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ON GRAPHS WITH EDGE-TRANSITIVE AUTOMORPHISM GROUPS

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In [4], Goldschmidt considered groups G with finite subgroups M_1 and M_2 and the following three properties:

(i) $G = \langle M_1, M_2 \rangle$.

(ii) No non-trivial normal subgroup of G is contained in $M_1 \cap M_2$.

(iii) $|M_i/M_1 \cap M_2| = 3$ for $i = 1,2$.

He was able to give the exact structure (the isomorphism classes) of all possible pairs of subgroups M_1 and M_2. In his proof he used a graph theoretical approach:

Any group G with properties (i) and (ii) operates as an edge-transitive group of automorphisms on a graph Γ whose vertex set is

$$\{M_1x / x \in G\} \dot{\cup} \{M_2x \mid x \in G\}$$

and where two vertices are adjacent iff they have non-empty intersection. G operates on Γ by right multiplication, the vertex-stabilizers in G are conjugate to M_1 or M_2, and the edge-stabilizers are conjugate to $M_1 \cap M_2$ (see [4, (2.6)]).

Since G is a homomorphic image of the amalgamated product of M_1 and M_2 with respect to $M_1 \cap M_2$, one can study this amalgamated product and the corresponding graph Γ. Serre [9] has shown in this case that Γ is a tree. Hence the above problem leads to the investigation of edge-transitive groups of automorphisms of the trivalent tree with finite vertex-stabilizers.

We use this method to investigate a more general situation. We make the following hypotheses.

Hypothesis A. Let G be a group and M_1 and M_2 be finite subgroups of G such that:

(1) $G = \langle M_1, M_2 \rangle$.

(2) No non-trivial normal subgroup of G is contained in $M_1 \cap M_2$.

(3) $|M_i/M_1 \cap M_2| = 2^{n_i} + 1$, $n_i \geq 1$, $i = 1,2$ and $\max\{n_1, n_2\} > 1$.

(4) There exists a normal subgroup N_i in M_i such that

$$N_i/R \simeq L_2(2^{n_i})' \quad \text{for } R = \cap_{x \in M_i} (M_i \cap M_j^x) \text{ and } \{i,j\} = \{1,2\}.$$

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Hypothesis B. Let Γ be a connected graph and G be an edge-transitive group of automorphisms of Γ such that for $\alpha \in \Gamma$:

(a) G_α is finite.

(b) $|\Delta(\alpha)| = 2^{n_\alpha} + 1$, $n_\alpha \geq 1$ and $\max\{n_\alpha, n_\beta\} > 1$ for $\beta \in \Delta(\alpha)$.

(c) There exists a normal subgroup N_α in G_α such that $N_\alpha^{\Delta(\alpha)} \simeq L_2(2^{n_\alpha})'$.

Here G_α denotes the stabilizer of α in G, $\Delta(\alpha)$ the set of vertices adjacent to α, and $N_\alpha^{\Delta(\alpha)}$ the permutation group on $\Delta(\alpha)$ induced by N_α. Any graph in this paper is undirected and without loops and multiple edges.

The condition $\max\{n_1, n_2\} > 1$ (resp. $\max\{n_\alpha, n_\beta\} > 1$) only excludes cases treated in [4], and condition (b) and (c) imply that N_α is transitive on $\Delta(\alpha)$.

Let q, q_1 and q_2 be powers of 2, and let $\mathrm{Aut}(L_2(q_1)) \int \mathrm{Aut}(L_2(q_2))$ be the wreath product of $\mathrm{Aut}(L_2(q_1))$ with $\mathrm{Aut}(L_2(q_2))$ with respect to the natural permutation representation of $L_2(q_2)$. We define:

$\mathscr{L} = \{L_2(q_1) \times L_2(q_2),\ \mathrm{Aut}(L_2(q_1)) \int \mathrm{Aut}(L_2(q_2)),\ \max\{q_1, q_2\} > 1;\ L_3(q), Sp_4(q), G_2(q), q > 2;\ U_4(q), {}^3D_4(q), J_2\}$.

Let X be a group in $\mathscr{L}$. If X is not the wreath product, then X contains exactly two conjugacy clases of maximal 2-local subgroups which contain Sylow 2-subgroups of X. Let X_1 and X_2 be representatives for these two classes in X. If X is the wreath product, then there exist exactly two classes of 2-local subgroups which contain Sylow 2-subgroups of X and fulfil (3) and (4) of Hypothesis A. In this case let X_1 and X_2 be representatives for these classes.

Definition. A pair of groups $\{M_1, M_2\}$ is parabolic of type X for $X \in \mathscr{L}$, if for $i = 1,2$,

(*) X is not the wreath product, and M_i is isomorphic to a subgroup of $N_{Aut(X)}(X_i)$ which contains X_i, or

(**) X is the wreath product, and M_i is isomorphic to a subgoup of X_i which contains $X_i \cap L_2(q_1)' \int L_2(q_2)'$.

A pair of groups $<M_1, M_2>$ is parabolic of type J, if for $i = 1,2$ there exists a normal subgroup X_i in M_i such that:

(i) $|M_i/X_i| \leq 2$.

(ii) $X_1/O_2(X_1) \simeq L_2(4)$, $O_2(X_1) \simeq Q_8 * D_8$ and $C_{M_1}(O_2(X_1)) \leq O_2(X_1)$.

(iii) $X_2 = BO_2(X_2), B \simeq C_3 \times \Sigma_3$, $O_2(X_2)$ is special of order 2^6, and the 3-elements in $O^{2'}(X_2)$ operate fixed point freely on $O_2(X_2)$.

Note that all groups in $\mathscr{L}$ fulfil Hypothesis A with respect to X_1 and X_2. But these are not all the known examples.

The simple group J_3 has (up to notation and conjugation) two pairs of subgroups M_1 and M_2 for which Hypothesis A holds, in one case they are parabloic of type J_2, in the other case parabolic of type $L_3(4)$.

But as the following theorems show, the examples in $\mathscr{L}$ give the pattern for all possible examples.

THEOREM 1. *Assume Hypothesis A. Then one of the following holds (possibly after interchanging* 1 *and* 2*):*

(a) $M_i \simeq H \leq \mathrm{Aut}(L_2(2^{n_1}))$, $i = 1,2$.

(b) $\{M_1, M_2\}$ *is parabolic of type* X *for some* X *in* $\mathscr{L}$.

(c) $\{M_1, M_2\}$ *is parabolic of type* J.

(d) $n_1 > 1$, $O_2(M_1)$ *is elmentary abelian,* $M_1/O_2(M_1) \simeq H \leq \mathrm{Aut}(L_2(2^{n_1}))$, *and* $O_2(M_1)$ *is isomorphic to a submodule of the natural permutation* $GF(2)$*-module for* $M_1/O_2(M_1)$; $n_2 = 1$, $M_2 = N_{M_1}(S)W$ *for* $S \in Syl_2(M_1 \cap M_2)$ *and a normal subgroup* W *of* M_2 *which is isomorphic to* Σ_3.

As a special case we get from Theorem 1 and [3]:

COROLLARY 1. *Assume Hypothesis A, and suppose that G is finite and that*

$$M_i = N_G(O_2(M_i)) \quad \textit{for } i = 1,2.$$

Then $\{M_1, M_2\}$ *is parabolic of type* X *for some* $X \in \mathscr{L}$, *or* $G = M_jO(G)$ *for some* $j \in \{1,2\}$.

A graph Γ is locally (G, s)-transitive with respect to a group G of automorphisms of Γ, if for every $\alpha \in \Gamma$, G_α is transitive on the arcs of length k starting at α for $k \leq s$ and s is maximal with this property.

THEOREM 2. *Assume Hypothesis B. Then* Γ *is locally* (G, s)*-transitive, and one of the following holds for* $\Lambda = \{G_\alpha, G_\beta\}$:

(a) $s = 2$, *and* $G_\delta \simeq H \leq \mathrm{Aut}(L_2(2^{n_\alpha}))$ *for* $\delta = \alpha, \beta$.

(b) $s = 3$, *and* Λ *is parabolic of type* $L_2(2^{n_\alpha}) \times L_2(2^{n_\alpha})$.

(c) $s = 3$, *and* Λ *is parabolic of type* $\mathrm{Aut}(L_2(2^{n_\alpha})) \int \mathrm{Aut}(L_2(2^{n_\beta}))$.

(d) *(possibly after interchanging* α *and* β*)* $s = 3$, $n_\beta = 1$, $O_2(G_\alpha)$ *is elementary abelian,* $G_\alpha/O_2(G_\alpha) \simeq H \leq \mathrm{Aut}(L_2(2^{n_\alpha}))$, *and* $O_2(G_\alpha)$ *is isomorphic to a submodule of the natural permutation* $G(2)$*-module for* $G_\alpha/O_2(G_\alpha)$; $G_\beta = N_{G_\alpha}(S)W$ *for* $S \in Syl_2(G_{\alpha\beta})$ *and a normal subgroup* W *of* G_β *isomorphic to* Σ_3.

(e) $s = 4$, *and* Λ *is parabolic of type* $L_3(2^{n_\alpha})$.

(f) $s = 5$, *and* Λ *is parabolic of type* $U_4(2^{n\alpha})$, $Sp_4(2^{n\alpha})$, *or* J.

(g) $s = 7$, *and* Λ *is parabolic of type* $G_2(2^{n\alpha})$, *or* ${}^3D_4(2^{n\alpha})$.

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1. Group theoretical results

Hypothesis I. Let G be a finite group such that

(a) $C_G(O_2(G)) \leq O_2(G)$ *and*

(b) $G/O_2(G) \simeq L_2(2^n)$, $n \geq 1$.

DEFINITION. Let V be a faithful $GF(2)$-module for $L_2(2^n)$ and T be a Sylow 2-subgroup of $L_2(2^n)$.

V is a natural module for $L_2(2^n)$ iff $|C_V(T)|^2 = |V| = 2^{2n}$.

V is an orthogonal module for $L_2(2^n)$ iff $|C_V(T)|^4 = |V| = 2^{2n}$.

