,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1. \tag{3.13}$$

Step 3.12.2:. If (v_2, v_1) are such that $0 \leq -v_2 \leq f(u_2) \leq v_1 \leq f(u_1)$. Then (2.12) implies that

$$p_{t_1}(y_1+v_1,y_2+v_2) \leq p_{t_1}(y_1-v_2,y_2-v_1).$$

Integrating over $A^-(u_2,u_1) \setminus A^-(u_2,u_2)$ we have that

$$\int_{A^-(u_1,u_2)\setminus A^-(u_2,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1 \leq \int_{A^-(u_2,u_1)\setminus A^-(u_2,u_2)} p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1. \tag{3.14}$$

Clearly (3.12) follows from (3.13) and (3.14). □

Lemma 3.3. *Suppose* $0 \le x_1 \le x_2$ *and* $0 \le y_1 \le y_2$. *Then*

$$\Phi_1^+(x_1,y_2)\Phi_1(x_2,y_1)+\Phi_1^+(x_2,y_2)\Phi_1(x_1,y_1)\ge \Phi_1^+(x_1,y_1)\Phi_1(x_2,y_2)+\Phi_1^+(x_2,y_1)\Phi_1(x_1,y_2), \tag{3.15}$$

and

$$\Phi_1^+(x_1,y_2)\Phi_1(x_2,y_1)+\Phi_1^+(x_1,y_1)\Phi_1(x_2,y_2)\ge \Phi_1^+(x_2,y_2)\Phi_1(x_1,y_1)+\Phi_1^+(x_2,y_1)\Phi_1(x_1,y_2). \tag{3.16}$$

Proof. As before, one easily sees that (3.15) is equivalent to

$$\int_0^a\int_{u_1}^a\Bigg[\hat{q}_{t_1}(x_1-u_1,x_2-u_2)\int_{A(u_2,u_1)}p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1+ \hat{q}_{t_1}(x_1-u_2,x_2-u_1)\int_{A(u_1,u_2)}p_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1\Bigg]du_2du_1\ge \int_0^a\int_{u_1}^a\Bigg[\hat{q}_{t_1}(x_1-u_1,x_2-u_2)\int_{A(u_2,u_1)}p_{t_1}(y_2-v_2,y_1-v_1)dv_2dv_1+ \hat{q}_{t_1}(x_1-u_2,x_2-u_1)\int_{A(u_1,u_2)}p_{t_1}(y_2-v_2,y_1-v_1)dv_2dv_1\Bigg]du_2du_1,$$

where

$$\hat{q}_{t_1}(x_1-u,x_2-v)=\hat{p}_{t_1}(x_1-u,x_2-v)+\hat{p}_{t_1}(x_1-v,x_2-u), \tag{3.17}$$

for all $u,v\in\mathbb{R}$.

In the same way, (3.16) is equivalent to

$$\int_0^a\int_{u_1}^a\Bigg[\hat{p}_{t_1}(x_1-u_1,x_2-u_2)\int_{A(u_2,u_1)}q_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1+ \hat{p}_{t_1}(x_1-u_2,x_2-u_1)\int_{A(u_1,u_2)}q_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1\Bigg]du_2du_1\ge \int_0^a\int_{u_1}^a\Bigg[\hat{p}_{t_1}(x_2-u_1,x_1-u_2)\int_{A(u_2,u_1)}q_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1+ \hat{p}_{t_1}(x_2-u_2,x_1-u_1)\int_{A(u_1,u_2)}q_{t_1}(y_1-v_2,y_2-v_1)dv_2dv_1\Bigg]du_2du_1,$$

where

$$q_{t_1}(x_1-u,x_2-v)=p_{t_1}(x_1-u,x_2-v)+p_{t_1}(x_1-v,x_2-u) \tag{3.18}$$

for all $u, v \in \mathbb{R}$. One easily shows that (2.10), (2.11), and (2.14) hold if we replace p_t by q_t and $\hat{p}_t$ by $\hat{q}_t$.

Let $0 \le x_1 \le x_2$ and $0 \le -v_2 \le v_1$. Then

$$\begin{aligned}(x_1 - v_1)^2 + (x_2 - v_2)^2 \le (x_1 + v_2)^2 + (x_2 + v_1)^2 \Leftrightarrow \\ -v_2\,(x_1 + x_2) \le v_1\,(x_1 + x_2).\end{aligned}$$

Using the fact that for all $t > 0$, p_t is a symmetric non–increasing function, we have

$$p_t(x_1 - v_1, x_2 - v_2) \ge p_t(x_1 + v_2, x_2 + v_1).$$

This inequality and (2.12) imply that

$$q_t(x_1 - v_2, x_2 - v_1) \ge q_t(x_1 + v_1, x_2 + v_2).$$

Hence, (3.15) and (3.16) follow from the proof of Lemma 2.1 and Lemma 2.2. □

4. Proof of Theorem 2.1, general case

Let us assume that (2.4), (2.5), (2.6), (2.7) are true for $n-1$, where $n \ge 2$. For any Borel set $A \subset \mathbb{R}^2$ we define

$$I_A(x,y,z,w) = \int_A p_{t_n}(x - v_2, y - v_1)\Phi^+_{n-1}(z, v_1)\Phi_{n-1}(w, v_2)dv_2dv_1. \tag{4.1}$$

Proof of (2.4):. Fubini's theorem implies that (2.4) is equivalent to

$$\begin{aligned}&\int_0^a \int_{u_1}^a \big[\hat{p}_{t_n}(x_1 - u_1, x_2 - u_2)I_{A(u_2,u_1)}(y_2, y_1, u_1, u_2) + \\ &\hat{p}_{t_n}(x_1 - u_2, x_2 - u_1)I_{A(u_1,u_2)}(y_2, y_1, u_2, u_1)\big]\, du_2du_1 \le \\ &\int_0^a \int_{u_1}^a \big[\hat{p}_{t_n}(x_1 - u_1, x_2 - u_2)I_{A(u_2,u_1)}(y_1, y_2, u_1, u_2) + \\ &\hat{p}_{t_n}(x_1 - u_2, x_2 - u_1)I_{A(u_1,u_2)}(y_1, y_2, u_2, u_1)\big]\, du_2du_1.\end{aligned}$$

Thus as in the proof of Lemma 2.1 it is enough to prove that for all $0 \le y_1 \le y_2$ and $0 \le u_1 \le u_2$,

$$I_{A(u_2,u_1)}(y_2, y_1, u_1, u_2) \le I_{A(u_2,u_1)}(y_1, y_2, u_1, u_2), \tag{4.2}$$

and

$$\begin{aligned}&I_{A(u_2,u_1)}(y_2, y_1, u_1, u_2) + I_{A(u_1,u_2)}(y_2, y_1, u_2, u_1) \le \\ &I_{A(u_2,u_1)}(y_1, y_2, u_1, u_2) + I_{A(u_1,u_2)}(y_1, y_2, u_2, u_1).\end{aligned} \tag{4.3}$$

We begin with the proof of (4.2). As before, we break the proof into steps.

