FOURIER-STIELTJES TRANSFORMS WITH SMALL SUPPORTS

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1. Let G be a locally compact abelian group and S a closed subset of the character group $G^{\wedge}$. If S is sufficiently "small", it is natural to expect that any finite complex measure μ on G with Fourier-Stieltjes transform $\hat{\mu}$ vanishing off S will be absolutely continuous. As the simplest case, one knows that

(1.1) if S has finite Haar measure, every μ with $\hat{\mu} = 0$ off S is absolutely continuous,

since $\hat{\mu}$ is then integrable [4]. Deeper examples are provided by[2] the F. and M. Riesz theorem (where $G^{\wedge}$ is the integer group Z and S the non-negative integers) and Bochner's generalization of that result (where $G^{\wedge} = Z^n$ and S is the positive orthant) [4]. In both these results S has the property that for all $\hat{g}$ in $G^{\wedge}$

(1.2) $S \cap (\hat{g} - S)$ has finite measure;

the purpose of the present note is to point out that (1.2) alone insures something suggesting absolute continuity, specifically that $\mu * \mu$ is then absolutely continuous for every measure μ with $\hat{\mu}$ vanishing off S.

(In case G is metric, even[3] $|\mu| * |\mu|$ is absolutely continuous. Since there are examples [5] of (non-negative) singular measures μ on the circle group with $\mu * \mu$ absolutely continuous, we are of course still quite far from concluding that (1.2) implies absolute continuity.)

Our proof is mainly measure-theoretic and depends basically on disintegration of measures [1], [2]; just about the only fact from harmonic analysis that is needed is (1.1) itself. Indeed the result comes from the observations that (1.2) says that certain sections of $S \times S$ (by cosets of the antidiagonal of $G^{\wedge} \times G^{\wedge}$) have finite measure, and that on each of these sections $(\mu \times \mu)^{\wedge}$ is the transform of a measure on G which is closely related to $|\mu| * |\mu^*|$—a fact which appears from a disintegration of $\mu \times \mu$.

Since all proofs of the F. and M. Riesz theorem and Bochner's theorem depend (in one way or another) on the fact that there S is a proper subsemigroup of $G^{\wedge}$, one might hope to obtain the full analogue of these results using such an hypothesis; as will be seen, our proof seems unsuited to producing such a result. However, the approach can be combined with the F.

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[2] In our references to these results below we always have in mind only that half which yields the absolute continuity of μ.

[3] $|\mu|$ denotes the usual absolute value (total variation) measure associated with μ.

and M. Riesz theorem for the real line to obtain some variants of Bochner's theorem (§3).

In what follows we shall frequently use μ for both a measure and the corresponding integral, with $\mu(f) = \int f\, d\mu$. For simplicity we shall take the Fourier-Stieltjes transform to be defined without the usual conjugation, so that $\hat{\mu}(\hat{g}) = \mu(\hat{g}) = \int \hat{g}\, d\mu$. We shall also frequently multiply a measure μ by a function $h : h\mu(f) = \int fh\, d\mu$.

The author is indebted to Karel de Leeuw for his introduction to disintegration and several other techniques used below.

2. Theorem. *Let S be a closed subset of $G^\wedge$ for which $S \cap (\hat{g} - S)$ has finite Haar measure for a dense set of $\hat{g}$ in $G^\wedge$. If μ and ν are finite complex measures on G with Fourier-Stieltjes transforms vanishing off S, then $\mu * \nu$ is absolutely continuous, and if G is metrizable, even $|\mu| * |\nu|$ is absolutely continuous.*

As an example, when $G^\wedge = Z$ we might take $S = \{(-1)^j n_j : j \geq 1\}$, where $\{n_j\}$ is a non-decreasing sequence of positive integers with $\lim\, (n_{j+1} - n_j) = \infty$. (Then any n in Z has only finitely many representations $n = n_i \pm n_j$, which implies $(n - S) \cap S$ is finite.)

We shall first assume G is metric and σ-compact (hence satisfies the second axiom of countability), and then indicate a reduction to this case.