Note that this definition is compatible with the usual definition of a natural (resp. orthogonal) $L_2(2^n)$ $GF(2)$-module. If $X \simeq L_2(2^n)$ and V is a natural (resp. orthogonal) $L_2(2^n)$-module for X, we simply write V is a natural (orthogonal) module for X.

We assume Hypothesis I for the lemmata (1.1)–(1.7).

(1.1) *Let* $O_2(G)$ *be elementary abelian of order* 2^{2n}. *Then* $O_2(G)$ *is a natural or orthogonal module for* $G/O_2(G)$, *and* $O_2(G)$ *is a natural module, if and only if all elements in* $O_2(G)^{\#}$ *are conjugate in* G.

Proof. See [1, 4.3].

(1.2) $|O_2(G)| \geq 2^{2n}$.

Proof. See [2, Hilfssatz].

(1.3) *Let* T *be a Sylow* 2-*subgroup of* G, *and suppose that* $O_2(G)$ *is elementary abelian,* $Z(G) = 1$ *and*

(i) $[G, O_2(G)] = O_2(G)$, or

(ii) $O_2(G) = \langle C_{O_2(G)}(T)^G \rangle$.

Then the following statements are equivalent:

(a) $O_2(G)$ is direct sum of natural modules for $G/O_2(G)$.

(b) $[O_2(G), T, T] = 1$.

(c) $|C_{O_2(G)}(T)|^2 = |O_2(G)|$.

(d) All non-trivial elements of odd order in G operate fixed-point-freely on $O_2(G)$.

Proof. Note that $G = <T, t>$ for any element $t \in G \setminus N_G(T)$ (see (3.1)); thus

$$C_{O_2(G)}(T) \cap C_{O_2(G)}(t) = 1 \quad \text{and} \quad |C_{O_2(G)}(T)|^2 \leq |O_2(G)|.$$

Set $V = [G, O_2(G)]$. It follows from [5, Theorem 8.2] that the three statements are equivalent for V in place of $O_2(G)$. If $V \neq O_2(G)$, then

$$O_2(G) = VC_{O_2(G)}(T),$$

and from $|C_V(T)|^2 = |V|$, we get $|C_{O_2(G)}(T)|^2 > |O_2(G)|$ and $Z(G) \neq 1$, a contradiction.

(1.4) *Suppose that an element of order three in G operates fixed-point-freely on $O_2(G)$. Then $O_2(G)$ is elementary abelian and direct sum of natural modules for $G/O_2(G)$, or $n = 1$.*

Proof. See [5, Theorem 8.2].

(1.5) *Let $Z(G)$ be elementary abelian and $O_2(G)/Z(G)$ be a natural module for $G/O_2(G)$. Then $O_2(G)$ is elementary abelian, or $n = 1$.*

Proof. We may assume that $Z(G)$ has order 2. If $Z(G)$ contains all involutions of $O_2(G)$, then $O_2(G) \simeq Q_8$ and $n = 1$.

If $Z(G)$ does not contain all involutions of $O_2(G)$, then by (1.1) all elements in $xZ(G)$ for $x \in O_2(G) \setminus Z(G)$ are involutions. But this implies that all elements in $O_2(G)^{\#}$ are involutions, and $O_2(G)$ is elementary abelian.

(1.6) [2]. *Let T be a Sylow 2-subgroup of G, and suppose that no non-trivial characteristic subgroup of T is normal in G. Then the following hold:*

(a) *T has class 2.*

(b) *$Z(O_2(G))/Z(G)$ is a natural module, and $[G, O_2(G)] \leq Z(O_2(G))$.*

(c) *There exists $\alpha \in \mathrm{Aut}(T)$ such that $T = Z(O_2(G))^{\alpha} O_2(G)$.*

(1.7) *Assume the hypothesis of* (1.6). *Then*

$$< Z(O_2(G))^{\alpha} \ / \ \alpha \in \mathrm{Aut}(T),\ o(\alpha)\ odd >$$

is a normal subgroup of G.

Proof. Define $Q = O_2(G)$, $Z = Z(Q)$ and $\Delta = \{Z^{\alpha} / \alpha \in \mathrm{Aut}(T),$ $Z^{\alpha} \leq Q\}$, and let β be an automorphism of T of odd order. From (1.6) we get

$$[< \Delta >, G] \leq Z \leq < \Delta >.$$

So it suffices to show $Z^\beta \in \Delta$.

Assume $Z^\beta \notin \Delta$. Let γ be any automorphism of T such that $Z^\gamma \not\leq Q$. Then $Z^{\gamma^{-1}} \not\leq Q$, and $|Z/C_Z(Z^\gamma)| = |Z^\gamma/C_{Z^\gamma}(Z)| = 2^n$, since $Z/Z(G)$ is a natural module for $G/O_2(G)$ by (1.6). In particular we have $Z^\gamma Q = T$ and $|Q/C_Q(Z^\gamma)| = |Z/C_Z(Z^\gamma)|$.

Let d be a p-element in $G \setminus N_G(T)$, p an odd prime. Then d is fixed-point-free on $Z/Z(G)$ (see (1.3)(d)) and $G = \langle Z^\beta, Z^{\beta d}\rangle Q$. Set

$$Q_0 = C_Q(Z^\beta) \cap C_Q(Z^{\beta d}).$$

Then $Q = Q_0 Z$ and $Q_0 \cap Z = Z(G)$, in particular Q_0 is normal in G. Therefore we have $[Z^\beta, T] = [Z^\beta, Z] = [Z, T]^\beta = [Z, Z^\beta]^\beta$, which implies

$$(*) \qquad [Z^\beta, Z]^\beta = [Z^\beta, Z].$$

From $(*)$ we get $[Z^{\beta^2}, Z^\beta] \neq 1$. Assume that $[Z^{\beta^2}, Z] \neq 1$. Then $T = Z^{\beta^2} Q$ and

$$Z^{\beta^2} \not\subseteq Z \cup Q_0 Z^\beta,$$

but in T/Q_0 the only maximal elementary abelian subgroups are the images of Z and Z^β.

So we have $Z^{\beta^2} \in \Delta$. Since β has odd order, we may assume that $\Delta^{\beta^2} \neq \Delta$. Pick $B \in \Delta^{\beta^2} \setminus \Delta$, then $T = BQ$ and

$$[Z^{\beta^2}, BQ_0 Z] = [Z^{\beta^2}, Q_0] \leq Q_0 \cap Z = Z(G).$$

On the other hand $(*)$ implies $[Z^{\beta^2}, T] = [Z^{\beta^2}, Z^\beta] = [Z^\beta, Z] \not\leq Z(G)$. This contradiction shows the assertion.

Hypothesis II. Let G be a group and M_1 and M_2 finite subgroups of G such that for $i = 1,2$:

(a) $O^{2'}(M_i/O_2(M_i)) \simeq L_2(2^{n_i})$, $n_i \geq 1$.

(b) $M_1 \cap M_2 = N_{M_1}(S) = N_{M_2}(S)$ for $S \in Syl_2(M_1 \cap M_2)$.

(c) No non-trivial normal subgroup of $O^{2'}(M_i)$ is normal in $O^{2'}(M_j)$, $j \neq i$.

We assume Hypothesis II for the lemmata (1.8)–(1.11).

Notation. $Q_i = O_2(M_i)$, $Z_i = Z(Q_i)$, $L_i = O^{2'}(M_i)$, $\overline{L_i} = L_i/Q_i$, $S \in Syl_2(M_1 \cap M_2)$, K_i is a complement for S in $N_{L_i}(S)$. In addition we choose K_1 and K_2 such that $K = K_1K_2$ is a subgroup of odd order.

(1.8) (a) $J(S) \not\leq Q_1 \cap Q_2$.

(b) $S = Q_1Q_2$, *or* $Q_1 = Q_2 = 1$.

Proof. Part (a) is obvious. The structure of $L_2(2^n)$ (see (3.1)) implies that $\overline{K_i}$ is transitive on $\overline{S}^{\#}$ $(i = 1,2)$. This yields (b).

(1.9) *Suppose that* $C_{L_1}(Q_1) \not\leq Q_1$. *Then* $O^2(L_1) \simeq L_2(2^{n_1})'$, *and one of the following holds:*

(a) $O^2(L_2) \simeq L_2(2^{n_2})'$, *S is elementary abelian, and* $|S| = 2^{n_1}$ or $2^{n_1+n_2}$.

(b) $n_1 = 1$, *and* Q_2 *is elementary abelian and non-central in* $O^2(L_2)Q_2$.

Proof. If $Q_1 = 1$ or $Q_2 = 1$, then S has order 2^{n_1}, and S is elementary abelian, since Sylow 2-subgroups of $L_2(2^n)$ are elementary abelian. Thus we may assume $Q_1 \neq 1 \neq Q_2$.

Suppose first that $O_2(O^2(L_1)) \neq 1$. Then from [6, V 25.7] we get

$$S \cap O^2(L_1) \simeq Q_8 \quad \text{and} \quad \Omega_1(Z_2) \leq Q_1.$$

Hence $\Omega_1(Z_2)$ is normal in M_1 and M_2 and therefore $\Omega_1(Z_2) = 1$, but this contradicts $Q_2 \neq 1$.

Assume now $O^2(L_1) \simeq L_2(2^{n_1})'$. Then $\phi(Q_2) \leq Q_1$, and $\phi(Q_2)$ is normal in L_1 and L_2. This implies $\phi(Q_2) = 1$.

Assume $n_1 > 1$. Then $K_1 \neq 1$ and $C_S(K_1) = Q_1$. From (1.8)(b) we get $[S, K_1] \leq Q_2$, and the structure of $\mathrm{Aut}(L_2(2^n))$ implies $[L_2, K_1] \leq Q_2$. Hence $C_{Z_2}(K_1)$ is normal in L_1 and L_2 and must be trivial. But then

$$Z_2 \cap Z(S) \cap Q_1 = 1,$$

and $Q_1 = 1$ or $Z(S) \not\leq Q_2$. The first case contradicts the assumption. In the second case we get as above $O^2(L_2) \simeq L_2(2^{n_2})'$ and $[Q_2, O^2(L_2)] = 1$. Thus $Q_1 \cap Q_2$ is normal in L_1 and L_2 and must be trivial. This proves assertion (a).