Step 4.2.1:. Let (v_2, v_1) be such that $0 \le v_2 \le f(u_2) \le v_1 \le f(u_1)$. By (2.10),

$$p_{t_n}(y_2 - v_2, y_1 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2) \le$$
$$p_{t_n}(y_1 - v_2, y_2 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2).$$

Thus

(4.4)
$$\int_0^{f(u_2)}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_2 - v_2, y_1 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2)dv_1dv_2$$
$$\le \int_0^{f(u_2)}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_1 - v_2, y_2 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2)dv_1dv_2.$$

Step 4.2.2:. Let (v_2, v_1) be such that $-f(u_2) \le v_2 \le 0$ and $0 \le v_1 \le f(u_1)$. From (2.11) we see that

$$p_{t_n}(y_2 - v_2, y_1 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2) \le$$
$$p_{t_n}(y_1 - v_2, y_2 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2).$$

Then

(4.5)
$$\int_{-f(u_2)}^{0}\int_0^{f(u_1)} p_{t_n}(y_2 - v_2, y_1 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2)dv_1dv_2$$
$$\le \int_{-f(u_2)}^{0}\int_0^{f(u_1)} p_{t_n}(y_1 - v_2, y_2 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2)dv_1dv_2.$$

Step 4.2.3:. Let (v_2, v_1) be such that $0 \le v_2 \le v_1 \le f(u_2)$. Recall that

$$p_{t_n}(y_2 - v_2, y_1 - v_1) \le p_{t_n}(y_1 - v_2, y_2 - v_1).$$

By the induction hypothesis for (2.4)

$$\Phi^+_{n-1}(u_1, v_2)\Phi_{n-1}(u_2, v_1) \le \Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2).$$

Hence

$$p_{t_n}(y_2 - v_2, y_1 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2)+$$
$$p_{t_n}(y_1 - v_2, y_2 - v_1)\Phi^+_{n-1}(u_1, v_2)\Phi_{n-1}(u_2, v_1) \le$$
$$p_{t_n}(y_1 - v_2, y_2 - v_1)\Phi^+_{n-1}(u_1, v_1)\Phi_{n-1}(u_2, v_2)+$$
$$p_{t_n}(y_2 - v_2, y_1 - v_1)\Phi^+_{n-1}(u_1, v_2)\Phi_{n-1}(u_2, v_1).$$

Integrating we find that

$$
\begin{aligned}
(4.6)\qquad &\int_{A^+(u_2,u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1\\
&\le \int_{A^+(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1.
\end{aligned}
$$

Now clearly (4.4), (4.5) and (4.6) imply (4.2).

We next prove (4.3).

Step 4.3.1:. Let (v_2,v_1) be such that $0\le v_2\le v_1\le f(u_2)$. Then the induction hypothesis for (2.6) implies that

$$
\begin{aligned}
&\Phi^+_{n-1}(u_1,v_2)\Phi_{n-1}(u_2,v_1)+\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1)\le\\
&\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2).
\end{aligned}
$$

From (2.10) we conclude that

$$
\begin{aligned}
p_{t_n}(y_1-v_2,y_2-v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\
&\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)\,\big\}+\\
p_{t_n}(y_2-v_2,y_1-v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,v_2)\Phi_{n-1}(u_2,v_1)+\\
&\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1)\,\big\}\ \ge\\
p_{t_n}(y_2-v_2,y_1-v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\
&\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)\,\big\}+\\
p_{t_n}(y_1-v_2,y_2-v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,v_2)\Phi_{n-1}(u_2,v_1)+\\
&\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1)\,\big\}\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(4.7)\qquad &\int_{A^+(u_2,u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1\\
&+\int_{A^+(u_2,u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1\\
&\le \int_{A^+(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1\\
&+\int_{A^+(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1.
\end{aligned}
$$

Step 4.3.2:. Let (v_2,v_1) be such that $0\le -v_2\le v_1\le f(u_2)$. One easily proves that

$$
(4.8)\qquad p_{t_n}(y_1+v_1,y_2+v_2)\ge p_{t_n}(y_1+v_2,y_2+v_1).
$$

This inequality, and (2.11) imply

$$\begin{aligned}
p_{t_n}(y_1-v_2,y_2-v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\
&\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)\,\big\}+\\
p_{t_n}(y_2+v_2,y_1+v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,-v_2)\Phi_{n-1}(u_2,-v_1)+\\
&\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1)\,\big\}\ \geq\\
p_{t_n}(y_2-v_2,y_1-v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\
&\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)\,\big\}+\\
p_{t_n}(y_1+v_2,y_2+v_1)\ &\big\{\,\Phi^+_{n-1}(u_1,-v_2)\Phi_{n-1}(u_2,-v_1)+\\
&\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1)\,\big\}\,.
\end{aligned}$$

Thus as in the previous step we have

$$\begin{aligned}
(4.9)\qquad &\int_{A^-(u_2,u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1\\
&+\int_{A^-(u_2,u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1\\
&\leq\int_{A^-(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1\\
&+\int_{A^-(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1.
\end{aligned}$$

Step 4.3.3:. Let $(v_2,v_1)\in A^+(u_2,u_1)$ be such that $0\leq v_2\leq v_1$. Then the induction hypothesis for (2.4) and (2.5) imply

$$\frac{\Phi^+_{n-1}(u_1,v_1)}{\Phi_{n-1}(u_1,v_1)}\geq\frac{\Phi^+_{n-1}(u_2,v_1)}{\Phi_{n-1}(u_2,v_1)}\geq\frac{\Phi^+_{n-1}(u_2,v_2)}{\Phi_{n-1}(u_2,v_2)}.$$

Then (2.11) implies

$$\begin{aligned}
&p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\
&p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1)\geq\\
&p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\
&p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1).
\end{aligned}$$

Thus

(4.10)

$$\int_0^{f(u_2)}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_1dv_2$$
$$+\int_{f(u_2)}^{f(u_1)}\int_0^{f(u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_1dv_2$$
$$\le\int_0^{f(u_2)}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_1dv_2$$
$$+\int_{f(u_2)}^{f(u_1)}\int_0^{f(u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_1dv_2.$$

Step 4.3.4:. Let $(v_2,v_1)\in A^-(u_2,u_1)$ be such that $0\le -v_2\le v_1$. Thanks to (2.11) and (4.8) we have

$$p_{t_n}(y_1-v_2,y_2-v_1)\ge p_{t_n}(y_2-v_2,y_1-v_1),$$

and

$$p_{t_n}(y_2+v_2,y_1+v_1)\ge p_{t_n}(y_1+v_2,y_2+v_1).$$

Thus

$$\begin{gathered}p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\ p_{t_n}(y_2+v_2,y_1+v_1)\,\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1)\ge\\ p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\\ p_{t_n}(y_1+v_2,y_2+v_1)\,\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1).\end{gathered}$$

This gives

(4.11)

$$\int_{-f(u_2)}^{0}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_1dv_2$$
$$+\int_{-f(u_1)}^{-f(u_2)}\int_0^{f(u_2)} p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_1dv_2$$
$$\le\int_{-f(u_2)}^{0}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_1dv_2$$
$$+\int_{-f(u_1)}^{-f(u_2)}\int_0^{f(u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_1dv_2,$$

and inequality (4.3) follows from (4.7), (4.9), (4.10), and (4.11).