Let Δ be the diagonal of $H = G \times G$, H_0 the closed subgroup $\{0\} \times G$ of H. Evidently

$$(g_1, g_2) = (g_1, g_1) + (0, g_2 - g_1)$$

shows

$$H = \Delta \oplus H_0, \tag{2.1}$$

where (as is easily seen) we have a topological direct sum of these closed subgroups of H. Thus

$$H^\wedge = H_0^\perp \oplus \Delta^\perp, \tag{2.2}$$

where $H_0^\perp$ (resp. $\Delta^\perp$) is the subgroup of $H^\wedge = G^\wedge \times G^\wedge$ orthogonal to H_0 (resp. Δ). Trivially $H_0^\perp = G^\wedge \times \{0\}$, while $\Delta^\perp = \{(\hat{g}, -\hat{g}) : \hat{g} \in G^\wedge\}$.

Let π denote the projection of H onto H_0 given by (2.1); we shall also denote the induced map of measures by π. Since H satisfies the second axiom of countability, we can disintegrate [1], [2] the measure $|\mu| \times |\nu|$ on $H = G \times G$ relative to the map π. That is, we have a map

$$h \to \lambda_h'$$

of H_0 into measures (of norm ≤ 1) on H, with each λ_h' carried by $\pi^{-1}(h)$, a map which is measurable in the sense that

(2.3) $h \to \lambda_h'(f)$ is measurable for each f in $C_0(H)$, with

$$(|\mu| \times |\nu|)(f) = \int_{H_0} \lambda_h'(f)\eta(dh), \qquad f \in C_0(H), \tag{2.4}$$

where $\eta = \pi(|\mu| \times |\nu|)$. Alternatively we may write

$$|\mu| \times |\nu| = \int_{H_0} \lambda_h' \, \eta(dh). \tag{2.4$'$}$$

Of course the usual monotonicity arguments show (2.3) and (2.4) continue to hold for f a bounded Baire (= Borel) function. In particular, if we write $\mu \times \nu = g(|\mu| \times |\nu|)$, where g is a unimodular Baire function, we have

$$\mu \times \nu(f) = |\mu| \times |\nu|(gf) = \int_{H_0} \lambda_h'(gf)\eta(dh)$$

$$= \int_{H_0} (g\lambda_h')(f)\eta(dh)$$

so that we may write

$$\mu \times \nu(f) = \int_{H_0} \lambda_h(f)\eta(dh) \tag{2.5}$$

for each bounded Baire f on H, where each λ_h is a complex measure of norm ≤ 1 carried by $\pi^{-1}(h)$.

Let us write $\delta^\perp$, $h^\perp$ for generic elements of $\Delta^\perp$, $H_0^\perp$. Since λ_h is carried by $\pi^{-1}(h) = h + \Delta$, on which $\delta^\perp$ is constant, (2.5) applied to the general character $\delta^\perp + h^\perp$ of $H^\wedge$ yields

$$(\mu \times \nu)^\wedge(\delta^\perp + h^\perp) = \int_{H_0} \lambda_h(h^\perp)\langle h, \delta^\perp\rangle\eta(dh). \tag{2.6}$$

But by (2.1), (2.2) we can identify $\Delta^\perp$ with $H_0^\wedge$, and so for a fixed $h^\perp$ the function

$$f : \delta^\perp \to (\mu \times \nu)^\wedge(\delta^\perp + h^\perp)$$

is precisely the Fourier-Stieltjes transform of the measure

$$\lambda_h(h^\perp)\eta(dh) \tag{2.7}$$

by (2.6).

Now f vanishes unless $\delta^\perp + h^\perp \in S \times S$, since

$$(\mu \times \nu)^\wedge(\hat{g}_1, \hat{g}_2) = \hat{\mu}(\hat{g}_1)\hat{\nu}(\hat{g}_2).$$

Writing $\delta^\perp = (\hat{g}, -\hat{g})$, $h^\perp = (\hat{g}_1, 0)$, $\delta^\perp + h^\perp \in S \times S$ amounts to $\hat{g} + \hat{g}_1 \in S$, $-\hat{g} \in S$, or $\hat{g} \in (-\hat{g}_1 + S) \cap (-S)$, and by hypothesis this last set has finite measure for $\hat{g}_1$ lying in a dense subset of $G^\wedge$. Let F be the corresponding (dense) set of elements $h^\perp = (\hat{g}_1, 0)$ in $H_0^\perp = G^\wedge \times \{0\}$. With our fixed $h^{\perp}$ now taken in F, the image of $(-\hat{g}_1 + S) \cap (-S)$ under the topological isomorphism $\hat{g} \to (\hat{g}, -\hat{g})$ of $G^{\wedge}$ onto $\Delta^{\perp}$ has finite Haar measure in $\Delta^{\perp}$, so that f vanishes off a set of finite Haar measure.