Now assume $n_1 = 1$. Then (b) holds, or Q_2 is central in $O^2(L_2)Q_2$, and with the above argument (a) holds.

(1.10) *Suppose that* M_1 *and* M_2 *are conjugate in* G. *Then one of the following holds for* $i = 1,2$:

(a) $O^2(L_i) \simeq L_2(2^{n_1})'$, *and S is elementary abelian of order* 2^{2n_1} *or* 2^{n_1}.

(b) Q_i *is elementary abelian of order* 2^{2n_1} or 2^{3n_1}, *and* $Q_i/Z(L_i)$ *is a natural module for* $\overline{L_i}$.

Proof. Pick $g \in G$ such that $M_1^g = M_2$. Then $\langle S, S^g \rangle \leq M_2$ and $S = S^{gm}$ for some $m \in M_2$, since S is a Sylow 2-subgroup of M_2. Hence we may choose $g \in N_G(S)$.

If $C_{L_i}(Q_i) \not\leq Q_i$ for $i \in \{1,2\}$, then (1.9) yields assertion (a). Thus we assume $C_{L_i}(Q_i) \leq Q_i$ and can apply (1.6).

Set $\{i,j\} = \{1,2\}$. If $Z_i \leq Q_j$, then $[Z_iZ_j, L_i] \leq Z_i$ and $[Z_iZ_j, L_j] \leq Z_j$, and Z_iZ_j is normal in L_1 and L_2, a contradiction. Hence $Z_i \not\leq Q_j$, and the operation of K on S (see (3.1)) yields

$$S = Z_iQ_j, Q_j = C_{Q_j}(Z_i)Z_j \quad \text{and} \quad |Q_j/C_{Q_j}(Z_i)| = |Z_j/Z(S)| = 2^{n_j}.$$

Let d be an element of odd order in $L_j \setminus N_{L_j}(S)$ and

$$Q_0 = C_{Q_j}(Z_i) \cap C_{Q_j}(Z_i^d).$$

Then

$$L_j = \langle Z_i, Z_i^d \rangle Q_j, \quad Q_j = Q_0 Z_j \quad \text{and} \quad Q_0 \cap Z_j = Z(L_j).$$

In particular $L_j = C_{L_j}(Q_0)Q_0$, and $Z_j/Z(L_j)$ is a natural module for $\overline{L_j}$.

Now set $j = 1$ and $i = 2$. Assume that $[Q_0^g, Z_1] \neq 1$. Then

$$[Z_2, Z_1] = [Q_0^g, Z_1] \leq Z_1 \cap Q_0^g = Z(S) \cap Q_0^g = Z_2 \cap Q_0^g = Z(L_2).$$

This contradicts the operation of Z_1 on $Z_2/Z(L_2)$.

We have shown that $Q_0^g \leq C_{Q_1}(Z_2)$. Since $Q_0 \cap Q_0^g$ is normal in L_1 and L_2, we get $Q_0 \cap Q_0^g = 1$, and the operation of K_1 yields $C_{Q_1}(Z_2) = Q_0 Q_0^g$ or $Q_0 = 1$. In particular $|Q_0| = 1$ or 2^{n_1}, and Q_0 is elementary abelian. This implies assertion (b).

(1.11) *Suppose that $C_{L_i}(Q_i) \leq Q_i$ for $i = 1,2$. Then one of the following holds:*

(a) $J(S) \nsubseteq Q_1 \cup Q_2$, $Z(J(S)) = Z(S)$, $Z(L_i) \neq 1$, *and $Z_i/Z(L_i)$ is a natural module for L_i ($i = 1,2$).*

(b) $Z_1 = Z(L_1)$.

(c) $Z_2 = Z(L_2)$.

(d) *S has class 2, and $Z_i/Z(L_i)$ is a natural module for $\overline{L_i}$ ($i = 1,2$). Moreover, if $Z(L_1) = 1$ or $Z(L_2) = 1$, then $Q_i = Z_i$, and Q_i is a natural module for $\overline{L_i}$ ($i = 1,2$).*

Proof. Assume $Z_1 \neq Z(L_1)$ and $Z_2 \neq Z(L_2)$. If the hypothesis of (1.6) holds in M_1, we get (d) for $i = 1$ and $Z(S) = Z(J(S))$. This shows $J(S) \not\leq Q_2$ and (d) for $i = 2$, too.

Thus we may assume additionally that M_1 and M_2 do not fulfil the hypothesis of (1.6) and that (without loss) $J(S) \not\leq Q_1$. We apply the techniques in [2]. Define $B = C_s(Z(J(S)))$ and $\tilde{L}_1 = \langle B^{L_1} \rangle$. Then Baumann's argument [2, (6)] shows that $Z(J(S)) = XZ(S)$, where X is a normal subgroup of $\tilde{L}_1$. This yields $B = C_S(X)$ and $B \in Syl_2(\tilde{L}_1)$.

If $J(S) \leq Q_2$, then $C_{L_2}(Z(J(S))) = B$ is normal in L_2, and no non-trivial characteristic subgroup of B is normal in L_1. Now (1.7) applied to $\tilde{L}_1$ and $L_2 = N_{L_2}(B)$ yields a contradiction.

So we may assume $J(S) \not\leq Q_2$. As above $B \in Syl_2(\langle B^{L_2} \rangle)$, and [2, (6)] implies that $[S, Z(J(S))]$ is normal in L_1 and L_2. Hence we get $Z(J(S)) = Z(S)$.

An application of Baumann's techniques in [2, (1), (10)] yields assertion (a).

For the next two lemmata suppose that $X = L_2(2^m)$. Let V be a natural $GF(2^m)$-module for X, and denote by V^σ the conjugate of V by $\sigma \in Gal(GF(2^m))$. If $\sigma \neq 1$, then V and V^σ are non-isomorphic $GF(2^m)$-modules.

For $S \leq X$ and an X-module W we define

$$[W,S] = [W,S,1] \quad \text{and} \quad [W,S,n] = [[W,S,n-1],S]$$

for $n \geq 2$.

(1.12) *Let W be a non-trivial irreducible $GF(2^m)$-module for X. Then there exist $n \in \mathbf{N}$ and $\sigma_1, \ldots, \sigma_n \in Gal(GF(2^m))$ such that $W = \otimes_{i=1}^{n} V^{\sigma_i}$, where $V^{\sigma_1}, \ldots, V^{\sigma_n}$ are pairwise non-isomorphic $GF(2^m)$-modules. Moreover, the following two statements for $S \in Syl_2(X)$ are equivalent:*

(a) $W = \otimes_{i=1}^{n} V^{\sigma_i}$.

(b) $[W,S,n] \neq 0$ *and* $[W,S,n+1] = 0$.

Proof. The first part of the assertion follows from [5, Theorem 8.2]. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ be a basis of V^{σ_i} $(1 \leq i \leq n)$ and

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ q_j & 1 \end{pmatrix} / 1 \leq j \leq 2^m, \{q_1, \ldots, q_{2^m}\} = GF(2^m) \right\}.$$

Set

$$d_j = \begin{pmatrix} 1 & 0 \\ q_j & 1 \end{pmatrix}.$$

Then d_j operates on V^{σ_i} in the following way:

$$e_1 d_j = e_1 \quad \text{and} \quad e_2 d_j = e_2 + q_j^{\sigma_i} e_1.$$

If $n = 1$, then W is a natural module, and (a) and (b) are equivalent. Hence we may assume $n > 1$.

Define $W_1 = \otimes_{i=1}^{n-1} V^{\sigma_i}$ and $w = w_1 \otimes e_2$ for $w_1 \in W_1$. Then

$$[wd_j, d_k] = [w, d_k]d_j$$

and

$$[w, d_j] = w_1 \otimes e_2 + (w_1 \otimes e_2)d_j = [w_1, d_j] \otimes e_2 + q_j^{\sigma_n}(w_1 \otimes e_1)d_j.$$

Hence

$$(*) \qquad [w, d_1, \ldots, d_r] = [w_1, d_1, \ldots, d_r] \otimes e_2 + \sum_{i=1}^{n} q_i^{\sigma_n}([w_1, d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_r] \otimes e_1)d_i.$$

Applying induction on n we get, from $(*)$,

$$[w, d_1, \ldots, d_{n+1}] = 0 \quad \text{and} \quad [W,S,n+1] = 0.$$

It remains to show that $[W,S,n] \neq 0$. Let $\tilde{W}$ be the natural permutation $GF(2^m)$-module for X. Then X operates on a basis $\{a_0, \ldots, a_{2^m}\}$ of $\tilde{W}$, and

$$W_S = \tilde{W} / < \sum_{i=0}^{2^m} a_i >$$

is an irreducible $GF(2^m)$-module, the Steinberg-module. Hence $W_S = \otimes_{i=1}^{m} V^{\sigma_i}$.

We first argue that $[W_S, S, m] \neq 0$. For this purpose we choose generators $d_1, \ldots, d_m$ for S and assume $a_0 S = a_0$. Then the operation of S on $\{a_1, \ldots, a_{2^m}\}$ yields

(**) $a_i \prod_{i \in \Lambda} d_i \neq a_i$ for any $a_i \neq a_0$ and $\Lambda \subseteq \{1, \ldots, m\}$, $\Lambda \neq \emptyset$.

Define $\Gamma_0 = \{a_1\}$ and $\Gamma_i = \Gamma_{i-1} \cup \{b_{i-1} d_i \,/\, b_{i-1} \in \Gamma_{i-1}\}$ for $i = 1, \ldots, m$. Then from (**) we get $\Gamma_{i-1} \cap \{b_{i-1} d_i \,/\, b_{i-1} \in \Gamma_{i-1}\} = \emptyset$. Hence

$$[a_1, d_1, \ldots, d_j] = \sum_{b_k \in \Gamma_j} b_k \quad \text{for } j \leq m \,;$$

in particular

$$[a_1, d_1, \ldots, d_m] = \sum_{i=1}^{2^m} a_i \notin < \sum_{i=0}^{2^m} a_i >$$

and $[W_S, S, m] \neq 0$.