Proof of (2.5). One easily sees that (2.5) is equivalent to

$$\int_0^a \int_{u_1}^a \big[\hat{p}_{t_n}(x_1-u_1,x_2-u_2)I_{A(u_2,u_1)}(y_1,y_2,u_1,u_2)+$$
$$\hat{p}_{t_n}(x_1-u_2,x_2-u_1)I_{A(u_1,u_2)}(y_1,y_2,u_2,u_1)\big]\,du_2du_1 \geq$$
$$\int_0^a \int_{u_1}^a \big[\hat{p}_{t_n}(x_2-u_1,x_1-u_2)I_{A(u_2,u_1)}(y_1,y_2,u_1,u_2)+$$
$$\hat{p}_{t_n}(x_2-u_2,x_1-u_1)I_{A(u_1,u_2)}(y_1,y_2,u_2,u_1)\big]\,du_2du_1.$$

As in the proof of Lemma 2.2, it is enough to prove that

$$I_{A(u_2,u_1)}(y_1,y_2,u_1,u_2) \geq I_{A(u_1,u_2)}(y_1,y_2,u_2,u_1), \tag{4.12}$$

for $u_2 \geq u_1$. Again, we break this into several steps.

Step 4.12.1:. Let (v_2,v_1) be such that $0 \leq v_2 \leq v_1 \leq f(u_2)$. Then the induction hypothesis for (2.4) and (2.7) imply

$$\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)+\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1) \leq$$
$$\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\Phi^+_{n-1}(u_1,v_2)\Phi_{n-1}(u_2,v_1),$$

and

$$\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2) \leq \Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2).$$

From (2.10) we conclude that

$$p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+$$
$$p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_1,v_2)\Phi_{n-1}(u_2,v_1) \geq$$
$$p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)+$$
$$p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1).$$

Thus

$$\begin{aligned} &\int_{A^+(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1 \\ &\leq \int_{A^+(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1. \end{aligned} \tag{4.13}$$

Step 4.12.2:. Let (v_2,v_1) be such that $0 \leq -v_2 \leq v_1 \leq f(u_2)$. Then the induction hypothesis for (2.2), (2.3), (2.4), and (2.7) imply that

$$\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)+\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1) \leq$$
$$\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)+\Phi^+_{n-1}(u_1,-v_2)\Phi_{n-1}(u_2,-v_1)$$

and

$$\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2) \leq \Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2).$$

Thus (2.12) implies

$$\int_{A^-(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1 \tag{4.14}$$
$$\leq \int_{A^-(u_2,u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1.$$

Step 4.12.3:. Let $(v_2,v_1)\in A^+(u_2,u_1)$ be such that $0\leq v_2\leq v_1$. Then the induction hypothesis for (2.4) and (2.5) imply

$$\frac{\Phi^+_{n-1}(u_1,v_1)}{\Phi_{n-1}(u_1,v_1)}\geq\frac{\Phi^+_{n-1}(u_2,v_2)}{\Phi_{n-1}(u_2,v_2)}.$$

Then

$$p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)\geq$$
$$p_{t_n}(y_2-v_2,y_1-v_1)\,\Phi^+_{n-1}(u_2,v_2)\Phi_{n-1}(u_1,v_1).$$

Thus

$$\int_0^{f(u_2)}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_2-v_1,y_1-v_2)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_2dv_1 \tag{4.15}$$
$$\leq\int_{f(u_2)}^{f(u_1)}\int_0^{f(u_2)} p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_2dv_1.$$

Step 4.12.4:. Let $(v_2,v_1)\in A^-(u_2,u_1)$ be such that $0\leq -v_2\leq v_1$. Then

$$\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)\geq\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1).$$

Hence (2.12) implies

$$p_{t_n}(y_1-v_2,y_2-v_1)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)$$
$$\geq p_{t_n}(y_1+v_2,y_2+v_1)\,\Phi^+_{n-1}(u_2,-v_2)\Phi_{n-1}(u_1,-v_1),$$

and

$$\int_{-f(u_1)}^{-f(u_2)}\int_0^{f(u_2)} p_{t_n}(y_2-v_1,y_1-v_2)\,\Phi^+_{n-1}(u_2,v_1)\Phi_{n-1}(u_1,v_2)dv_1dv_2 \tag{4.16}$$
$$\leq\int_{-f(u_2)}^{0}\int_{f(u_2)}^{f(u_1)} p_{t_n}(y_2-v_1,y_1-v_2)\,\Phi^+_{n-1}(u_1,v_1)\Phi_{n-1}(u_2,v_2)dv_1dv_2.$$

Inequality (4.12) follows from (4.13), (4.14), (4.15), and (4.16). $\square$

Proof of (2.6) and (2.7). One easily sees that (2.6) is equivalent to

$$\int_0^a \int_{u_1}^a \big[\hat{q}_{t_n}(x_1-u_1,x_2-u_2)I_{A(u_2,u_1)}(y_1,y_2,u_1,u_2)+$$
$$\hat{q}_{t_n}(x_1-u_2,x_2-u_1)I_{A(u_1,u_2)}(y_1,y_2,u_2,u_1)\big]\,du_2du_1 \leq$$
$$\int_0^a \int_{u_1}^a \big[\hat{q}_{t_n}(x_2-u_1,x_1-u_2)I_{A(u_2,u_1)}(y_1,y_2,u_1,u_2)+$$
$$\hat{q}_{t_n}(x_2-u_2,x_1-u_1)I_{A(u_1,u_2)}(y_1,y_2,u_2,u_1)\big]\,du_2du_1,$$

where $\hat{q}_{t_n}(x_1-u,x_2-v)$ is given by (3.17). Since the function $\hat{q}_t(x,y)$ satisfies (2.14), the proof of (2.5) implies the result.