As we saw f coincides with the transform of (2.7) so, by (1.1), (2.7) is absolutely continuous and

$$\lambda_h(h^{\perp})\eta(dh) = \lambda_h(h^{\perp})\eta_a(dh) \tag{2.8}$$

where η_a is the absolutely continuous component of the measure η on H_0. Thus

$$\int \lambda_h(h^{\perp})\langle h, \delta^{\perp}\rangle\eta(dh) = \int \lambda_h(h^{\perp})\langle h, \delta^{\perp}\rangle\eta_a(dh)$$

for all $\delta^{\perp}$ in $\Delta^{\perp}$ and all $h^{\perp}$ in the dense subset F of $H_0^{\perp}$, or

$$(\mu \times \nu)^{\wedge} = \left(\int \lambda_h \, \eta_a(dh)\right)^{\wedge}$$

on the dense subset $\Delta^{\perp} + F$ of $H^{\wedge}$, hence everywhere. So

$$\mu \times \nu = \int \lambda_h \, \eta_a(dh). \tag{2.9}$$

But (2.9) implies $\eta = \eta_a$; for if η_s is the singular component of η we have

$$\| \eta_a \| + \| \eta_s \| = \| \eta \| = \|(| \mu | \times | \nu |)\| = \|(| \mu \times \nu |)\| = \| \mu \times \nu \|,$$

while $\| \lambda_h \| \leq 1$ implies

$$\| \mu \times \nu \| = \left\| \int \lambda_h \, \eta_a(dh) \right\| \leq \| \eta_a \|,$$

so $\| \eta_s \| = 0$, $\eta_s = 0$.

For any Baire set E in H_0 we have $\eta(E) = | \mu | \times | \nu |(E + \Delta)$; since $(g_1, g_2) \in E + \Delta$ is equivalent to $(0, g_2 - g_1) \in E$, we have

$$\eta(E) = \iint \varphi_E(g_2 - g_1) \, | \mu | \, (dg_1) \, | \nu | \, (dg_2)$$

where φ_E is the characteristic function of E, and thus, with

$$| \mu |^{\sim}(F) = | \mu |(-F),$$

we have

$$\eta = | \mu |^{\sim} * | \nu |.$$

Since $| \mu |^{\sim} = | \mu^* |$, where $\mu \to \mu^*$ is the usual involution, we conclude that $| \mu^* | * | \nu | = \eta$ is absolutely continuous. But μ^* is another measure with transform vanishing off S, since $(\mu^*)^{\wedge} = \hat{\mu}^{-}$, so that $| \mu^{**} | * | \nu | = | \mu | * | \nu |$ is absolutely continuous, as desired. (Of course this implies $\mu * \nu$ is absolutely continuous.)

Now if G is metric but not σ-compact we can find an open σ-compact subgroup G_1 of G carrying $|\mu|$ and $|\nu|$; applying disintegration to the measure $|\mu| \times |\nu|$ on $H_1 = G_1 \times G_1$ and the map $\pi \mid H_1$ then yields the decomposition

$$(2.10) \qquad |\mu| \times |\nu| (f) = \int_{\{0\}\times G_1} \lambda_h(f)\eta(dh) = \int_{H_0} \lambda_h(f)\eta(dh)$$

(since $(\pi \mid H_1)(|\mu| \times |\nu|) = \pi(|\mu| \times |\nu|) = \eta$ is carried by the subgroup $\{0\} \times G_1$ of $H_0 = \{0\} \times G$), valid for all bounded Baire f on H_1. Since the measures λ_h are now carried by H_1, (2.10) continues to hold for all bounded locally Baire f on H, and the remainder of the proof applies.

It remains to obtain the absolute continuity of $\mu * \nu$ when G is not metric. Suppose in that case that the singular component $(\mu * \nu)$. of $\mu * \nu$ does not vanish. We can find a Baire set E in G, of Haar measure zero, which carries $(\mu * \nu)_s$, and an open σ-compact subgroup G_0 of G containing E. Now [3, G, p. 287] there is a compact subgroup K of G_0 for which G_0/K (and so G/K) is metrizable while E is a union of cosets of K and (as can be seen from the proof of [3, E, p. 285]) has as its image in G_0/K a Baire subset of G_0/K.