Now let W be a counterexample to $[W, S, n] \neq 0$ such that n is maximal. We have just proved $n < m$. Hence there exists $\sigma \in Gal(\mathrm{GF}(2^m)) \setminus \{\sigma_1, \ldots, \sigma_n\}$, and $W \otimes V^\sigma$ is not a counterexample. Pick

$$\hat{w} = w \otimes v \in W \otimes V^\sigma, \quad w \in W \text{ and } v \in V^\sigma,$$

such that $[\hat{w}, d_1, \ldots, d_{n+1}] \neq 0$. Then

$$v = k_1 e_1 + k_2 e_2 \quad (k_1, k_2 \in GF(2^m)),$$

and $[W, S, n+1] = 0$ and (*) imply

$$0 \neq [\hat{w}, d_1, \ldots, d_{n+1}] = k_2 \sum_{i=1}^{n+1} q_i^{\sigma_{n+1}}([w, d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n+1}] \otimes e_1) d_i.$$

But this is only possible, if

$$[w, d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n+1}] \neq 0 \quad \text{for some } i \in \{1, \ldots, n+1\},$$

which shows that W is not a counterexample.

(1.13) *Let S be a Sylow 2-subgroup of X and W be an irreducible $GF(2)$-module for X. Suppose that*

(a) $[W, S, 4] = 0$, *and*

(b) $|W| = 2^{2m+2r}$, $0 < r < m$.

Then $m = 3r$ and $[W, S, 3] \neq 0$.

Proof. Set $\tilde{W} = W \otimes GF(2^m)$. Then (a) holds for $\tilde{W}$ and $\dim \tilde{W} = 2(m+r)$. On the other hand $\tilde{W} = \otimes_{i=1}^{n} \tilde{W}^{\sigma_i}$, where $\sigma_1, \ldots, \sigma_n \in$

$\mathrm{Gal}(GF(2^m))$, $m = na$ ($a \in \mathbf{N}$), and $\hat{W}$ is an irreducible $GF(2^m)$-module (see [7, (30.11)]). Now (1.12) implies $\dim \hat{W} = 2^k$, $k \leq 3$; hence $2^{k-1}m/a = m + r$. This yields $k = 3$ and $a = 3$.

2. Graph theoretical results

(2.0) *Hypothesis.* Let Γ be a graph and G be a group of automorphism of Γ.

Notation. The notation differs only slightly from that in [4].

We write $\alpha \in \Gamma$, if α is a vertex of Γ, and $\gamma \subseteq \Gamma$, if γ is a set or ordered tuple of vertices.

For $\alpha \in \Gamma$ and $\gamma \subseteq \Gamma$ G_α is the stabilizer of α in G and G_γ is the pointwise stabilizer of γ in G. $\Delta(\alpha)$ is the set of vertices adjacent to α. An arc of length n is an ordered $(n+1)$-tuple of vertices $(\alpha_0, \ldots, \alpha_n)$, where $n > 0$, $\alpha_i \in \Delta(\alpha_{i+1})$ for $0 \leq i \leq n-1$ and $\alpha_i \neq \alpha_j$ for $i \neq j$ and $(i,j) \neq (0,n)$.

A line is an ordered set $\{\alpha_i \ / \ i \in \mathbf{Z}\}$ of vertices such that $\alpha_i \in \Delta(\alpha_{i+1})$ for $i \in \mathbf{Z}$ and $\alpha_i < \alpha_j$ iff $i < j$; here again $\alpha_i \neq \alpha_j$ for $i \neq j$.

For an arc $\gamma = (\alpha_0, \ldots, \alpha_n)$ we define

$$\Delta_L(\gamma) = \Delta(\alpha_0) \setminus \{\alpha_1\} \quad \text{and} \quad \Delta_R(\gamma) = \Delta(\alpha_n) \setminus \{\alpha_{n-1}\}.$$

γ is left (resp. right) singular, if G_γ is not transitive on $\Delta_L(\gamma)$ (resp. $\Delta_R(\gamma)$); otherwise it is left (resp. right) regular, and γ is regular, if γ is left and right regular. Let X be a set of vertices. By (X, n) (resp. (n, X)) we denote the set of arcs of length n whose left (resp. right) endpoint is in X. If $\alpha \in \Gamma$ is in the same G-orbit as α', we say that α is conjugate to α' (under G).

(2.1) [4, 2.3]. *Suppose that Γ is connected, G_α is transitive on $\Delta(\alpha)$ and G_β is transitive on $\Delta(\beta)$ for some pair of adjacent vertices α, β. Then G is edge-transitive on Γ.*

(2.2) *Suppose that Γ is a tree. Then Γ is a bipartite graph.*

The proof is obvious.

(2.3) [4, 2.6]. *Suppose that Γ is a tree, α_1 and α_2 are adjacent vertices, P_i is a subgroup of G fixing α_i $(i = 1,2)$ and*

$$(P_1)_{\alpha_2} = (P_2)_{\alpha_2} = P_1 \cap P_2.$$

Then $\langle P_1, P_2 \rangle_{\alpha_1} = P_i$ $(i = 1,2)$.

(2.4) *Suppose that N is an edge-transitive subgroup of G. Then $G = G_{\alpha\beta}N$ for adjacent vertices α and β of Γ.*

The proof is obvious.

(2.5) *Let Γ be a tree and G be edge-transitive on Γ, and let α_1 and α_2 be adjacent vertices. Suppose that the following hold:*

(a) *No proper normal subgroup of G is edge-transitive on Γ.*

(b) *N_{α_i} is a normal subgroup of G_{α_i} transitive on $\Delta(\alpha_i)$ $(i = 1,2)$.*

Then

$$G_{\alpha_1\alpha_2} = (G_{\alpha_1\alpha_2} \cap N_{\alpha_1})(G_{\alpha_1\alpha_2} \cap N_{\alpha_1}).$$

Proof. Set $N = \langle N_{\alpha_1}(G_{\alpha_1\alpha_2} \cap N_{\alpha_2}),\ N_{\alpha_2}(G_{\alpha_1\alpha_2} \cap N_{\alpha_1})\rangle$. Then (2.1) and (2.4) imply that N is edge-transitive on Γ and $G = G_{\alpha_1\alpha_2}N$. Hence N is normal in G and $G = N$ by (a). Now the assertion follows from (2.3).

(2.6) [4, 2.12]. *Suppose that G is edge-transitive on Γ and that there exist non-regular arcs. Let s be the smallest integer for which a non-regular arc of length s exists, and let $\mathcal{O}$ and $\mathcal{N}$ be the two G-orbits of vertices of Γ(allowing $\mathcal{O} = \mathcal{N}$ if G is vertex-transitive). Then G is transitive on $(\mathcal{O}, m)$ and $(\mathcal{N}, m)$ for $m \leq s$, and one of the following holds:*

(a) *There are no left or right regular arcs of length greater than $s-1$.*

(b) *s is odd, $\mathcal{O} \neq \mathcal{N}$, and if notation is chosen so that the elements of $(\mathcal{O}, s)$ are right singular, then every regular arc of length greater than $s-1$ is in $(o, 2n)$ for some n, and the elements in $(m, \mathcal{N})$ (resp. $(\mathcal{N}, m)$) are right (resp. left) singular for $m \geq s$.*

The integer s in (2.6) is called the singularity of Γ.

(2.7) *Let Γ be a tree, $s \in \mathbf{N}$ and p be a prime. Suppose that the following hold for $\alpha \in \Gamma$:*

(a) *G_α is finite.*

(b) *G_α is transitive on all arcs of length s starting at α.*

(c) *Stabilizers of arcs of length s are p'-groups.*

(d) $|\Delta(\alpha)| = 1 + p^{n_\alpha}$, $n_\alpha \geq 1$.

Then $s \in \{1,2,3,4,5,7,9,13\}$.

Proof. Let T be a Sylow p-subgroup of $G_{\alpha\beta}$, $\beta \in \Delta(\alpha)$, and

$$\gamma = (\alpha, \beta, \alpha_2 \ldots \alpha_t)$$

be an arc of length $t \leq s-1$. Then (d) and an easy inductive argument yield

$$T_\gamma \in Syl_p(G_\gamma),$$

and T_γ is transitive on $\Delta(\alpha_t) \setminus \{\alpha_{t-1}\}$. This observation enables us to apply the proof in [10].

DEFINITION. An n-translation on a line ℓ is a permutation x on ℓ such that $\alpha_i^x = \alpha_{i+n}$ for all $i \in \mathbf{Z}$ and $\alpha_i \in \ell$.

A track is a pair (T, τ) where T is a line and τ is a 2-translation on T.

A K-track is a triple (T, τ, K) where (T, τ) is a track and K is a subgroup of G_T which is normalized by τ.

(2.8) *Suppose that Γ is a tree and α and β are adjacent vertices in Γ. Let K be a subgroup in $G_{\alpha\beta}$,*

$$x \in N_{G_\alpha}(K) \setminus G_\beta \quad and \quad y \in N_{G_\beta}(K) \setminus G_\beta.$$

Then there is a K-track (T, xy, K) with $\alpha, \beta \in T$.

The proof is the same as in [4, 2.10].

DEFINITION. Let $\gamma = (\alpha_0, \ldots, \alpha_n)$ be an arc of Γ and K be a subgroup of G_γ. We define $S_{\gamma,K}$ to be the set of subgroups $X \neq 1$ of G_γ such that:

(1) $K \leq N_G(X)$.

(2) $N_G(X)_{\alpha_0}$ is a transitive on $\Delta(\alpha_0)$, and $N_G(X)_{\alpha_n}$ is transitive on $\Delta(\alpha_n)$.

(3) $N_G(X)_{\alpha_i}$ normalizes $\Delta(\alpha_i) \cap \gamma$ for $0 < i < n$.

(4) There exists $x \in N_G(X)$ with $\alpha_0^x = \alpha_n$.

(2.9) *Suppose that Γ is a tree, $\gamma = (\alpha_0, \ldots, \alpha_n)$ is an arc of Γ and $X \in S_{\gamma,K}$. Set $N = N_G(X)$, and let $\tilde{\Gamma}$ be the graph with vertex set α_0^N where two vertices α and α' are adjacent, if and only if they have distance n in Γ. Assume that one of the following holds:*

(i) $n = 2$.