On the other hand, for any Borel subset A of $\mathbb{R}$ we define

$$\bar{I}_A(x,y,z,w)=\int_A q_{t_n}(x-v_2,y-v_1)\Phi^+_{n-1}(z,v_1)\Phi_{n-1}(w,v_2)dv_2dv_1, \tag{4.17}$$

where $q_{t_n}(x_1-u,x_2-v)$ is given by (3.18). Then (2.7) is equivalent to

$$\int_o^a \int_{u_1}^a \big[\hat{p}_{t_n}(x_1-u_1,x_2-u_2)\bar{I}_{A(u_2,u_1)}(y_1,y_2,u_1,u_2)+$$
$$\hat{p}_{t_n}(x_1-u_2,x_2-u_1)\bar{I}_{A(u_1,u_2)}(y_1,y_2,u_2,u_1)\big]\,du_2du_1 \leq$$
$$\int_0^a \int_{u_1}^a \big[\hat{p}_{t_n}(x_1-u_1,x_2-u_2)\bar{I}_{A(u_2,u_1)}(y_2,y_1,u_1,u_2)+$$
$$\hat{p}_{t_n}(x_1-u_2,x_2-u_1)\bar{I}_{A(u_1,u_2)}(y_2,y_1,u_2,u_1)\big]\,du_2du_1.$$

Since the function $q_t(x,y)$ satisfies (2.10), (2.11), (2.12), and (4.8), the proof of (2.4) implies the result. This completes the proof of Theorem 2.1. ☐

5. The hot–spots results

In this section we will prove Theorem 1.1, Theorem 1.2 and Corollary 1.1. We begin by proving Theorem 1.1. For this, fix $t>0$ and consider the sequence of times $t_i=\frac{it}{n}$, where $1\leq i\leq n$. By the continuity of the Brownian paths we have that for any domain D,

$$\begin{aligned} P_z\{\,\tau_D>t\,\} &= \lim_{m\to\infty} P_z\{\,B_s\in D_m, 0\leq s\leq t\,\} \\ &= \lim_{m\to\infty}\lim_{n\to\infty} P_z\{\,B_{\frac{it}{n}}\in D_m, i=1,\dots,n\,\}, \end{aligned} \tag{5.1}$$

where D_m is an increasing sequence of domains whose union is D and $\overline{D}_m \subset D_{m+1}$. In our case we may choose each domain D_m satisfying the hypotheses of Theorem 1.1. The Markov property of the Brownian motion implies

$$P_z\{\,B_{\frac{it}{n}}\in D_m, i=1,\dots,n\,\} = \int_{D_m}\cdots\int_{D_m}\prod_{i=1}^n p_{\frac{t}{n}}(z_{i-1}-z_i)\,dz_1\dots dz_n,$$

where $z=z_0$. From (2.8) and (2.9) in Corollary 2.1, it follows that

(i) if $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} \leq \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}} \tag{5.2}$$

for any $t > 0$, and that

(ii) if $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| \leq |x_1|$, then

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} \leq \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}}, \tag{5.3}$$

for all $t > 0$.

It remains to prove that the inequality in (5.2) is strict. First observe that for each fix $t > 0$ the function

$$\Psi(z, t) = \frac{P_z\{\tau_{D^+} > t\}}{P_z\{\tau_D > t\}}$$

is real analytic in D since both $P_z\{\tau_{D^+} > t\}$ and $P_z\{\tau_D > t\}$ are strictly positive solutions of the heat equation in D. Therefore, if $\Psi(z, t)$ is constant in a sub–interval of the vertical cross section at $(x, 0)$, $\Psi(z, t)$ must be constant in the whole cross section. Given that $\Psi(z, t) = 0$ for all $z = (x, 0) \in D$, it follows that $\Psi(z, t)$ cannot be constant on any vertical cross section. Hence, we must have strict inequality in (5.2). In particular, for each $t > 0$ arbitrarily fixed, the function

$$\Psi(z, t) = \frac{P_z\{\tau_{D^+} > t\}}{P_z\{\tau_D > t\}}$$

cannot have a maximum at an interior point of D^+. This completes the proof of Theorem 1.1.

We now prove Theorem 1.2. Recall that for any bounded domain

$$\lim_{t\to\infty} e^{\lambda_1 t} P_z\{\tau_D > t\} = \varphi_1(z) \int_D \varphi_1(w) dw, \tag{5.4}$$

for each fix $z \in D$. Applying this to the domains D and D^+, it follows from Theorem 1.1 that

(i) if $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then

$$\Psi(z_1) \leq \Psi(z_2), \tag{5.5}$$

and

(ii) if $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| < |x_1|$, then

$$\Psi(z_1) \leq \Psi(z_2). \tag{5.6}$$

Notice that the function

$$\Psi(z) = \frac{\varphi_2(z)}{\varphi_1(z)}$$

is also real analytic in D since both $\varphi_1(z)$ and $\varphi_2(z)$ are real analytic in D and $\varphi_1(z)$ is strictly positive in D. Given that the nodal line of φ_2 is the intersection of the x-axis with the domain we have

$$\Psi(z) = 0 \text{ for all } z = (x, 0) \in D.$$

Thus we must have strict inequality in (5.5). The symmetry of the domain implies that φ_2 can be chosen such that

$$\Psi(x, y) = -\Psi(x, -y),$$

for all $(x, y) \in D$. Then (1.10) immediately follows from the strict inequality in (5.5). This completes the proof of Theorem 1.2.

Finally, we prove Corollary 1.1. As above, we observe that $\Psi(z)$ is real analytic in D. This time we also assume that D has piecewise smooth boundary. Then $\Psi(z)$ has first partial derivatives which are continuous up to the boundary except at the non-smooth boundary points of D and its normal derivative satisfies (1.3); see [20]. Recall that $b = f(0)$. Suppose we have $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_1)$ in D such that $x_1 < x_2$ and $\Psi(z_1) = \Psi(z_2)$. Then, by real analyticity, there exists a smooth function ψ such that

$$\Psi(x, y) = \psi(y) \text{ for all } (x, y) \in D^+.$$

Fix $0 < y_0 < b$. By (1.9) and (1.10) we have that

$$\frac{\partial}{\partial y}\Psi(x, y) = \psi'(y) > 0, \text{ for all } (x, y) \in D.$$

Then if ∂D is smooth at the boundary point (x_0, y_0), we have

$$\frac{\partial}{\partial y}\Psi(x_0, y_0) > 0 \text{ and } \frac{\partial}{\partial x}\Psi(x_0, y_0) = 0.$$

However, by (1.3) Ψ satisfies the Neumann boundary conditions on D. Thus the vertical component of the normal to D at (x_0, y_0) is zero and D must be a rectangle. We conclude that strict inequality holds in (1.11) unless D is a rectangle, and that the maximum and minimum of Ψ on $\overline{D}$ are achieved at the points where the y-axis meets ∂D and, except for the rectangle, at no other point. This completes the proof of the Corollary 1.1.