Let ρ denote the canonical homomorphism of G onto G/K (and also the induced map of measures). Since ρE is a Baire subset of the open subgroup G_0/K of G/K it is a Baire subset of G/K, and if m denotes Haar measure of G_0/K, m_0 that of G_0, then evidently for some $c > 0$, $m(F) = cm_0(\rho^{-1}F)$ for all Baire $F \subset G_0/K$, so $m(\rho E) = cm_0(\rho^{-1}\rho E) = cm_0 E = 0$. Thus ρE has Haar measure zero in G_0/K, hence in G/K.

Since $(\mu * \nu)_s \neq 0$ there is some $\hat{g}_0$ in $G^\wedge$ for which

$$0 \neq (\mu * \nu)_s^\wedge(\hat{g}_0) = (\mu * \nu)_s(\hat{g}_0) = (\hat{g}_0(\mu * \nu)_s)^\wedge(0).$$

Let $\lambda = \hat{g}_0(\mu * \nu)$, so that $\lambda_s = (\hat{g}_0(\mu * \nu))_s = \hat{g}_0 \cdot (\mu * \nu)_s$ is a measure carried by E, and $\rho\lambda_s$ is a measure carried by ρE, hence a singular measure if it does not vanish; and $\rho\lambda_s$ cannot vanish since $(G/K)^\wedge = K^\perp$, $(\rho\lambda_s)^\wedge = \lambda_s^\wedge \mid K^\perp$, and $\hat{\lambda}_s(0) \neq 0$. Since the absolutely continuous component λ_a of λ has an absolutely continuous image under ρ, we conclude that $\rho\lambda = \rho\lambda_a + \rho\lambda_s$ is not absolutely continuous.

But $\lambda = \hat{g}_0 \cdot (\mu * \nu) = (\hat{g}_0\, \mu) * (\hat{g}_0\, \nu)$, so $\rho\lambda = \rho(\hat{g}_0\, \mu) * \rho(\hat{g}_0\, \nu)$. We thus have two measures $\mu' = \rho(\hat{g}_0\, \mu)$, $\nu' = \rho(\hat{g}_0\, \nu)$ on the metric group G/K for which $\mu' * \nu'$ is not absolutely continuous, while the transform $\mu'^\wedge = (\hat{g}_0\, \mu)^\wedge \mid K^\perp$ vanishes at $k^\perp \in K^\perp = (G/K)^\wedge$ unless $\hat{\mu}(\hat{g}_0 + k^\perp) \neq 0$, so certainly vanishes unless $\hat{g}_0 + k^\perp \in S$; of course the same applies to $\nu'^\wedge$. So the transforms of μ' and ν' vanish off

$$S_1 = K^\perp \cap (S - \hat{g}_0).$$

Since $K^\perp$ is open subgroup of $G^\wedge$ ($G^\wedge/K^\perp = K^\wedge$, and K is compact), Haar measure of $K^\perp$ is just the restriction of that of $G^\wedge$, and in order to see that

$$(2.11) \qquad (k^\perp - S_1) \cap S_1$$

has finite Haar measure in $K^\perp$ for a dense set of $k^\perp$ in $K^\perp$ we need only note that (2.11) is contained in

$$(k^\perp - S + \hat{g}_0) \cap (S - \hat{g}_0) \tag{2.12}$$

and show (2.12) has finite Haar measure in $G^\wedge$ for a dense set of $k^\perp$. The Haar measure of (2.12) is the same as that of

$$(k^\perp + 2\hat{g}_0 - S) \cap S,$$

which is finite whenever $k^\perp + 2\hat{g}_0$ lies in a dense subset F of $G^\wedge$, i.e., when $k^\perp$ lies in the dense subset $F - 2\hat{g}_0$ of $G^\wedge$, and since $K^\perp$ is open $K^\perp \cap (F - 2\hat{g}_0)$ is certainly dense in $K^\perp$.

