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SPLITTING THEOREMS FOR QUADRATIC RING EXTENSIONS

BY

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1. Introduction

Let R be a regular Noetherian ring (all rings are commutative, with identity) and let $S \supset R$ be a module-finite extension algebra. It is an open question whether $R \hookrightarrow S$ splits as a map of R-modules, i.e., whether the copy of R in S has an R-module complement E such that $S = R \oplus_R E$. This is known if R contains a field, and also if S_m has a big Cohen-Macaulay module for every maximal ideal m of S (see [2]). The question can be reduced to the case where S is a domain (see [2]).

We shall show here that when S is a domain such that the extension of fraction fields is quadratic the answer is affirmative: In fact, it suffices that R be supernormal and locally factorial, where "supernormal" means that the Serre conditions R_2 and S_3 hold (see [7, p. 124]). The main case is where R is of mixed characteristic 2.

Moreover, we give an interesting almost "generic" counterexample when the condition R_2 is weakened: In this example, the ring is a factorial *complete* local domain of mixed characteristic 2 which is a hypersurface. The most difficult feature of this example is to prove factoriality after completion: This is achieved by representing the hypersurface as a ring of invariants and calculating group cohomology (cf. [1], [2]).

It has recently been shown [6] that the direct summand conjecture has the same homological consequences (i.e., implies the same standard homological conjectures) as does the existence of big Cohen-Macaulay modules. This focuses increased attention on the direct summand conjecture. Further discussion of the conjectures may be found in [3], [4], [5], [6], [8], [9] and [11].

2. The Splitting Theorems

(2.1) THEOREM. *Let R be a locally factorial Noetherian domain which satisfies R_2 and S_3, e.g., a regular Noetherian domain, and let S be a*

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module-finite extension algebra such that the degree of the fraction field L of S over the fraction field K of R is two. Then $R \to S$ splits.

Proof. Let

$$S^{**} = \{f \in L: \text{height}\{r \in R: rf \in S\} \geqslant 2\}$$

where, for this purpose, height $R = +\infty$. S^{**}, as an R-module, is in fact the double dual of S into R, so that it is a module-finite R-algebra, and since $R \subset S \subset S^{**}$ it suffices to show that S^{**} can be retracted to R. Henceforth, we may assume that S is reflexive as an R-module (replacing S by S^{**}). We next observe:

(2.2) LEMMA. *Let R be a Noetherian domain which is R_2 and S_3 and let S be a R-reflexive module-finite extension algebra of R. Then S/R is a reflexive R-module.*

Proof. If $\dim R \leqslant 2$ then, passing to the case where R is local, we see that we may assume that R is a regular local ring of dimension less than or equal to 2. The fact that S is reflexive implies that S has depth $\min\{\dim R, 2\}$ and so is free over R. Moreover, if m is the maximal ideal of R, $1 \notin mS$, which means that 1 is part of a minimal and, hence, free basis for S over R, so that S/R is R-free.

If $\dim R \geqslant 3$ we may assume that R is local and it suffices to prove that every R-sequence of length 2 is an (S/R)-sequence. Let x, y be an R-sequence of length 2. Let an overbar denote reduction modulo R in S. If $x\bar{s} = 0$, $xs \in R$, whence the integral element s is in the fraction field of R. Since R is normal, $s \in R$, i.e., $\bar{s} = 0$.

Now suppose $y\bar{t} = x\bar{s}$. We must show that $\bar{t} \in x(S/R)$. We know that $yt - xs = r \in R$. We claim that $r \in (x, y)R$. For if $r \notin (x, y)R$ then since R is S_3 all associated primes of (x, y) have height 2, and we will still have $r \notin (x, y)R_P$ after localizing at a suitable prime P among these. But then R_P has dimension 2 and so R_P is a direct summand of S_P and $(x, y)R_P$ is contracted from $(x, y)S_P$. Since $r = yt - xs \in (x, y)S \subset (x, y)S_p$, this is a contradiction.

Thus, we can write $r = ya - xb$ for suitable $a, b \in R$, and we then have $yt - xs = r = ya - xb$ and so $y(t - a) = x(s - b)$ in S. Hence, $t - a = xs'$ (since S is reflexive) and $\bar{t} - \overline{t - a} = x\overline{s'}$, as required. This completes the proof of Lemma 2.2.