(ii) *$\Delta(\alpha_i) \cap \gamma$ is the set of fixed points of X in $\Delta(\alpha_i)$ for $0 < i < n$.*

Then the following hold:

(a) *α_0 has the same valency in $\tilde{\Gamma}$ as in Γ.*

(b) *N is vertex-transitive on $\tilde{\Gamma}$.*

Proof. Let r be the valency of α_0 in Γ. As N_{α_0} operates transitively on $\Delta(\alpha_0)$, we get $n_1, \ldots, n_r \in N_{\alpha_0}$, $n_1 = 1$, and $\gamma_i = \gamma^{n_i}$ such that

$$\gamma_i \cap \gamma_j = \{\alpha_0\} \quad \text{for } i \neq j.$$

Let β be a vertex of $\tilde{\Gamma}$ adjacent to α_0. Then by definition there exists a unique arc $\gamma' = (\alpha_0, \ldots, \beta)$ of length n in Γ. It suffices to prove

$$\gamma' \in \{\gamma_1, \ldots, \gamma_r\}.$$

After conjugation with a properly chosen element of $\{n_1^{-1}, \ldots, n_r^{-1}\}$ we may assume that

$$n \geq |\gamma \cap \gamma'| \geq 1.$$

Set $\gamma \cap \gamma' = (\alpha_0, \ldots, \alpha_k)$. If (i) holds, there exists $\gamma^g = (\beta\ \beta_1\ \beta_2)$, $g \in N$, and since N_β is transitive on $\Delta(\beta)$, we may assume

$$\gamma \cap \gamma^g \supseteq \{\alpha_1\} \quad \text{and} \quad \alpha_1 = \alpha_1^g.$$

Hence $\gamma' = \gamma$, since N_{α_1} leaves invariant $\{\alpha_0, \alpha_2\}$.

Now assume that (ii) holds. Then $\Delta(\alpha_k) \cap \gamma = \Delta(\alpha_k) \cap \gamma'$ and $\gamma = \gamma'$.

(2.10) [4, (2.11)]. *Suppose that (T, τ, K) is a K-track in a tree Γ and G_α is finite for all $\alpha \in T$. For any $U \leq G$ let T_U be the set of all fixed points of U in T. Then either $T_U = T$ or T_U is a finite subarc of T.*

3. Point stabilizers with $L_2(2^n)$-sections

(3.0) *Hypothesis.* Let Γ be a tree and G be a group of automorphisms of Γ such that for $\alpha \in \Gamma$ the following hold:

(i) G is edge-transitive on Γ.

(ii) No proper normal subgroup of G is edge-transitive on Γ.

(iii) G_α is finite.

(iv) $|\Delta(\alpha)| = 2^{n_\alpha} + 1$, $n_\alpha \geq 1$, and there exists a normal subgroup N_α of G_α such that $O_2(G_\alpha) \leq N_\alpha$, $N_\alpha / O_2(G_\alpha) \simeq L_2(2^{n_\alpha})$, and N_α is transitive on $\Delta(\alpha)$.

Throughout this paper we use the following facts about $L_2(2^n)$ and its operation on $2^n + 1$ symbols.

(3.1) *Let S be a Sylow 2-subgroup of N_α and K be a complement for S in $N_{N_\alpha}(S)$. Then the following hold:*

(a) *All elements in $S \setminus O_2(G_\alpha)$ have exactly one fixed point in $\Delta(\alpha)$.*

(b) *K is cylcic, $|K| = 2^{n_\alpha} - 1$, and all elements in $K^\#$ fix exactly 2 points in $\Delta(\alpha)$; and $C_{N_\alpha}(K) \leq KO_2(G_\alpha)$ if $K \neq 1$.*

(c) *K operates transitively on $(S/O_2(G_\alpha))^\#$.*

(d) *$|N_{N_\alpha}(K) \,/\, KN_{O_2(G_\alpha)}(K)| = 2$ if $K \neq 1$.*

(e) *If z is an involution in $N_\alpha \setminus O_2(G_\alpha)$, then z is conjugate in N_α to an element of $N_{N_\alpha}(K)$.*

(f) *If $K \neq 1$ and P is a 2-subgroup of N_α, then*

$$\langle K, P\rangle O_2(G_\alpha) = N_\alpha \quad \text{or} \quad \langle K, P\rangle \leq N_{N_\alpha}(K) O_2(G_\alpha).$$

(g) *$N_\alpha \cap G_\beta = N_{N_\alpha}(S^g)$ for $\beta \in \Delta(\alpha)$ and suitable $g \in N_\alpha$.*

(h) $N_\alpha = \langle S, g\rangle$ *for* $g \in N_\alpha \setminus N_{N_\alpha}(S)$.

(3.2.) *For* $\delta \in \Gamma$ *define* $L_\delta = O^{2'}(G_\delta)$. *Suppose that* $\beta \in \Delta(\alpha)$. *Then the following hold:*

(a) $L_\alpha = N_\alpha$, *and* $G_\alpha = G_{\alpha\beta}L_\alpha$.

(b) $G_{\alpha\beta} = (G_{\alpha\beta} \cap O^2(L_\alpha))(G_{\alpha\beta} \cap O^2(L_\beta))$.

(c) $G_{\alpha\beta} = KO_2(G_{\alpha\beta})$, *K a subgroup of odd order.*

(d) If $O_2(G_\alpha) \neq 1$, *then*

$$O_2(G_\alpha)O_2(G_\beta) \in Syl_2(G_{\alpha\beta}) \quad \text{and} \quad Syl_2(G_{\alpha\beta}) \subseteq Syl_2(G_\alpha).$$

(e) *No non-trivial normal subgroup of* L_α *(resp.* $O^2(L_\alpha)$*) is normal in* $(L_\beta$*resp.* $O^2(L_\beta))$.

Proof. With the Frattini argument we get $G_\alpha = G_{\alpha\beta}N_\alpha$, and (2.5) implies

$$G_{\alpha\beta} = (G_{\alpha\beta} \cap N_\alpha)(G_{\alpha\beta} \cap N_\beta).$$

Pick $T \in Syl_2(N_\beta \cap G_{\alpha\beta})$. Since $N_\beta \cap G_{\alpha\beta}$ and $N_\alpha \cap G_{\alpha\beta}$ are 2-closed and normal in $G_{\alpha\beta}$, the structure of $\mathrm{Aut}(L_2(2^{n_\alpha}))$ implies $T \leq N_\alpha$, hence (a) and (c) hold.

The normal subgroup $O^2(L_\alpha)$ is also transitive on $\Delta(\alpha)$, therefore a further application of (2.5) yields (b).

Let X be a normal subgroup of L_α (resp. $O^2(L_\alpha)$) which is also normal in L_β*(resp.* $o^2(L_\beta)$*)*. Then $X \leq G_{\alpha\beta}$, and (2.1) implies that X fixes every edge and thus every vertex in Γ, so $X = 1$, and (e) is proved.

In particular, $O_2(G_\alpha) = O_2(G_\beta) = 1$ or $O_2(G_\alpha) \neq O_2(G_\beta)$. In the second case we may assume $O_2(G_\alpha) \not\leq O_2(G_\beta)$ and get (d) from (a) and (3.1)(c).

We now fix some notation for the remainder of the paper:

(3.3) *Notation.* $Q_\delta = O_2(G_\delta)$,

$$Z_\delta = \langle Z(S) \cap Q_\delta \,/\, S \in Syl_2(G_\delta)\rangle,$$

$L_\delta = O^{2'}(G_\delta)$ and $\overline{L_\delta} = L_\delta / Q_\delta$ for $\delta \in \Gamma$; $|\gamma|$ denotes the length of an arc γ of Γ.

We fix $\alpha \in \Gamma$, $\beta \in \Delta(\alpha)$, $S = O_2(G_{\alpha\beta})$ and a complement K for S in $G_{\alpha\beta}$, and set $K_\delta = K \cap L_\delta$ for $\delta \in \Gamma$.

(T, τ, K) is a K-track with $\alpha, \beta \in T$, s is the singularity of Γ, and $\mathcal{O}$ and $\mathcal{N}$ are the G-orbits on Γ (allowing $\mathcal{O} = \mathcal{N}$, if G is vertex-transitive).

We set $T = (\ldots\alpha_{-i}\ldots\alpha_o\ldots\alpha_i\ldots)$, $i \in \mathbf{N}$, $\alpha_o = \alpha$ and $\alpha_1 = \beta$, and we then identify the vertices in T with their indices such that

$$T = (\ldots -i\ldots 0\ldots i\ldots),$$

$\alpha = 0$, $\beta = 1$, and $G_{\alpha_i} = G_i$, $Z_{\alpha_i} = Z_i$, $K_{\alpha_i} = K_i$, $n_{\alpha_i} = n_i$ etc. for $\alpha_i \in T$.

For $i \in T$ we define $b_i = \max\{|j - i| \,/\, j \in T \text{ and } Z_i \leq G_j\}$, if such a maximum exists, and $b_i = \infty$ otherwise. Note that in the case $b_i < \infty$, $i - b_i$ and $i + b_i$ are not only integers but also vertices in T and $Z_i \leq G_{i-b_i}$ or $Z_i \leq G_{i+b_i}$. Suppose $Z_i \leq G_{i-b_i}$ (resp. G_{i+b_i}); then (3.1)(a) and (3.2) imply $Z_i \leq Q_k$ for $i - b_i < k \leq i$ (resp. $i \leq k < i + b_i$).

(3.4) *Suppose that $n_0 > 1$ and $n_1 > 1$. Then*

(a) $T = C_\Gamma(K)$ *and*

(b) $C_{G_j}(K) \leq G_T$ *for* $j \in T$.

Proof. Assume that $T \neq C_\Gamma(K)$. Then there exists $\varrho \in C_\Gamma(K)$ and an arc

$$\gamma = (\varrho, \varrho_1 \ldots \varrho_n)$$

such that $\varrho_n \in T$ *and* $\varrho_{n-1} \notin T$. Therefore $K \leq G_\gamma$, and K fixes three vertices in $\Delta(\varrho_n)$, a contradiction to (3.1)(b). Assume that $X = C_{G_j}(K) \not\leq G_T$. Then there exist $k \in T$ and $k' \in \Delta(k) \cap T$ such that $X \leq G_k$ and $X \not\leq G_{k'}$. Now (3.1)(b) and (3.2)(a) yield a contradiction.