We end the paper by stating some generalizations of the above results which follow from our arguments. In [8], we used techniques on multiple integrals similar to those used in this paper to obtain inequalities for ratios of probabilities similar to the results in Theorem 1.1. These inequalities were proved not only for the Laplacian but also for certain Schrödinger operators. In the same way the techniques in this paper, when put together with the arguments in [8], give the following results.

THEOREM 5.1. *Let D be a bounded domain in $\mathbb{R}^2$ which is symmetric and convex with respect to both axes. Let V be a bounded non–negative C^2 function defined on D which is symmetric relative to both coordinate axes and with the property that*

$$\frac{\partial^2 V}{\partial x \partial y}(x,y) \geq 0, \tag{5.7}$$

for all $(x,y) \in D$.

(i) *If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then*

$$\frac{E_{z_1}\{\exp(-\int_0^t V(B_s)ds), \tau_{D^+} > t\}}{E_{z_1}\{\exp(-\int_0^t V(B_s)ds), \tau_D > t\}} \leq \frac{E_{z_2}\{\exp(-\int_0^t V(B_s)ds), \tau_{D^+} > t\}}{E_{z_2}\{\exp(-\int_0^t V(B_s)ds), \tau_D > t\}} \tag{5.8}$$

for any $t > 0$. If, in addition, the function V is real analytic in D then strict inequality holds in (5.8) *and the function*

$$\Psi_V(z,t) = \frac{E_z\{\exp(-\int_0^t V(B_s)ds), \tau_{D^+} > t\}}{E_z\{\exp(-\int_0^t V(B_s)ds), \tau_D > t\}},$$

for each $t > 0$ arbitrarily fixed, cannot have a maximum at an interior point of D^+.

(ii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| \leq |x_1|$, then*

$$\frac{E_{z_1}\{\exp(-\int_0^t V(B_s)ds), \tau_{D^+} > t\}}{E_{z_1}\{\exp(-\int_0^t V(B_s)ds), \tau_D > t\}} \leq \frac{E_{z_2}\{\exp(-\int_0^t V(B_s)ds), \tau_{D^+} > t\}}{E_{z_2}\{\exp(-\int_0^t V(B_s)ds), \tau_D > t\}},$$

for each $t > 0$ arbitrarily fixed.

Next, we consider the eigenvalue problems

$$\begin{cases} -\Delta\varphi + V\varphi = \lambda\varphi, \text{ in } D, \\ \qquad\qquad \varphi = 0, \text{ on } \partial D, \end{cases} \tag{5.9}$$

and

$$\begin{cases} -\Delta\varphi + V\varphi = \lambda\varphi, \text{ in } D^+, \\ \qquad\qquad \varphi = 0, \text{ on } \partial D^+, \end{cases} \tag{5.10}$$

and denote their first eigenvalues and eigenfunctions by $\lambda_{1,D}^V$, $\varphi_{1,D}^V$ and λ_{1,D^+}^V, φ_{1,D^+}^V, respectively. Following the arguments in [8] and the proof of Theorem 1.2, we obtain the following result.

Theorem 5.2. *Let D and V be as in the statement of Theorem 5.1. Set $\Psi_{D,V}(z) = \varphi^V_{1,D^+}(z)/\varphi^V_{1,D}(z)$, for $z \in D^+$.*

(i) *If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then*

$$\Psi_{D,V}(z_1) \leq \Psi_{D,V}(z_2). \tag{5.11}$$

If, in addition, the potential V is real analytic in D, then strict inequality holds in (5.11) *and the function $\Psi_{D,V}(z)$ cannot have a maximum at an interior point of D^+.*

(ii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| < |x_1|$, then*

$$\Psi_{D,V}(z_1) \leq \Psi_{D,V}(z_2). \tag{5.12}$$

Theorem 5.1 will follow from a new version of Theorem 2.1 for the relevant multiple integrals that arise from the Feynman–Kac formula. For this, one follows the argument for the case $V = 0$ making the appropriate changes as in [8]. For the sake of completeness and since the argument may not be completely straightforward to those not familiar with these techniques, we present the proof.

First we introduce some new notation. For $(x, y) \in \mathbb{R}^2$, $\{t_k\}_{k=1}^{\infty} \subset (0, \infty)$, and $n \geq 2$, we define the following functions in $\mathbb{R}^2$:

$$\Psi_{0,V}(x, y) = \Psi^+_{0,V}(x, y) = \exp\{-t_1 V(x, y)\},$$

$$\Phi_{1,V}(x, y) = \int_D p_{t_1}(x_1 - x, x_2 - y)\, \Psi_{0,V}(x_1, x_2)\, dx_1 dx_2,$$

$$\Phi^+_{1,V}(x, y) = \int_{D^+} p_{t_1}(x_1 - x, x_2 - y)\, \Psi^+_{0,V}(x_1, x_2)\, dx_1 dx_2,$$

$$\Psi_{n-1,V}(x, y) = \exp\{-t_n V(x, y)\}\ \Phi_{n-1,V}(x, y),$$

$$\Psi^+_{n-1,V}(x, y) = \exp\{-t_n V(x, y)\}\ \Phi^+_{n-1,V}(x, y),$$

$$\Phi_{n,V}(x, y) = \int_D p_{t_n}(x_1 - x, x_2 - y)\, \Psi_{n-1,V}(x_1, x_2)\, dx_1 dx_2,$$

and

$$\Phi^+_{n,V}(x, y) = \int_{D^+} p_{t_n}(x_1 - x, x_2 - y)\, \Psi^+_{n-1,V}(x_1, x_2)\ dx_1 dx_2.$$

Notice that for all $(x, y) \in \mathbb{R}^2$

$$\Phi_{n,V}(x, y) = \Phi_{n,V}(-x, y) = \Phi_{n,V}(x, -y),$$

and

$$\Phi^+_{n,V}(x, y) = \Phi^+_{n,V}(-x, y).$$

The next result implies Theorem 5.1.