We can now complete the proof of Theorem (2.1) easily. Since we have reduced to the case where S is reflexive the lemma implies that S/R is reflexive. Since the field extension is quadratic, S has torsion-free rank two over R and so S/R has torsion-free rank one. Since R is locally factorial and factoriality is equivalent to the freeness of rank one reflexives (for a normal Noetherian domain), we have that S/R is a rank one projective, whence $0 \to R \to S \to S/R \to 0$ splits, Q.E.D.

We obtain the following rather odd corollary:

(2.3) PROPOSITION. *Let R be a locally factorial R_2, S_3 Noetherian domain and suppose $w^2 \in (4, x^2)R$, where $x \in R$. Then $w \in (2, x)R$.*

Proof. If char $R = 2$ this is immediate from the normality of R: the case $x = 0$ is trivial, while if $x \neq 0$, $(w/x)^2 \in R$ implies $w/x \in R$. Assume char $R \neq 2$ and $w^2 = 4u + x^2v$, $u, v \in R$. Let $\sqrt{v}$ denote some square root of v in an extension domain of R. Then the elements $(w \pm x\sqrt{v})/2$ are in the fraction field of $R[\sqrt{v}]$ and are integral over R since their sum is w and their product is $(w^2 - x^2v)/4 = u$. By Theorem (2.1), there is an R-linear retraction

$$f\colon R[\sqrt{v}, (w + x\sqrt{v})/2] \to R,$$

and

$$\begin{aligned} w &= f(w) \\ &= f(w + x\sqrt{v}) - f(x\sqrt{v}) \\ &= 2f((w + x\sqrt{v})/2) - xf(v) \in (2, x)R, \end{aligned}$$

Q.E.D.

Of course, what we really used about R here is that it is a direct summand of every quadratic integral extension.

The conclusion of Proposition (2.3) does not seem obvious even when R is regular (of mixed characteristic 2) in the ramified case.

3. A Counterexample

Our objective here is to show that the condition R_2 in Theorem (2.1) cannot be relaxed: even if the local ring is complete and a hypersurface.

Let A be a regular Noetherian factorial domain in which $2A$ is a nonzero proper prime ideal (e.g. A might be $\mathbf{Z}$, $\mathbf{Z}_{(2)}$, or the completion of $\mathbf{Z}_{(2)}$, the 2-adic numbers). Let $S = A[X, W, U, V]$, and let $R = S/FS$, where X, W, U, V are indeterminates and $F = W^2 - 4U - X^2V$. Let x, w, u, v be the images of X, W, U, V in R. We note the following facts:

(1) R is a hypersurface (hence R is Gorenstein and, in particular, Cohen-Macaulay, which implies S_3).

(2) R is factorial. To see this, note that 2 is a prime element of R, for $R/2R \cong (A/2A)[X, W, U, V]/(W^2 - X^2V)$. Hence, localizing at the element 2 does not affect factoriality. But

$$R[1/2] \cong A[1/2][X, W, V],$$

since $F = 0$ may be solved for U when $1/2$ is in the ring.

(3) By construction, $w^2 \in (2, x^2)R$. But $w \notin (2, x)R$. In fact

$$R/(2, x) \cong (A/2A)(W, U, V,]/(W^2).$$

(4) Hence, R admits a quadratic extension domain of which R is not a direct summand, by Proposition (2.3).

This example is also cited in [10].

We now want to modify the example so that R is a *complete* local domain. We henceforth assume that $A = \Delta$, a complete discrete valuation ring in which $2 \neq 0$ generates the maximal ideal (e.g., Δ might be the 2-adic integers).

Let $\hat{S} = \Delta[[X, W, U, V]]$ and $\hat{R} = \hat{S}/F$, where $F = W^2 - 4U - X^2V$, as before. Thus, $\hat{R}$ is the m-adic completion of R in the case $A = \Delta$, with $m = (2, x, w, u, v)$. Remarks (1), (3) and (4) above remain essentially unchanged (replacing "[ ]" by "[[ ]]") but the proof of factoriality (2) is no longer valid, because $R[1/2]$ is smaller than $\Delta[1/2][[X, W, V]]$ (localization on Δ does not commute with adjunction of power series indeterminates). Nonetheless:

(3.1) THEOREM. *$\hat{R}$ is a complete local factorial hypersurface which admits a quadratic extension domain of which $\hat{R}$ is not a direct summand.*

The proof, by the remarks above, reduces to showing that $\hat{R}$ is factorial. We conclude with a demonstration of this fact.