Thus by the metric case, applied to G/K, μ', ν', and S_1 we conclude that $\mu' * \nu'$ is absolutely continuous, a contradiction which shows $(\mu * \nu)_s = 0$ and completes our proof.

Remark. When G is a compact connected metric group the measure $|\mu| * |\nu|$ of our theorem is equivalent to Haar measure (when $\mu, \nu \neq 0$).

For then (1.2) implies the transform f of (2.7) has finite support for any $h^\perp$, and choosing $h^\perp$ so that $f \neq 0$, we have (2.7) simply a multiple of Haar measure by a non-zero trigonometric polynomial p. Thus Haar measure is absolutely continuous with respect to η, since otherwise $\eta E = 0$ for a set E of positive Haar measure, so that $p^{-1}(0)$ has positive Haar measure, which is easily seen to be impossible.

(Indeed, when G is a torus T^k it is simple to see that a non-zero trigonometric polynomial p has $p^{-1}(0)$ of measure zero using Fubini's theorem and the fact that, as a function of a single coordinate, p vanishes identically or has finitely many zeroes. The general case easily reduces to this one, since if $\hat{g}_1, \cdots, \hat{g}_n$ are the characters involved in p, the map $\rho : g \to (\langle g, \hat{g}_1\rangle, \cdots \langle g, \hat{g}_n\rangle)$ has ρG a closed connected subgroup of T^n, hence another torus T^k, on which p appears as a non-zero tiigonometric polynomial p' $(p = p' \circ \rho)$, so that $p'^{-1}(0)$ has Haar measure zero. Since the Haar measure of $T^k = \rho G$ is the image of that of G, $p^{-1}(0)$ has Haar measure zero.)

3. Disintegration can also be used to yield some variants of Bochner's theorem, when combined with the F. and M. Riesz theorem for the line R.

Indeed let

$$\tau : G^\wedge \to R$$

be a non-constant representation, and let S be a closed subset of $G^\wedge$ contained in $\tau^{-1}(R_+)$, R_+ the non-negative reals, with

$$\tau^{-1}(r - R_+) \cap S \tag{3.1}$$

of finite Haar measure in $G^\wedge$ for all real r.

Then if G is metric any measure μ on G with Fourier-Stieltjes transform vanishing off S is absolutely continuous.[4]

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[4] When G is compact this follows from the Helson-Lowdenslager argument (see [4]);

We shall consider only the case in which G is also σ-compact; the reduction to that case proceeds as before. Let

$$\sigma : R \to G$$

be the map dual to τ. If we replace the diagonal in our earlier argument by the (closed) subgroup

$$\Delta = \{(r, \sigma(r)) : r \in R\}$$

(isomorphic to R) of $R \times G$, and set $H_0 = \{0\} \times G$, then

$$(r, g) = (r, \sigma(r)) + (0, g - \sigma(r))$$

leads easily to the (topological) direct sum decomposition

$$H = R \times G = \Delta \oplus H_0 \tag{3.2}$$

so

$$H^\wedge = R \times G^\wedge = \Delta^\perp \oplus H_0^\perp,$$

where $\Delta^\perp = \{(-\tau(\hat{g}), \hat{g}) : \hat{g} \in G^\wedge\}$, $H_0^\perp = R \times \{0\}$.

Let ν be a measure on R with $\hat{\nu} = 0$ off R_+, $\| \nu \| = 1$. Let π be the projection of H onto H_0 given by (3.2), and $\eta = \pi(| \nu | \times | \mu |)$. As before we obtain

$$| \nu | \times | \mu | = \int_{H_0} \lambda_h' \, \eta(dh),$$

$$\nu \times \mu = \int_{H_0} \lambda_h \, \eta(dh),$$

where λ_h, λ_h' are carried by $\pi^{-1}(h) = h + \Delta$, $\lambda_h = g\lambda_h'$, $| g | \equiv 1$. Writing $\delta^\perp + h^\perp$ for the generic element of $\Delta^\perp \oplus H_0^\perp = (R \times G)^\wedge$, we have

$$(\nu \times \mu)^\wedge(\delta^\perp + h^\perp) = \int \lambda_h(h^\perp)\langle h, \delta^\perp \rangle \eta(dh). \tag{3.3}$$