(3.5) *Suppose that $\gamma = (m \ldots r)$ is a right (resp. left) singular subarc of T. Then $O_2(G_\gamma)$ fixes every element in $\Delta(r)$ (resp. $\Delta(m)$).*

Proof. If $K = 1$, then $n_m = n_r = 1$ and $|\Delta(m)| = |\Delta(r)| = 3$, and the assertion is obvious.

Assume that $K \neq 1$ and that γ is right singular. By way of contradiction we may additionally assume that $O_2(G_\gamma) \not\leq Q_r$. From (3.1)(a) we get that no element in $O_2(G_\gamma) \setminus Q_r$ fixes an element in $\Delta(r) \setminus \gamma$. On the other hand $K \leq G_\gamma$ and K has orbits of length 1 and $2^{n_r} - 1$ on $\Delta(r) \setminus \gamma$ (see (3.1)(b)). This yields that G_γ is transitive on $\Delta(r) \setminus \gamma$, contradicting the hypothesis.

We will use (3.5) in the following without reference.

4. The case $|G_T| \equiv 1$ (2)

(4.0) *Hypothesis and notation.* (3.0) and (3.3) hold, and in addition:

(a) $n_0 > 1$ and $n_1 > 1$.

(b) $Z_0 \neq 1 \neq Z_1$.

(c) $s \equiv 1$ (2) and $s \geq 5$.

(d) $|G_T| \equiv 1$ (2).

(e) γ is a regular subarc of maximal length r in T such that $Q = O_2(G_\gamma) \neq 1$.

(4.1) *Assume that $Q_1 \cap Q_{-1}$ is normal in G_0. Then the following hold:*

(a) *$Q_0/Q_1 \cap Q_{-1}$ is elementary abelian of order 2^{2n_1}.*

(b) $Q_0 = [Q_0, Q_1][Q_0, Q_{-1}](Q_1 \cap Q_{-1})$.

(c) *If* Z_0 *is a natural module for* $\overline{L_0}$ *and* $[Q_1 \cap Q_{-1}, L_0] \leq Z_0$, *then* $Q_1 \cap Q_{-1}$ *is elementary abelian.*

Proof. Set $A = Q_1 \cap Q_{-1}$. We apply (3.2). Since Sylow 2-subgroups of $\overline{L_1}$ (and $\overline{L_{-1}}$) are elementary abelian of order 2^{n_1}, we get $\phi(Q_0) \leq A$ and $|Q_0 / A| \leq 2^{2n_1}$. Hence Q_0 / A is elementary abelian, and the operation of K_1 and K_{-1} on Q_0/A yields

$$Q_0 \cap Q_1 = A \quad \text{or} \quad Q_0/A = (Q_0 \cap Q_1) / A \times (Q_0 \cap Q_{-1}) / A.$$

In the first case $G_{(-1\,0\,1\,2)} = K(Q_0 \cap Q_1) = KA$, and $(-1\,0\,1\,2)$ is not (left-) regular, a contradiction to $s \geq 5$.

Thus the second case holds. If $[Q_1, Q_0 \cap Q_{-1}] \leq A$, then $Q_0 \cap Q_{-1}$ is normal in $<Q_1, Q_{-1}>Q_0 = L_0$ and $A = Q_0 \cap Q_{-1} = Q_0 \cap Q_1$, a contradiction. Hence we have

$$[Q_1, Q_0 \cap Q_{-1}] \not\leq A$$

and with the same argument

$$[Q_{-1}, Q_0 \cap Q_1] \not\leq A.$$

Now again the operation of K_1 and K_{-1} implies assertion (b).

Assume now that Z_0 is natural and $[A, L_0] \leq Z_0$. By (1.3),

$$A = C_A(K_0) \times Z_0 \quad \text{and} \quad \phi(A) = \phi(C_A(K_0)).$$

On the other hand $\phi(A)$ is normal in a Sylow 2-subgroup S of L_0. Thus

$$\phi(A) \cap Z(S) \neq 1,$$

which contradicts $\phi(A) \cap Z(S) \leq \phi(A) \cap Z_0 = 1$.

Without loss of generality we may assume $\gamma = (0 \ldots r)$. Note that by (2.10), γ has finite length and subarcs of T of length greater than r have stabilizers of odd order. We will use this last fact without reference.

(4.2) (a) $|Q| = 2^{n_0}$.

(b) $r \equiv 0$ (2), $s - 1 \leq r$, and $r = s - 1$ *or* $\tilde{\gamma} \in (o, r)$ $(0 \in o)$ *for every maximal regular arc* $\tilde{\gamma}$ *in* Γ.

(c) $|N_{G_i}(K) / K| = 2$ *and* $C_{G_i}(K) \leq K$ *for* $i \in T$.

(d) *For* $i \in T$, $x \in N_{G_i}(K) \setminus K$ *and* $m \in \mathbb{N}$, x *interchanges the two vertices* $i + m$ *and* $i - m$ *of distance* m *from* i *in* T.

Proof. We have $Q \leq G_0$ but $Q \cap Q_0 = 1$. The operation of K on Q ((3.1)(c)) yields (a). Assertion (b) follows from (2.6) and the maximality of r, and (c) and (d) are consequences of (3.1) and (3.4).

(4.3) $b_1 \in \{r/2 - 1, r/2\}$.

Proof. Set $b = b_1 + 1$, and pick $x \in N_{G_1}(K) \setminus K$. Then $Z_1^x = Z_1$, and by (4.1)(d),

$$C_T(Z_1) = (-(b-2)\dots b).$$

Therefore Z_1 is in G_b but not in Q_b, and the maximality of r yields

$$|C_T(Z_1)| = 2b - 2 \leq r \quad \text{and} \quad b_1 \leq r/2.$$

Now assume $r/2 > b$. For $\tau^* \in \langle\tau\rangle$ with $1^{\tau^*} = 2b - 1$ we get

$$C_T(Z_1^{\tau^*}) = (b\dots 3b-2)$$

and $[Q, Z_1^{\tau^*}] = 1$, as $2b - 1 < r$. Hence $\langle Z_1, Z_1^{\tau^*}, K\rangle \leq N_G(Q) = N$, and N_b operates transitively on $\Delta)b)$. We choose $z \in Z_1 \setminus Q_b$. From (3.1)(e) we get that z normalizes K^u for suitable $u \in N_b$. Together with (3.1)(a) and (3.4)(a) this implies that

$$\gamma^* = (r^{uz}\dots(b+1)^{uz}\; b\; (b+1)^u\dots r^u)$$

or

$$\gamma^{**} = (r^u\dots(b+1)^u\; b\; (b+1)^{uz}\dots r^{uz})$$

is a subarc of T^u. As γ^* and γ^{**} are stabilized by K^uQ, the maximality of r implies $|\gamma^*| = |\gamma^{**}| = 2(r-b) \leq r$ and $r/2 \leq b$, a contradiction.

(4.4) $b_0 \in \{r/2 - 2, r/2 - 1, r/2\}$.

Proof. Set $b = b_0 + 2$. Then $C_T(Z_2) = (-(b-4)\dots b)$, and we get the assertion with the same argument as in (4.3).

(4.5). *One of the following holds:*

(a) $[Z_1, Z_{b_1+1}] \leq Z_1 \cap Z_{b_1+1}$.

(b) $r = s - 1$, $[Z_0, Z_{b_0}] \neq 1$, *and* b_0 *is in the same G-orbit as* 0 *(i.e.,* (a) *holds with the roles of* 0 *and* 1 *interchanged).*

Proof. Set $h = b_1 + 1$, $R = [Z_1, Z_h]$, $X = [Z_0, Z_{b_0}]$, and assume that (a) does not hold. Then $R \neq 1$, $b_h = b_0 < b_1$, and h is in the same G-orbit as 0, in particular $b_1 \equiv 1$ (2).

Suppose that b_0 is in the same G-orbit as 0. Then $Z_0 \neq Z(L_0)$ and $X \neq 1$. From (4.3) and (4.4) we get

(1) $r/2 - 2 \leq b_0 = b_1 - 1 < r/2$.

As $X \leq Z_0 \cap Z_{b_0}$ and $|Z_0| = |Z_{b_0}|$, (1.3) implies

(2) $Z_0/Z(L_0)$ is a natural module for $\overline{L_0}$.

Assume $r \leq s$. Then (4.2)(b) yields $r = s - 1$, and assertion (b) follows. Therefore we may assume

(3) $s < r$.

Assume $Z(L_h) \neq 1$. We have $[Z_1, Z(L_h)] = 1$ and $Z(L_h) \leq Z_{h+1} \cap Z_{h-1}$. Hence by (1), $Z(L_h)$ stabilizes the subarc $(0 \ldots 2h)$ of length r in T, and (4.2)(a) implies $Z(L_h) = Q$ and $|Z(L_h)| = 2^{n_0}$. Together with (2) we get

$$|Z(S) \cap Z_0| = 2^{2n_0} \quad \text{for } S \in Syl_2(G_0 \cap G_1).$$

On the other hand (3.2)(e) implies $Z(L_0) \cap Z(L_1) = 1$, hence

$$|Z_1| \geq 2^{3n_0} \quad \text{and} \quad |Q_h \cap Z_1| \geq 2^{2n_0}.$$

Thus $Q_h \cap Q_{-(h-2)} \cap Z_1 \neq 1$, and $Q_h \cap Q_{-(h-2)} \cap Z_1$ stabilizes $(-(h-1) \ldots h+1)$ of length r, where $h+1$ is odd. This contradicts (3) and (2.6). Since h is in the same G-orbit as 0, we have shown together with (2):

(4) $Z(L_0) = 1$, and Z_0 is a natural module for $\overline{L_0}$.

The subgroup X stabilizes $(-b_0 \ldots 2b_0)$ of length $3b_0$, and the maximality of r implies $3b_0 \leq r$. From (1) and (3) we get

(5) $b_0 = r/2 - 2, \quad b_1 = r/2 - 1$ and $r = 8$ or 12,

or

(6) $b_0 = 2$, $b_1 = 3$ and $r = 6$.