The key point is that $\hat{R}$ may be viewed as the ring of invariants of an action of a cyclic group G of order 2 (with generator, say, σ) acting on a formal power series ring $T = \Delta[[x, y, z]]$: there is a unique continuous action such that $\sigma(x) = x$, $\sigma(y) = -y$ and $\sigma(z) = z + xy$. It is clear that x, $v = y^2$, $z + \sigma(z) = 2z + xy = w$ and $z\sigma(z) = z(z + xy) = u$ are fixed by G. Map $\Delta[[X, W, U, V]]$ continuously into T over Δ by sending X, W, U, V to x, w, u, v. Since $w^2 = 4u + x^2v$ in T, F is killed and we obtain a continuous Δ-homomorphism $\hat{R} \to T^G \hookrightarrow T$. Denote the image of $\hat{R}$ by $\Delta[[x, w, u, v]]$. Then T is integral over Im $\hat{R}$, the degree of the extension of fraction fields is two, and the same is true for T^G and T. It follows that T^G is contained in the fraction field of Im $\hat{R}$ and integral over it. Krull dim $T = 4$ implies Krull dim (Im $\hat{R}$) $= 4$. Since $\hat{R}$ is itself a four-dimensional normal domain (for $R = \Delta[X, W, U, V]/F$ is a normal excellent domain), the surjection of $\hat{R} \twoheadrightarrow$ Im $\hat{R}$ is an isomorphism. Thus, Im $\hat{R}$ is normal and Im $\hat{R} = T^G$.

The map $\hat{R} \to T$ therefore permits us to identify $\hat{R}$ with T^G, and it will suffice to show that T^G is factorial.

For any commutative ring with identity C let C^* denote the multiplicative group of units in C. Then $T^* = \Delta^*. (1 + I)$, where $I = (x, y, z)T$, and T^* is in fact the direct sum (or product) of Δ^* and $1 + I$.

Let $r \in T^G$ be a nonzero nonunit. rT factors uniquely, in T, into prime principal ideals, say

$$rT = \prod_{j=1}^{k} (s_jT)^{m_j}$$

where the s_jT are distinct. Since G stabilizes rT, G permutes $\{s_1T, \ldots, s_kT\}$ and this set breaks up into G-orbits. If s_iT and s_jT are in the same orbit, $m_i = m_j$. If there are h G-orbits and I_λ denotes the product of the ideals in the λth orbit, then

$$rT = I_1 \cdots I_k$$

is the unique (except for order) factorization of rT into G-stable principal ideals which cannot be so factored further. If it were the case that each I_λ is generated by an invariant we would be done: these invariants would give the factorization of r in T^G (up to an invariant unit). The situation, however, is not quite this simple.

Let $I_\lambda = t_\lambda T$. Then we shall show:

(3.2) *For all but evenly many, say 2ν, values of λ, t_λ may be chosen to be G-invariant, while the remaining 2ν factors are all associates of y in T.*

It then follows easily that if

$$L = \{\lambda\colon 1 \leqslant \lambda \leqslant h,\, t_\lambda T \neq yT\},$$

then r has the unique factorization (in T^G) $r = \alpha(y^2)^\nu \Pi_{\lambda \in L}\, t_\lambda$, where α is a unit of T^G. (Note: a unit of T which is in T^G is evidently a unit of T^G.)

In order to prove (3.2), let $t \in T$ be a nonzero nonunit which generates a G-stable ideal. Thus, if $G = \{1, \sigma\}$, $\sigma(t) = \alpha_\sigma t$, where α_σ is a unit of T, and $\sigma(\alpha_\sigma t) = \sigma(\alpha_\sigma)\alpha_\sigma t = T$, i.e., $\sigma(\alpha_\sigma) = \alpha_\sigma^{-1}$. Under these circumstances we shall prove that one of two facts holds:

(1) tT is of the form ryT, where $r \in T^G$. (Then $\sigma(ry) = -ry$.)
(2) tT is of the form rT, where $r \in T^G$.

In fact, the element $\alpha_\sigma \in T^*$ represents an element of $H^1(G, T^*)$. As remarked earlier,

$$T^* = \Delta^* \times (1 + I), \quad \text{where } I = (x, y, z)T.$$

Thus, $H^1(G, T^*) \cong H^1(G, \Delta^*) \times H^1(G, 1 + I)$. *We shall show in the next section that* $H^1(G, 1 + I) = 0$ (see Theorem (4.3)). Let us assume this for the moment. Then the only elements of $H^1(G, \Delta^*)$, since G acts trivially on Δ, are given by the α_σ such that $(\alpha_\sigma)^2 = 1$, i.e., $\alpha_\sigma = \pm 1$. Thus $H^1(G, T^*) = \{\pm 1\}$, and this says that given α_σ we can find $\beta_\sigma \in T^*$ such that $\alpha_\sigma = \pm\sigma(\beta_\sigma)^{-1}$. If we replace t by $\beta_\sigma t = t_1$ then $\sigma(t_1) = t_1$. If the sign is $+$, we are in Case (2). If the sign is $-$ we shall show that $t_1 = ry$, where r is invariant. In fact, it suffices to show that $t_1 \in yT$, for if $t_1 =$