Now $h^\perp = (r, 0)$, $\delta^\perp = (-\tau(\hat{g}), \hat{g})$, and $(\nu \times \mu)^\wedge(r_1, \hat{g}_1) = \hat{\nu}(r_1) \cdot \hat{\mu}(\hat{g}_1)$ vanishes unless $(r_1, \hat{g}_1) \in R_+ \times S$, so (3.3) vanishes unless

$$\delta^\perp + h^\perp = (r - \tau(\hat{g}), \hat{g}) \in R_+ \times S,$$

i.e., unless $\hat{g} \in S$ and $\tau(\hat{g}) \in r - R_+$, or $\hat{g}$ lies in (3.1). Since (3.3), as a

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nevertheless our argument can yield variations which do not follow from that approach. For example, with $G^\wedge = Z^2$ and

$$S = \{(m, n) : m\sqrt{2} - n \geq 0, \text{ and } m\sqrt{2} - \log(1 + m) \geq n \text{ when } m \geq 0\}$$

we can obtain the same assertion. Here (3.1) fails (τ can only be taken as $(m, n) \to m\sqrt{2} - n$) but (3.1) always lies in a sector of opening $< \pi$, so Bochner's theorem can be used in place of (3.1) where that is used in the argument which follows.

function of $\delta^{\perp}$, is the transform of

$$\lambda_h(h^{\perp})\eta(dh)$$

we conclude exactly as before that η is an absolutely continuous measure on $H_0 \equiv \{0\} \times G$.

But now we can also conclude that η-almost all the measures λ_h are absolutely continuous with respect to m_h, Haar measure of $\Delta(\approx R)$ translated to the coset $\pi^{-1}(h) = h + \Delta$ of Δ. Once we have seen this our proof will be complete; for if F is of Haar measure zero in $\Delta \oplus H_0$, which we can view as the product space, then $F \cap (h + \Delta)$ is of measure zero m_h, (so $\lambda_h(F) = 0 = \lambda_h'(F)$), except for h in a set E of Haar measure zero in H_0, whence

$$|\nu| \times |\mu| (\varphi_F) = \int_{H_0 \setminus E} \lambda_h'(\varphi_F)\eta(dh) + \int_E \lambda_h'(\varphi_F)\eta(dh) = 0 + 0.$$

Thus $|\nu| \times |\mu|$ is absolutely continuous on $R \times G$, so that $|\mu|$, its projection on G, is absolutely continuous.

In order to obtain the desired absolute continuity of almost all λ_h mod η note that (3.1) is void for $r < 0$ since $S \subset \tau^{-1}(R_+)$. As we have seen, (3.3) vanishes unless, when we write $\delta^{\perp} = (-\tau(\hat{g}), \hat{g})$, we have $\hat{g}$ in (3.1); so certainly (3.3) vanishes when (3.1) is void; hence

$$(3.4) \qquad 0 = \int_{H_0} \lambda_h(h^{\perp})\langle h, \delta^{\perp}\rangle\eta(dh) \quad \text{when} \quad h^{\perp} = (r, 0), \qquad r < 0,$$

for all $\delta^{\perp}$ in $\Delta^{\perp}$.

Since $H = R \times G$ is metric and σ-compact, $C_0(H)$ is separable with a countable dense subset E; by Lusin's theorem, for $\varepsilon > 0$ we can find a closed set $K = K_\varepsilon$ in the closed carrier of η for which $\eta(H_0 \setminus K) < \varepsilon$ and on which

$$(3.5) \qquad h \rightarrow \lambda_h(f)$$

is continuous for all f in E, hence with (3.5) continuous on K for all f in $C_0(H)$ since the λ_h are bounded in norm. Evidently it is sufficient to show λ_h is absolutely continuous with respect to m_h for all h in $K = K_\varepsilon$, since $\eta(H_0 \setminus \bigcup K_{1/n}) = 0$.