As Z_0 is a natural module and $Z_0 \leq Q_1$, (3.2)(e) yields $C_{L_i}(Q_i) \leq Q_i$ for $i = 0,1$. Therefore we can apply (1.11). If (1.11)(d) holds, then $|L_0| = 2^{3n_0}$ and $s < 5$, a contradiction. Thus we get together with (4):

(7) $Z_1 = Z(L_1)$ and $|Z_1| = 2^{n_0}$.

Now (7) and (4) imply $X = C_{Z_0}(Z_{b_0}) = Z_1 = C_{Z_{b_0}}(Z_0) = Z_{b_0-1}$, and the operation of $<\tau>$ yields $b_0 = 2$. Together with (5) we have proved:

(8) $b_0 = 2$, $b_1 = 3$, $r = 6$ or $b_0 = 2$, $b_1 = 3$, $r = 8$.

Set $V = <Z_0^{G_1}>$ and $A = Q_1 \cap Q_{-1}$. From (8) we get $Z_0 \leq A$ and $V \leq Q_1$, and from (4) and (7), $[V, Q_1] = Z_1 = Z(L_1) \leq Z_0$. The operation of K_0 yields

$$|VQ_0/Q_0| = 2^{n_0} \quad \text{and} \quad <V, V^{\tau^{-1}}>Q_0 = L_0.$$

We now apply (4.1). Then $Q_0 \cap Q_1 \leq VA$, and $V' \leq Z_0$ and (1.3) imply that Q_0/A is direct sum of natural modules for $\overline{L_0}$. Let d be an element of order three in L_0; then (1.3),(4) and (4.1) yield:

(9) Q_0/A is direct sum of natural modules for $\overline{L_0}$, $|Q_0/A| = 2^{2n_1}$, and $A = C_{Q_0}(d) \times Z_0$.

Assume $r = 6$; then $|L_0|_2 = 2^{3n_0}2^{2n_1}$ and $Q_1 \cap Q_{-1} = Z_0$. This implies (by (9)) that $C_{Q_0}(d) = 1$, and, from (1.4), Q_0 is elementary abelian and a direct sum of natural modules. But then $Q_0 = Z_0$ and $b_0 = 1$ which contradicts (8).

Note that we got this last contradiction with the help of (1.4) where $n_0 > 1$ is assumed. We will see in Section 5 that for $n_0 = 1$ another possibility arises which does not lead to a contradiction.

We may now assume $r = 8$. Set $L = \langle V^{\tau^{-1}}, V\rangle$, then $LQ_0 = L_0$ and $[A, L] = Z_0$. Hence $[O^2(L_0), A] = Z_0$, and (4) and (9) imply

$$A = C_{Q_0}(K_0) \times Z_0.$$

Set $D = C_{Q_0}(K_0)$ and pick $t_0 \in N_{O^2(L_0)}(K) \setminus G_1$ and $t_1 \in N_{L_1}(K) \setminus G_0$. Then t_0 normalizes K_0 and therefore D; hence

$$[D, t_0] \leq [D, O^2(L_0)] \cap D = Z_0 \cap D = 1.$$

According to (2.8) and (3.4) we may assume $t_0 t_1 = \tau$ and $t_1^2 \in G_T$. Thus τ normalizes $D \cap D^{t_1}$, and $|G_T| \equiv 1\ (2)$ implies $D \cap D^{t_1} = 1$. On the other hand $r = 8$ and $Q^{\tau^{-1}}$ and $Q^{\tau^{-2}}$ are contained in A. But the K-invariant subgroups of A of order 2^{n_0} are in D or Z_0. In the second case they are L_0-conjugates of Z_1 (by (4)). Hence $b_1 = 3$ implies

$$\langle Q^{\tau^{-1}}, Q^{\tau^{-2}}\rangle \leq D.$$

It follows that $Q^{\tau^{-1}t_1^{-1}} = Q^{\tau^{-2}}$ and $Q^{\tau^{-1}} \leq D \cap D^{t_1}$, a contradiction.

From now on we may suppose that b_0 is in the same G-orbit as 1. (4.3) and (4.4) yield:

(10) $b_0 = r/2 - 2$ and $b_1 = r/2$.

In particular Z_1 stabilizes the arc $(-(h-2)\ldots h)$ of length r. Then (4.2)(a) implies $|Z_1| = 2^{n_0}$, and K operates transitively on $Z_1^{\#}$. We get:

(11) $Z_1 = Z(L_1)$, $|Z_1| = 2^{n_0}$ and $X = 1$.

Assume that $r \leq s$. Then there exists a maximal regular subarc of T starting at 1. So we are allowed to interchange the rôles of 0 and 1, and from (4.3), we get $b_0 \geq r/2 - 1$, a contradiction to (10). We have shown:

(12) $s < r$.

Assume that $b_1 = 3$. Then (10) yields $b_0 = 1$ and $r = 6$. Together with (12) and (2.6) we get $|L_0|_2 = 2^{3n_0}2^{2n_1}$. In addition, by (4.1) we have

$$L_1 = \langle Z_0, Z_2\rangle Q_1,\ |Q_1/Q_0 \cap Q_2| = 2^{2n_0},\ Q_1 = (Z_0 \cap Q_1)(Z_2 \cap Q_1)(Q_0 \cap Q_2),$$
$$|Q_0 \cap Q_2| = 2^{n_0}2^{n_1} \quad \text{and} \quad Z_0 \cap Z_2 = Z_1.$$

This yields $|Q_0 / Z_0| = 2^{n_1}$. On the other hand

$$Q_0 = C_{Q_0}(K_1)Z_0 \quad \text{and} \quad [L_0, Q_0] \leq Z_0.$$

As $K = K_1K_0$ normalizes $C_{Q_0}(K_1)$, this implies $Q_0 = C_{Q_0}(K)Z_0$, contradicting (4.2)(c). So we have shown:

(13) $b_1 \geq 5$.

Pick $y \in Z_h$ and $x \in Z_1$, and let k be minimal in $(-(b_1-5)\ldots 3)$ such that k is fixed by y. Then (2.6) implies that x stabilizes

$$((-(b_1-5))^{y^{-1}}\ldots k\ldots 1), \quad \text{if } k \leq 1,$$

and

$$(1\ldots k\ (k-1)^{y^{-1}}\ldots(-(b_1-5))^{y^{-1}}), \quad \text{if } k > 1,$$

and that $[x,y]$ and therefore R stabilizes $(-(b_1-5)\ldots h+b_0)$. Hence $R \leq Q_1$, since $b_1 \geq 5$, and (1.3), (11) and (3.2)(e) imply that Z_h is a natural module for $\overline{L_h}$. Then $Z_h = Z_{h-1}Z_{h+1}$, and Z_{h-1} and Z_{h+1} stabilize the vertex 2. On the other hand $h = b_0 + 3$ by (10), and $Z_h \not\leq Q_3$, a contradiction to (3.1)(a).

(4.6) *Suppose that* $1 \neq [Z_1, Z_{b_1+1}] \leq Z_1 \cap Z_{b_1+1}$. *Then one of the following holds.*

(a) $b_0 = b_1 = 1$, $r = s-1 = 4$ *and:*

(a1) Q_0 *and* Q_1 *are elementary abelian of order* 2^{3n_0};

(a2) $|Z(L_0)| = |Z(L_1)| = 2^{n_0}$ *and* $n_0 = n_1$;

(a3) $Q_i/Z(L_i)$ *is a natural module for* $\overline{L_i}$ $(i = o, 1)$.

(b) $b_0 = 3$, $b_1 = 2$, $r = s-1 = 6$, $n_0 = 3n_1$ *and:*

(b1) $Z_0 = Z(L_0)$, $|Z_0| = 2^{n_1}$, *and* Q_0 *is special of order* 2^{9n_1};

(b2) Z_1 *is a natural module for* $\overline{L_1}$, Q_1/Z_1 *is special, and* $(Q_1/Z_1)/Z(L_1/Z_1)$ *is a direct sum of three natural modules for* $\overline{L_1}$.

(c) $b_0 = 3$, $b_1 = 2$, $r = s-1 = 6$, $n_0 = n_1$ *and:*

(c1) $Z_0 = Z(L_0)$, $|Z_0| = 2^{n_0}$, *and* Q_0 *is special of order* 2^{5n_0};

(c2) Q_1 *is special, and* Z_1 *and* $(Q_1/Z_1)/Z(L_1/Z_1)$ *are natural modules for* $\overline{L_1}$.

Proof. Set $h = b_1 + 1$ and $R = [Z_1, Z_{b_1+1}]$. Then R is contained in $Z_1 \cap Z_{b_1+1}$ and stabilizes $\gamma' = (-(h-2)\ldots(h+b_h))$. The length of γ' is $2b_1 + b_h$, and the maximality of r implies:

(1) $2b_1 + b_h \leq r$.

First suppose that h is in the same G-orbit as 1. Then (1) and (4.3) imply:

(2) $b_1 = 2$ and $r = 6$, and γ' is a maximal regular subarc of T.

Now (4.2)(b) yields $r = s-1$, since γ and γ' are not in the same set $(\mathcal{O}, r)$ (resp. $(\mathcal{N}, r)$), and $|Q_2| = 2^{2n_0}2^{3n_1}$. From $[R, Z_1] = 1$ we know that R is central in a Sylow 2-subgroup of $G_2 \cap G_3$ and therefore is contained in Z_2. Pick

$$t \in N_{G_2}(K) \setminus K.$$

Then (4.2)(d) and (3.1) imply $R^t = R$ and $R \leq Z(L_2)$. Hence $Z(L_3) \cap R = 1$ ((3.2)(e)), and from $[R, Z_1] = [R, Z_3] = 1$, (1.3) and (1.11) we derive that either $Z_2 = Z(L_2)$ or $Z_i/Z(L_i)$ is a natural module and $|Z_i| = 2^{3n_0}$ for $i = 2, 3$. In the second case $n_0 = n_1$, $Q_2 = Z_2Z_1Z_3$ and $Z_2 = RZ(L_1)Z(L_3)$. It follows that $[Z_2, Q_j] \leq Z(L_j)$ for $j = 1,3$, and $Z(L_1)Z(L_3)$ is a normal subgroup of L_2. Now (1.5) implies that $Z_1Z_3/Z(L_1)Z(L_3)$ is elementary abelian which contradicts $[Z_1, Z_3] = R \not\leq Z(L_1)Z(L_3)$.