yr, $r \in T$, then $\sigma(t_1) = -y$, and $\sigma(y) = -y$ imply $\sigma(r) = r$. But yT is a G-stable ideal of T and $T/yT \cong \Delta[[x, z]]$ is a *trivial* G-module ($\sigma(x) = x$, $\sigma(z) = z + xy \equiv z$ modulo yT), whence the image $\bar{t}_1$ of t_1 modulo yT is both fixed by and negated by σ. Thus, $\bar{t}_1 = 0$, and y divides t_1.

We return now to the situation where $t = t_\lambda$ is one of the generators of G-stable ideals I_λ in the factorization of rT. We have shown that each t_λ is, up to a unit, either an invariant r or of the form yr, where r is an invariant. In the second case, r must be a unit of T (and hence of T^G), for I cannot be factored further in T.

As before, let $L = \{\lambda\colon 1 \leq \lambda \leq h,\ t_\lambda T \neq yT\}$, and let μ be the number of λ not in L. Assume $t \in T^G$ for $\lambda \in L$. Then

$$r = \alpha y^{\mu} \prod_{\lambda \in L} t_\lambda,$$

where α is a unit of T. If μ were odd, we would have $\sigma(\alpha) = -\alpha$ which implies $y \mid \alpha$ in T, a contradiction. Hence, μ is even, say $\mu = 2\nu$, and $r = \alpha(y^2)^{\nu} \prod_{I_\lambda \neq yT} t_\lambda$. $\alpha \in T^G$ (since $y^2 \in T^G$) and then α must be a unit of T^G. The factoriality of T^G is now clear: it remains only to prove that $H^1(G, 1 + I) = 0$, which we shall accomplish in Section 4 (Theorem (4.3)).

4. Vanishing of Group Cohomology

Throughout this section, G is a multiplicative group of order 2 with generator σ. When G acts on a domain Λ we shall always mean that G acts by ring automorphisms. If $\lambda \in \Lambda$, $N(\lambda)$, the norm of λ, is $\lambda\sigma(\lambda)$. If V is a G-stable subgroup of Λ^*, $H^1(G, V)$ may be identified with

$$\{v \in V\colon N(v) = 1\}/\{v\sigma(v)^{-1}\colon v \in V\}$$

(4.1) Lemma. *Let Λ be a domain, I an ideal, and suppose G acts on Λ so that I is G-stable. Also, suppose that*

$$W = \{w \in \Lambda\colon w \equiv 1 \bmod I\}$$

is a subgroup of Λ^*. Then if $\lambda \in 2I$ and $1 + \lambda$ has norm 1, then there exists $w \in W$ such that $1 + \lambda = w^{-1}\sigma(w)$.

Proof. If $2 = 0$ this is clear, so suppose $2 \neq 0$. Then

$$(1 + \lambda)\sigma(1 + \lambda) = 1$$

implies

$$\lambda + \sigma(\lambda) + \lambda\sigma(\lambda) = 0 \quad \text{or} \quad 2 + \lambda = 2 + 2\lambda + \sigma(\lambda) + \lambda\sigma(\lambda),$$

i.e., $2 + \lambda = (2 + \sigma(\lambda))(1 + \lambda)$. But $\lambda = 2\mu$, $2 \neq 0$, whence

$$(1 + \mu) = (1 + \sigma(\mu))(1 + \lambda),$$

and we may choose $w^{-1} = 1 + \mu$, Q.E.D.