By (3.2) we can identify $H_0^{\perp} = R \times \{0\}$ with $\Delta^{\wedge}$, and $\Delta^{\perp}$ with $H_0^{\wedge}$; in fact let $\Delta_-^{\wedge}$ denote $(-R_+) \times \{0\}$. It will be convenient notationally to rewrite (3.2) as a direct product decomposition,

$$H = \Delta \times H_0$$

with $H^{\wedge} = \Delta^{\wedge} \times H_0^{\wedge} = H_0^{\perp} \times \Delta^{\perp}$; thus for $f \in L_1(H_0^{\perp}) = L_1(\Delta^{\wedge})$, $\hat{f}(\delta) = \int \langle \delta, h^{\perp}\rangle f(h^{\perp})\, dh^{\perp}$ can be unambiguously interpreted as a function

$$(\delta, h) \rightarrow \hat{f}(\delta)$$

on $H = \Delta \times H_0$. If we take f to be supported by $\Delta_-^\wedge$ then by (3.4) we have

$$0 = \iint \lambda_h(h^\perp)\langle h, \delta^\perp\rangle f(h^\perp)\eta(dh)\, dh^\perp$$
$$= \int \left(\int \hat{\lambda}_h(h^\perp) f(h^\perp)\, dh^\perp\right) \langle h, \delta^\perp\rangle \eta(dh)$$
$$= \int \lambda_h(\hat{f})\langle h, \delta^\perp\rangle \eta(dh)$$

(since $\int \hat{\lambda}_h f\, dh^\perp = \int \hat{f}\, d\lambda_h$) for all $\delta^\perp$ in $\Delta^\perp$. For $F \in L_1(\Delta^\perp) = L_1(H_0^\wedge)$ we thus have

$$0 = \iint \lambda_h(\hat{f})\langle h, \delta^\perp\rangle F(\delta^\perp)\eta(dh)\, d\delta^\perp,$$

or

$$0 = \int \lambda_h(\hat{f})\hat{F}(h)\eta(dh). \tag{3.6}$$

Since $h \to \lambda_h(\hat{f})$ is a bounded Baire function and such $\hat{F}$ are dense in $C_0(H_0)$, hence in $L_1(\eta)$, (3.6) holds for $\hat{F}$ any $L_1(\eta)$ and in particular, given any element h_0 of K,

$$0 = \int \lambda_h(\hat{f}) \left(\frac{\varphi_{K\cap V}(h)}{\eta(K \cap V)}\right) \eta(dh), \tag{3.7}$$

where V is any compact neighborhood of h_0 ($\eta(K \cap V) > 0$ since K is contained in the closed carrier of η). Now $\hat{f}$ coincides with an element of $C_0(H)$ on[5] $\Delta \times V$; thus $h \to \lambda_h(\hat{f})$ is continuous on $K \cap V$, so that (3.7) implies

$$\lambda_{h_0}(\hat{f}) = 0$$

for any h_0 in K, or

$$\int \lambda_{h_0}(h^\perp) f(h^\perp)\, dh^\perp = 0$$

for any f in $L_1(H_0^\perp) = L_1(\Delta^\wedge)$ carried by its negative half $\Delta_-^\wedge$. So $\hat{\lambda}_{h_0}(h^\perp) = 0$ if $h^\perp \in \Delta_-^\wedge$; translating λ_{h_0} and m_{h_0} from the coset $h_0 + \Delta$ to Δ, which we can identify with the real line R, we can thus conclude from the F. and M. Riesz theorem that λ_{h_0} is absolutely continuous with respect to m_{h_0}, completing our proof.

Finally we note one further application of the same sort.

Suppose (for example) μ is a measure on R^2 for which $\hat{\mu}(x', y') = 0$ when $x' \le 0$. Suppose

$$\int |\hat{\mu}(x', y')|\, dy' < \infty \tag{3.8}$$

for a dense set of x'. Then μ is absolutely continuous.

[5] Note that $\Delta \times V$ carries $\{\lambda_h : h \in V\}$ since we have replaced our direct sum decomposition by the direct product.

Indeed, if we write

$$\mu = \int \lambda_y \, \eta(dy) \tag{3.9}$$

where x, y are our coordinates in R^2, then

$$\hat{\mu}(x', y') = \int \hat{\lambda}_y(x') \langle y, y' \rangle \eta(dy)$$

so that by (3.8), for a dense set of x',

$$\hat{\lambda}_y(x') \eta(dy)$$

is absolutely continuous since its transform $y' \to \hat{\mu}(x', y')$ is integrable. As before this shows η is absolutely continuous; and as in the argument just concluded we obtain the fact that $\hat{\lambda}_y(x') = 0$ for all $x' \leq 0$ since $\hat{\mu}$ vanishes on that half plane. So just as before, (3.9) must be absolutely continuous.

References

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