THEOREM 5.3. *Suppose $0 \le x_1 \le x_2$ and $0 \le y_1 \le y_2$. Then for $n \ge 1$,*

$$\Phi^+_{n,V}(x_1, y_2)\Phi_{n,V}(x_2, y_1) \ge \Phi^+_{n,V}(x_1, y_1)\Phi_{n,V}(x_2, y_2), \tag{5.13}$$

$$\Phi^+_{n,V}(x_1, y_2)\Phi_{n,V}(x_2, y_1) \ge \Phi^+_{n,V}(x_2, y_2)\Phi_{n,V}(x_1, y_1), \tag{5.14}$$

$$\begin{aligned} &\Phi^+_{n,V}(x_1, y_2)\Phi_{n,V}(x_2, y_1) + \Phi^+_{n,V}(x_2, y_2)\Phi_{n,V}(x_1, y_1) \ge \\ &\Phi^+_{n,V}(x_1, y_1)\Phi_{n,V}(x_2, y_2) + \Phi^+_{n,V}(x_2, y_1)\Phi_{n,V}(x_1, y_2), \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} &\Phi^+_{n,V}(x_1, y_2)\Phi_{n,V}(x_2, y_1) + \Phi^+_{n,V}(x_1, y_1)\Phi_{n,V}(x_2, y_2) \ge \\ &\Phi^+_{n,V}(x_2, y_2)\Phi_{n,V}(x_1, y_1) + \Phi^+_{n,V}(x_2, y_1)\Phi_{n,V}(x_1, y_2). \end{aligned} \tag{5.16}$$

Proof. Let $0 \le x_1 \le x_2$ and $0 \le y_1 \le y_2$. For any Borel set $A \subset \mathbb{R}^2$ define

$$I_{A,V,n}(x, y, z, w) = \int_A p_{t_n}(x - v_2, y - v_1)\Psi^+_{n-1,V}(z, v_1)\Psi_{n-1,V}(w, v_2)dv_2dv_1,$$

where $x, y, z, w \in \mathbb{R}^2$. A straightforward computation shows that, if V is a C^2 function with the property that

$$\frac{\partial^2 V}{\partial x \partial y}(x, y) \ge 0,$$

for all $(x, y) \in D$, then

$$V(x_1, y_1) + V(x_2, y_2) \ge V(x_1, y_2) + V(x_2, y_1),$$

which is equivalent to

$$\begin{aligned} \Psi^+_{0,V}(x_1, y_2)\Psi_{0,V}(x_2, y_1) &\ge \Psi^+_{0,V}(x_1, y_1)\Psi_{0,V}(x_2, y_2) \\ &= \Psi^+_{0,V}(x_2, y_2)\Psi_{0,V}(x_1, y_1). \end{aligned} \tag{5.17}$$

On the other hand

$$\begin{aligned} &\Psi^+_{0,V}(x_1, y_2)\Psi_{0,V}(x_2, y_1) + \Psi^+_{0,V}(x_2, y_2)\Psi_{0,V}(x_1, y_1) = \\ &\Psi^+_{0,V}(x_1, y_2)\Psi_{0,V}(x_2, y_1) + \Psi^+_{0,V}(x_1, y_1)\Psi_{0,V}(x_2, y_2) = \\ &\Psi^+_{0,V}(x_1, y_1)\Psi_{0,V}(x_2, y_2) + \Psi^+_{0,V}(x_2, y_1)\Psi_{0,V}(x_1, y_2) = \\ &\Psi^+_{0,V}(x_2, y_2)\Psi_{0,V}(x_1, y_1) + \Psi^+_{0,V}(x_2, y_1)\Psi_{0,V}(x_1, y_2). \end{aligned} \tag{5.18}$$

After replacing $I_A(x, y, z, w)$ by $I_{A,V,n}(x, y, z, w)$, we can follow the proof of Theorem 2.1 in the general case to verify that (5.17) and (5.18) imply Theorem 5.3 for $n = 1$. Let us now assume that (5.13), (5.14), (5.15), and (5.16) hold for $n - 1$. Combining (5.17) and (5.13) (for $n - 1$) we obtain

$$\begin{aligned} &\Phi^+_{n-1,V}(x_1, y_2)\Phi_{n-1,V}(x_2, y_1) \exp\{-t_n V(x_1, y_2) - t_n V(x_2, y_1)\} \ge \\ &\Phi^+_{n-1,V}(x_1, y_1)\Phi_{n-1,V}(x_2, y_2) \exp\{-t_n V(x_1, y_1) - t_n V(x_2, y_2)\}. \end{aligned}$$

Hence

$$\Psi^+_{n-1,V}(x_1,y_2)\Psi_{n-1,V}(x_2,y_1) \geq \Psi^+_{n-1,V}(x_1,y_1)\Psi_{n-1,V}(x_2,y_2). \tag{5.19}$$

Furthermore (5.13), (5.14), (5.15), and (5.17) imply that

$$\begin{aligned}
&\Phi^+_{n-1,V}(x_1,y_2)\Phi_{n-1,V}(x_2,y_1)\, \exp\{-t_nV(x_1,y_2) - t_nV(x_2,y_1)\} + \\
&\Phi^+_{n-1,V}(x_2,y_2)\Phi_{n-1,V}(x_1,y_1)\, \exp\{-t_nV(x_2,y_2) - t_nV(x_1,y_1)\} \geq \\
&\Phi^+_{n-1,V}(x_1,y_1)\Phi_{n-1,V}(x_2,y_2)\, \exp\{-t_nV(x_1,y_1) - t_nV(x_2,y_2)\} + \\
&\Phi^+_{n-1,V}(x_2,y_1)\Phi_{n-1,V}(x_1,y_2)\, \exp\{-t_nV(x_2,y_1) - t_nV(x_1,y_2)\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Psi^+_{n-1,V}(x_1,y_2)\Psi_{n-1,V}(x_2,y_1) + \Psi^+_{n-1,V}(x_2,y_2)\Psi_{n-1,V}(x_1,y_1) \geq \\
&\Psi^+_{n-1,V}(x_1,y_1)\Psi_{n-1,V}(x_2,y_2) + \Psi^+_{n-1,V}(x_2,y_1)\Psi_{n-1,V}(x_1,y_2).
\end{aligned} \tag{5.20}$$

In a similar manner we obtain that

$$\Psi^+_{n-1,V}(x_1,y_2)\Psi_{n-1,V}(x_2,y_1) \geq \Psi^+_{n-1,V}(x_2,y_2)\Psi_{n-1,V}(x_1,y_1), \tag{5.21}$$

and

$$\begin{aligned}
&\Psi^+_{n-1,V}(x_1,y_2)\Psi_{n-1,V}(x_2,y_1) + \Psi^+_{n-1,V}(x_1,y_1)\Psi_{n-1,V}(x_2,y_2) \geq \\
&\Psi^+_{n-1,V}(x_2,y_2)\Psi_{n-1,V}(x_1,y_1) + \Psi^+_{n-1,V}(x_2,y_1)\Psi_{n-1,V}(x_1,y_2).
\end{aligned} \tag{5.22}$$

The proof of Theorem 2.1 in the general case can now be followed, step by step, to prove that if Theorem 5.3 is true for $n-1$, then it is true for n. $\square$

Unfortunately, as in [8], we are not able to immediately conclude that there is an eigenfunction $\varphi^V_{2,D}$ corresponding to the second eigenvalue $\lambda^V_{2,D}$ for the problem (5.9) such that $\varphi^V_{2,D}(z) = \varphi^V_{1,D^+}(z)$ for all $z \in D^+$. Hence, we are not able to conclude a statement similar to that in Theorem 1.2 for the operator

$$L_V = -\Delta + V, \tag{5.23}$$

even in the case of the unit disk. However, in the case of the interval $I = (-a,a)$ and $V(x)$ an even potential in I, it is known (see [8]) that there exists an eigenfunction $\varphi^V_{2,I}$ corresponding to the second eigenvalue $\lambda^V_{2,I}$ such that $\varphi^V_{2,I}(z) = \varphi^V_{1,I^+}(z)$, where $I^+ = (0,a)$. In this case, our arguments give the following result.