(4.2) LEMMA. *Let Δ be a domain such that 2Δ is a prime ideal. Let $\Lambda = \Delta[[s, t]]$, where s, t are formal power series indeterminates, and let G act continuously, fixing Δ, so that $\sigma(s) = -s$, $\sigma(t) = t$. Let $J = (s, t)\Lambda$ and $W = 1 + J \subset \Lambda^*$. Then $H^1(G, W) = 0$.*

Proof. Suppose $\lambda \in J$ and $N(1 + \lambda) = 1$, i.e.,

$$\lambda + \sigma(\lambda) + \lambda\sigma(\lambda) = 0.$$

Write $\lambda = \sum_{i=0}^{\infty} \lambda_i s^i$, where $\lambda_i = \lambda_i(t) \in \Delta[[t]]$. Then we have

$$\sum_{i=0}^{\infty} \lambda_i s^i + \sum_{i=0}^{\infty} \lambda_i(-s)^i + \sum_{i,j} \lambda_i\lambda_j s^i(-s^j) = 0$$

whence $2\lambda_0 + \lambda_0^2 = 0$. Since $2 \in J$ implies $2 = 0$, we must have $\lambda_0 = 0$.
At degree (in s) $2k > 0$ we get

$$2\lambda_{2k} + \sum_{i+j=2k} (-1)^j \lambda_i\lambda_j = 0$$

whence $\lambda_k^2 \in 2\Delta[[t]]$, a prime ideal of $\Delta[[t]]$. Thus, for all k, $\lambda_k \in 2\Delta[[t]]$, so that $\lambda \in 2J$, and $1 + \lambda$ is 0 in $H^1(G, W)$, by Lemma (4.1), Q.E.D.
We are now ready to prove the main result of this section.

(4.3) THEOREM. *Let Δ be a domain in which 2Δ is a prime ideal. Let $T = \Delta[[x, y, z]]$ and $I = (x, y, z)T$. Let $V = 1 + I$, a subgroup of T^*. Let $G = \{1, \sigma\}$ act on T so that σ is the unique continuous (in the I-adic topology) Δ-automorphism of T such that*

$$\sigma(x) = x,\ \sigma(y) = -y \quad \text{and} \quad \sigma(z) = z + xy.$$

Then $H^1(G, V) = 0$.

Proof. Let $U = 1 + xT \subset 1 + I = V$. We have a surjection

$$\pi\colon T \twoheadrightarrow \Delta[[s, t]] = \Lambda$$

by $\pi(f(x, y, z)) = f(0, s, t)$. Let G act on Λ as in Lemma (4.2) and let $W = 1 + (s, t)\Lambda$ as in Lemma (4.2). Then we have an exact sequence of G-modules

$$0 \to U \hookrightarrow V \overset{\pi}{\twoheadrightarrow} W \to 0.$$

Suppose we can show:

(*) if $u \in U$ and $u\sigma(u) = 1$, then there is a $v \in V$ such that $u = \sigma(v)v^{-1}$.

Then it will follow that $H^1(G, V) = 0$, for (*) simply says that in the piece

$$H^1(G, U) \overset{\alpha}{\to} H^1(G, V) \to H^1(G, W)$$

of the long exact sequence, the map α is 0, while we already know from Lemma (4.2) that $H^1(G, W) = 0$.

Before proving (*), we note that if $\theta \in I$ and $N(1 + \theta) = 1$ (i.e., $\theta + \sigma(\theta) + \theta\sigma(\theta) = 0$) then $\theta \in yT$. To see this, let $\Gamma = \Delta[[x]]$ and write $\theta = \Sigma_{i=0}^{\infty} \theta_i(z)y^i$, where $\theta_i(z) \in \Gamma[[z]]$. Then

$$\sum \theta_i(z)y^i + \sum \theta_i(z + yx)(-y)^i + \sum \theta_i(z)\theta_j(z + yx)y^i(-y)^j = 0,$$

and substituting $y = 0$ yields

$$2\theta_0(z) + \theta_0(z)^2 = 0 \Rightarrow \theta_0(z) = 0$$

($\theta_0(z) = -2 \Rightarrow 2 \in I \Rightarrow 2 = 0 \Rightarrow \theta_0(z) = 0$, whence $\theta_0(z) = 0$ in all cases). Thus, $\theta \in yT$, as claimed.

Now suppose $u \in U$ and $N(u) = 1$. Thus, $u = 1 + \theta$, where $\theta \in xT$. Now, by the above remarks, $\theta \in yT \Rightarrow \theta \in xT \cap yT = xy\,T$, so that $\theta = yf$, where $f \in xT$. Since $N(1 + yf) = 1$, we have

$$yf - y\sigma(f) - y^2 f\sigma(f) = 0$$

or, equivalently,

$$(\dagger) \qquad f - \sigma(f) = yf\sigma(f).$$

To complete the proof it suffices to construct by recursion on $i \geqslant 1$, a sequence of elements $a_1, a_2, \ldots, a_i, \ldots \in \Sigma_{j+k=i}\, \Gamma\, y^j z^k, \ldots$ such that if $a = \Sigma_{i=1}^{\infty} a_i$, then

$$(1 + a)(1 + yf) = 1 + \sigma(a)$$

or, equivalently,

$$(\#) \qquad (1 + a)fy = \sigma(a) - a,$$

for then $u = 1 + yf = (1 + a)^{-1}\sigma(1 + a)$ and $1 + a \in 1 + (y, z)T \subset V$.