Theorem 5.4. *Let $I = (-a,a)$, $I^+ = (0,a)$, and V be a continuous positive even function in I. If $x_1 \in I^+$ and $x_2 \in I^+$ with $x_1 < x_2$, then*

$$\frac{E_{x_1}\{\exp(-\int_0^t V(B_s)ds), \tau_{I^+} > t\}}{E_{x_1}\{\exp(-\int_0^t V(B_s)ds), \tau_I > t\}} \leq \frac{E_{x_2}\{\exp(-\int_0^t V(B_s)ds), \tau_{I^+} > t\}}{E_{x_2}\{\exp(-\int_0^t V(B_s)ds), \tau_I > t\}}. \tag{5.24}$$

For $x \in I^+$, *set* $\Psi_{I,V}(x) = \varphi^V_{2,I}(x)/\varphi^V_{1,I}(x)$. *If* $x_1 \in I^+$ *and* $x_2 \in I^+$ *with* $x_1 < x_2$, *then*

$$\Psi_{I,V}(x_1) \le \Psi_{I,V}(x_2). \tag{5.25}$$

If, in addition, the function V *is real analytic in* I, *then strict inequality holds in* (5.25). *Furthermore, the function* $\Psi_{I,V}(x)$ *attains its maximum in* $\overline{I^+}$ *at* a, *and only at* a.

As above, this theorem will follow from a new version of Theorem 2.1 for the relevant multiple integrals. Once again, we follow the arguments of the proof of Theorem 2.1 making the appropriate changes. As it turns out, for the one dimensional case one only needs symmetry on the potential. Again, for completeness, we present the argument. From now on we assume that $V(x) = V(-x)$ for all $x \in \mathbb{R}$. Set

$$\begin{aligned}
\Phi_1(x) &= \int_{-a}^{a} p_{t_1}(x_1 - x)\, \exp[-t_1\, V(x_1)\,]\, dx_1,\\
\Phi_1^+(x) &= \int_{0}^{a} p_{t_1}(x_1 - x)\, \exp[-t_1\, V(x_1)\,]\, dx_1,\\
\Phi_n(x) &= \int_{-a}^{a} p_{t_n}(x_1 - x)\ \exp[-t_n\, V(x_1)\,]\, \Phi_{n-1}(x_1)\, dx_1,\\
\Phi_n^+(x) &= \int_{0}^{a} p_{t_n}(x_1 - x)\ \exp[-t_n\, V(x_1)\,]\, \Phi_{n-1}^+(x_1)\, dx_1,
\end{aligned}$$

where

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

Lemma 5.1. *Suppose* $0 \le x_1 \le x_2 < a$. *Then for* $n \ge 1$

$$\Phi_n^+(x_2)\Phi_n(x_1) \ge \Phi_n^+(x_1)\Phi_n(x_2). \tag{5.26}$$

Proof. By Fubini's theorem and the symmetry of the interval $(-a, a)$, we see that (2.4) is equivalent to

$$\begin{aligned}
&\int_0^a \int_0^a \hat{p}_{t_n}(x_1 - u_1, x_2 - u_2)\ \exp[-\,t_n V(u_1) - t_n V(u_2)\,]\\
&\qquad\qquad\qquad\qquad \times \Phi_{n-1}^+(u_1)\Phi_{n-1}(u_2)\ du_1 du_2\\
&\le \int_0^a \int_0^a \hat{p}_{t_n}(x_2 - u_1, x_1 - u_2)\ \exp[-\,t_n V(u_1) - t_n V(u_2)\,]\\
&\qquad\qquad\qquad\qquad \times \Phi_{n-1}^+(u_1)\Phi_{n-1}(u_2)\ du_1 du_2,
\end{aligned}$$

where

$$\hat{p}_{t_n}(x - u, y - v) = p_{t_n}(x - u, y - v) + p_{t_n}(x - u, y + v).$$

Thus we must prove that

$$\begin{aligned}
&\int_0^a \int_{u_2}^a \Big\{ \hat{p}_{t_n}(x_1-u_1,x_2-u_2)\ \exp[-t_nV(u_1)-t_nV(u_2)] \\
&\qquad\qquad\qquad \times \Phi^+_{n-1}(u_1)\Phi_{n-1}(u_2) \\
&\quad + \hat{p}_{t_n}(x_1-u_2,x_2-u_1)\ \exp[-t_nV(u_1)-t_nV(u_2)] \\
&\qquad\qquad\qquad \times \Phi^+_{n-1}(u_2)\,\Phi_{n-1}(u_1) \Big\}\, du_2du_1 \\
\le &\int_0^a \int_{u_2}^a \Big\{ \hat{p}_{t_n}(x_2-u_1,x_1-u_2)\ \exp[-t_nV(u_1)-t_nV(u_2)] \\
&\qquad\qquad\qquad \times \Phi^+_{n-1}(u_1)\Phi_{n-1}(u_2) \\
&\quad + \hat{p}_{t_n}(x_2-u_2,x_1-u_1)\ \exp[-t_nV(u_1)-t_nV(u_2)] \\
&\qquad\qquad\qquad \times \Phi^+_{n-1}(u_2)\Phi_{n-1}(u_1) \Big\}\, du_2du_1.
\end{aligned}$$

As before, we proceed by induction. The case $n=1$ is exactly as the case $V=0$. Assuming $\Phi^+_{n-1}(u_1)\Phi_{n-1}(u_2) \ge \Phi^+_{n-1}(u_2)\Phi_{n-1}(u_1)$, it is enough to prove that

$$\hat{p}_{t_n}(x_1-u_1,x_2-u_2) \le \hat{p}_{t_n}(x_2-u_1,x_1-u_2), \tag{5.27}$$

and

$$\begin{aligned} &\hat{p}_{t_n}(x_1-u_1,x_2-u_2)+\hat{p}_{t_n}(x_1-u_2,x_2-u_1) \le \\ &\hat{p}_{t_n}(x_2-u_1,x_1-u_2)+\hat{p}_{t_n}(x_2-u_2,x_1-u_1), \end{aligned} \tag{5.28}$$

for all $0 \le x_1 \le x_2$ and $0 \le u_2 \le u_1$. Now by (2.10),

$$p_{t_n}(x_1-u_1,x_2-u_2) \le p_{t_n}(x_2-u_1,x_1-u_2).$$

On the other hand,

$$\begin{gathered} p_{t_n}(x_1-u_1,x_2+u_2) \le p_{t_n}(x_2-u_1,x_1+u_2) \Leftrightarrow \\ (x_2-u_1)^2+(x_1+u_2)^2 \le (x_1-u_1)^2+(x_2+u_2)^2 \Leftrightarrow \\ x_1(u_1+u_2) \le x_2(u_1+u_2), \end{gathered}$$

and (5.27) follows. Also, a simple computation shows that (5.28) is equivalent to

$$\exp(x_1u_1-x_2u_2)+\exp(x_1u_2-x_2u_1) \le \exp(x_2u_1-x_1u_2)+\exp(x_2u_2-x_1u_1),$$

and this follows from $x_1u_1-x_2u_2 \le x_2u_1-x_1u_2$ and $x_1u_2-x_2u_1 \le x_2u_2-x_1u_1$. This completes the proof of the lemma. □

It is an interesting problem to give conditions, of sufficiently general nature, that would yield an analogue of Theorem 5.4 for L_V in the two dimensional disk or the ball in $\mathbb{R}^n$, $n \ge 3$. Perhaps even more interesting, and challenging, would be to study the classical *hot-spots* conjecture of J. Rauch for the Neumann problem for L_V in the unit ball. As far as we know, there are no results in this direction outside of those for $V = 0$. We believe that studying the *hot-spots* problem for $-\Delta + V$ will shed new light on the *hot-spots* conjecture for the Laplacian as new techniques would have to be developed to deal with the potential.

Acknowledgment. We are very grateful to an anonymous referee for the careful reading of this paper and the many useful comments.

References

[1] G. Alessandrini, *Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains*, Comment. Math. Helv. **69** (1994), 142–154. MR 1259610 (95d:35111)

[2] R. Atar and K. Burdzy, *On Neumann eigenfunctions in lip domains*, J. Amer. Math. Soc. **17** (2004), 243–265 (electronic). MR 2051611 (2005b:35037)

[3] R. Bañuelos, *Lifetimes of conditioned Brownian motion and related estimates for heat kernels, eigenfunctions and eigenvalues in Euclidean domains*, MSRI Lectures, `http://www.math.purdue.edu/~banuelos/Papers/msrilec.pdf`.

[4] R. Bañuelos and K. Burdzy, *On the "hot spots" conjecture of J. Rauch*, J. Funct. Anal. **164** (1999), 1–33. MR 1694534 (2000m:35085)

[5] R. Bañuelos and M. Pang, *An inequality for potentials and the "hot-spots" conjecture*, Indiana Univ. Math. J. **53** (2004), 35–47. MR 2048182 (2005e:60172)

[6] ———, *Lower bound gradient estimates for solutions of Schrödinger equations and heat kernels*, Comm. Partial Differential Equations **24** (1999), 499–543. MR 1683048 (2000a:35031)

[7] R. Bañuelos, M. Pang, and M. Pascu, *Brownian motion with killing and reflection and the "hot-spots" problem*, Probab. Theory Related Fields **130** (2004), 56–68. MR 2092873 (2005j:60151)

[8] R. Bañuelos and P. J. Méndez-Hernández, *Sharp inequalities for heat kernels of Schrödinger operators and applications to spectral gaps*, J. Funct. Anal. **176** (2000), 368–399. MR 1784420 (2001f:35096)

[9] R. F. Bass and K. Burdzy, *Fiber Brownian motion and the "hot spots" problem*, Duke Math. J. **105** (2000), 25–58. MR 1788041 (2001g:60190)

[10] K. Burdzy, *The "hot–spots" problem in planar domains with one hole*, Duke Math. J. **129** (2005), 481–502. MR 2169871

[11] K. Burdzy and W. Werner, *A counterexample to the "hot spots" conjecture*, Ann. of Math. (2) **149** (1999), 309–317. MR 1680567 (2000b:35044)

[12] M. Cranston, *The influence of Kai Lai Chung on conditional Brownian motion and conditional gauge*, preprint.

[13] J. L. Doob, *Conditional Brownian motion and the boundary limits of harmonic functions*, Bull. Soc. Math. France **85** (1957), 431–458. MR 0109961 (22 #844)

[14] D. Jerison and N. Nadirashvili, *The "hot spots" conjecture for domains with two axes of symmetry*, J. Amer. Math. Soc. **13** (2000), 741–772 (electronic). MR 1775736 (2001f:35110)

[15] B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, Berlin, 1985. MR 810619 (87a:35001)

[16] A. D. Melas, *On the nodal line of the second eigenfunction of the Laplacian in $\mathbf{R}^2$*, J. Differential Geom. **35** (1992), 255–263. MR 1152231 (93g:35100)

[17] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, *The nodal line of the second eigenfunction of the Laplacian in $\mathbf{R}^2$ can be closed*, Duke Math. J. **90** (1997), 631–640. MR 1480548 (98m:35146)

[18] M. N. Pascu, *Scaling coupling of reflecting Brownian motions and the hot spots problem*, Trans. Amer. Math. Soc. **354** (2002), 4681–4702 (electronic). MR 1926894 (2003i:60141)

[19] L. E. Payne, *On two conjectures in the fixed membrane eigenvalue problem*, Z. Angew. Math. Phys. **24** (1973), 721–729. MR 0333487 (48 #11812)

[20] I. M. Singer, B. Wong, S.-T. Yau, and S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), 319–333. MR 829055 (87j:35280)

[21] R. G. Smits, *Spectral gaps and rates to equilibrium for diffusions in convex domains*, Michigan Math. J. **43** (1996), 141–157. MR 1381604 (97d:35037)

[22] D. You, *Sharp inequalities for ratios and partition functions of Schrödinger operators*, Potential Anal. **18** (2003), 219–250. MR 1953229 (2003m:35047)

Rodrigo Bañuelos, Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395, USA

E-mail address: `banuelos@math.purdue.edu`

Pedro J. Méndez-Hernández, Escuela de Matemática, Universidad de Costa Rica, San Pedro de Montes de Oca, San José, Costa Rica

E-mail address: `pmendez@cariari.ucr.ac.cr`