We can write, uniquely,

$$f = \sum_{i=0}^{\infty} f_i \quad \text{where} \quad f_i \in \sum_{j+k=i} \Gamma\, y^j z^k = T_i.$$

Note that each T_i is G-stable.

Since $f \in xT$, $f_i \in xT_i$, all i. Let $f_i = xf_i^*$. We choose $a_1 = f_0^* z$. Let $[t]_i$ denote the T_i-component of an element $t \in T$. Then

$$[(1 + a_1)fy]_1 = [\sigma(a_1) - a_1]_1.$$

In fact

$$[(1 + a_1)fy]_1 = [fy]_1 = f_0 y = f_0^* xy$$

while

$$[\sigma(a_1) - a_1]_1 = \sigma(a_1) - a_1 = \sigma(f_0^* z) - f_0^* z = f_0^*(z + xy) - f_0^* z = f_0^* xy.$$

Now suppose $n > 1$ and we have constructed $a_1, \ldots, a_{n-1}$, $a_i \in T_i$, such that if $A = a_1 + \cdots + a_{n-1}$, then

$$[(1 + A)fy]_d = [\sigma(a_d) - a_d]_d = \sigma(a_d) - a_d,\ 1 \leqslant d \leqslant n - 1.$$

Let $H = (1 + A)f$. Then $[H]_{d-1} \in T^G$, $1 \leqslant d \leqslant n - 1$, for $[H]_{d-1}y = [Hy]_d = \sigma(a_d) - a_d$ implies $\sigma([H]_{d-1}y) = -[H]_{d-1}y$ which implies $\sigma([H]_{d-1}) = [H]_{d-1}$.

We claim that $[H]_{n-1} \in T^G$ as well. To see this, note that

$$\begin{aligned} H - \sigma(H) &= (1 + A)f - (1 + \sigma(A))\sigma(f) \\ &= f - \sigma(f) + (A - \sigma(A))f + \sigma(A)(f - \sigma(f)) \\ &= (1 + \sigma(A))(f - \sigma(f)) + (A - \sigma(A))f \\ &= (1 + \sigma(A))f\sigma(f)y + (A - \sigma(A))f \quad (\text{by } \dagger) \\ &= f\sigma(B) \quad \text{where } B = -(1 + A)fy + \sigma(A) - A. \end{aligned}$$

Thus,

$$\begin{aligned} H_{n-1} - \sigma(H_{n-1}) &= [H - \sigma(H)]_{n-1} \\ &= [f\sigma(B)]_{n-1} \\ &= f_0\,\sigma(B)_{n-1} + f_1\sigma(B)_{n-2} + \cdots + f_{n-2}\,\sigma(B)_1 \end{aligned}$$

(for $B_0 = 0$). But our induction hypothesis was precisely that $B_d = 0$, $1 \leqslant d \leqslant n - 1$, and $\sigma(B)_i = \sigma(B_i)$. Thus, $H_{n-1} = \sigma(H_{n-1})$. Moreover, since $f \in xT$, $H \in xT$, and $H_{n-1} \in xT_{n-1}$, say $H_{n-1} = xg_{n-1}$. We also have then that $\sigma(g_{n-1}) = g_{n-1}$. Now let $a_n = g_{n-1}z \in T_n$.

Then

$$\begin{aligned} [(1 + a_1 + \cdots + a_n)fy]_n &= [(1 + A + a_n)fy]_n \\ &= [(1 + A)fy]_n + [a_n fy]_n \\ &= [(1 + A)fy]_n \\ &= [(1 + A)f]_{n-1}y \\ &= H_{n-1}y \\ &= g_{n-1}\,xy \\ &= g_{n-1}(z + xy) - g_{n-1}z \\ &= \sigma(a_n) - a_n, \end{aligned}$$

since $\sigma(g_{n-1}) = g_{n-1}$. Now, letting $a = \Sigma_{i=1}^{\infty} a_i$, we clearly have

$$(1 + a)fy = \sigma(a) - a,$$

since this holds for each graded component, Q.E.D.

Theorem (4.3) more than suffices to complete the proof of Theorem (3.1).

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