Thus we have shown $Z_2 = Z(L_2)$ and $Z(L_3) = 1$ by (3.2)(e). Hence Z_3 is a natural module for $\overline{L_3}$. In particular, $Z_3 = Z_2Z_4$ and $b_2 = 3$. Conjugation with τ^{-1} yields:

(3) $b_0 = 3$, $b_1 = 2$, $r = s - 1 = 6$, $Z_0 = Z(L_0)$, $|Z_0| = 2^{n_1}$, and Z_1 is a natural module for $\overline{L_1}$.

Since $s = 7$, the order of a Sylow 2-subgroup of L_0 is:

(4) $|L_0|_2 = 2^{3n_0}2^{3n_1}$.

Set $V = \langle Z_1^{G_0} \rangle$. Then (3) implies

$$V' = Z_0, \quad V/Z_0 \leq Z(Q_0/Z_0), \quad Q_1Z_4 \in Syl_2(L_1)$$

and

$$\langle Z_{-2}, Z_4 \rangle Q_1 = L_1.$$

We get

$$[Z_4, Q_1 \cap Q_2] \leq [V^{\tau}, Q_1 \cap Q_2] \leq Z_2$$

and

$$[\langle Z_4, Z_{-2} \rangle, Q_2 \cap Q_0] \leq Z_1.$$

Therefore $Q_0 \cap Q_2$ is normal in L_1, and by (4.1) and (1.3), $Q_1/Q_0 \cap Q_2$ has order 2^{2n_0} and is direct sum of natural modules for $\overline{L_1}$, in particular $n_1 \leq n_0$.

As we have seen above, $[O^2(L_1), Q_0 \cap Q_2] \leq Z_1$; on the other hand, nontrivial elements of odd order in $L_2(2^n)$ act fixed-point-freely on natural modules ((1.3)). This yields

$$C_{Q_1}(K_1) \leq Q_0 \cap Q_2, \quad Q_0 \cap Q_2 = C_{Q_1}(K_1) \times Z_1 \quad \text{and} \quad |C_{Q_1}(K_1)| = 2^{n_0}.$$

Set $D = C_{Q_1}(K_1)$. Then $Q_0 = VD$, and with the same arguments as in (4.1)(c) we conclude that D is elementary abelian. Hence:

(5) Q_0 is special, $n_1 \leq n_0$, and $(Q_1/Z_1)/Z(L_1/Z_1)$ is direct sum of natural modules for $\overline{L_1}$.

Since $Q_0 \cap Q_2$ has order $2^{n_0}2^{2n_1}$ and stabilizes $(-1\ldots3)$, a K-invariant subgroup of order 2^{n_0} stabilizes the maximal regular subarc $(-2\ldots4)$ in T. This subgroup must be D. In particular we have $[D, K] = D$ and therefore $[D, K_0] = D$, since K_1 centralizes D.

Let N be a normal subgroup of L_0 in Q_0 and $Z_0 < N$, and let t be an element in $N_{L_0}(K) \setminus G_1$. If $D \cap N \neq 1$, then the operation of K_0 on D yields $D \leq N$ and $[D, Q_1] = Z_1 \leq N$. Hence $DV = Q_0 = N$.

If $|N/Z_0| > 2^{2n_0}$, then $|Q_0/N| < 2^{2n_1} \leq 2^{2n_0}$, and (1.2) implies $[Q_0, L_0] \leq N$. Thus $D = [D, K_0] \leq N$ and $N = Q_0$.

Now let N/Z_0 be a minimal normal subgroup of G_0/Z_0. Since $D \leq [Q_0, L_0]$, we get with the above argument $[Q_0, L_0] = Q_0$ and $L_0 = L_0'$. If N/Z_0 is central in L_0/Z_0, then the 3-subgroup-lemma shows $[N, L_0] = 1$, a contradiction.

Now assume that N/Z_0 is not central. Then either $N = Q_0$ or N/Z_0 and Q_0/N are non-central factors of L_0. In the second case (4), (5) and (1.2) imply $n_0 = n_1$.

Assume the first case and $n_1 \neq n_0$. Then (5) implies

$$[Q_0, Q_1, Q_1, Q_1, Q_1] = 1.$$

Hence, from (1.13), we get $[Q_0, Q_1, Q_1, Q_1] \neq 1$ and $n_0 = 3n_1$. Together with (5) and (4) this yields assertion (b).

Assume $n_1 = n_0$. Then (5), (4) and (1.5) imply assertion (c).

Suppose now that h is in the same G-orbit as 0. Then (1), (4.3) and (4.4) yield:

(6) $b_1 = r/2 - 1$, $b_0 \leq 2$ and $r = 4$ or 8.

Assume that $r = 8$, then $b_0 = 2$ (by (4.4)), $\gamma' = (-2\ldots6)$ and $R^\tau = Q$. Therefore Z_2 is contained in G_4 but not in Q_4, and $[Z_2, Z_4] = R$. On the other hand, (4.2)(d) yields $\gamma'^t = \gamma'$ and $R^t = R$ for $t \in N_{G_2}(K) \setminus K$. This implies

$$R \leq Z(L_2) \quad\text{and}\quad [Z_2, L_2] \leq Z(L_2).$$

But then Z_2 centralizes $O^2(L_2)Q_2 = L_2$, and we get $[Z_2, Z_4] = 1$, a contradiction.

Assume that $r = 4$. If $b_0 = 2$, then Z_2 stabilizes γ. The action of K on Z_2 and (4.1)(a) imply $Q = Z_2$ and $|Z_2| = 2^{n_0}$. In particular Z_2 is central in L_2 and $R = 1$, a contradiction. Together with (6) we have shown:

(7) $b_0 = b_1 = 1$ and $r = 4$.

From $[R, Z_1] = [R, Z_2] = 1$ and (1.3) we get that $n_0 = n_1$ and that $Z_i/Z(L_i)$ is a natural module for $\overline{L_i}$ $(i = 1,2)$. Set $\{1,2\} = \{i,j\}$ and $n = n_0$, then we have $|L_i|_2 = 2^{4n}$, since $s = r + 1 = 5$. Now (1.2) implies

$$[Q_i, L_i] = Z_i \quad\text{and}\quad Q_i = C_{Q_i}(K_i)Z_i;$$

in particular, $|C_{Q_i}(K_i)| = 2^n$ and $C_{Q_i}(K_i) \cap Z(L_i) \neq 1$. On the other hand (3.2)(e) yields $Z(L_i) \cap Z(L_j) = 1$, and $Z(L_i)$ is a subgroup of Z_j. Hence the elements of K_j operate fixed-point-freely on $Z(L_i)$. Therefore

$$|Z(L_i)| = 2^n \quad\text{and}\quad C_{Q_i}(K_i) = Z(L_i),$$

and assertion (a) follows (after conjugation with τ^{-1}).

(4.7) *Suppose that* $[Z_1, Z_{b_1+1}] = 1$. *Then one of the following holds:*

(a) $b_1 + 1$ *is in the same G-orbit as 0.*

(b) $r = s - 1$, $[Z_0, Z_{b_0}] = 1$, *and* b_0 *is in the same G-orbit as* 1.

Proof. Set $b = b_1 + 1$ and assume that 1 is in the same G-orbit as b (we write $1 \sim b$). Then we have $Z_1 = Z(L_1)$ and $Z_b = Z(L_b)$, and (4.2)(b) and (4.3) imply that $b_1 \geq 2$, since b is odd. Therefore we get $Z_1 \leq Z_0$, and Z_1 stabilizes $(-b_0 \ldots b)$ in T; in particular:

(1) $b_0 \leq b - 2$.

First assume that $b_1 = r/2$. Then Z_b stabilizes the arc $\gamma' = (1 \ldots (r+1))$ in T of length r which has to be a maximal regular subarc of T. Now (2.6) and (4.2) imply $r = s - 1$. This allows us to interchange the rôles of 0 and 1 (and γ and γ').

Set $0 = 1'$ and $1 = 0'$. If $[Z_{1'}, Z_{b_{1'}+1'}] = 1$, we get assertion (b), or $b_{1'} + 1' \sim 1'$. In the second case we get as above $Z_{1'} = Z(L_{1'})$, a contradiction to (3.2)(e).

If $[Z_{1'}, Z_{b_{1'}+1'}] \neq 1$, we can apply (4.5) and (4.6) and get one of the following possibilities:

(2) $[Z_{0'}, Z_{b_{0'}+0'}] \neq 1$;

(3) $b_{0'}$ is odd.

Case (2) contradicts $[Z_1, Z_{b_1+1}] = 1$, and since $b_{0'} + 1$ is odd, case (3) can not occur.

Now we may assume that $b_1 = r/2 - 1$ and $b_0 = r/2 - 2$. Choose $\tau' \in \langle\tau\rangle$ such that $2^{\tau'} = r - 2$. Then QZ_b centralizes $E_b = \langle Z_2, Z_2^{\tau'}\rangle$, and $\overline{E_b} = \overline{L_b}$. As K normalizes E_b, we have $K \cap E_b = K_b$. Thus K_b centralizes QZ_b.

On the other hand QQ_0 is a Sylow 2-subgroup of G_0 and Z_bQ_1 is a Sylow 2-subgroup of G_1. The structure of $\mathrm{Aut}(L_2(2^n))$ implies

$$[L_0, K_b] \leq Q_0 \quad \text{and} \quad [L_1, K_b] \leq Q_1.$$

Hence $L_0 = C_{L_0}(K_b)Q_0$ and $L_1 = C_{L_1}(K_b)Q_1$, and, by (2.1), $C_G(K_b)$ is edge-transitive on Γ and $K_b = 1$, contradicting $n_1 > 1$.

(4.8) *Suppose that* $[Z_1, Z_{b_1+1}] = 1$. *Then one of the following holds.*

(a) $b_1 = 1, b_0 = 2, r = s - 1 = 4$ *and:*

(a1) $Z_0 = Z(L_0)$, $|Z_0| = 2^{n_0}$, Q_0 *is special, and* Q_0/Z_0 *is a direct sum of two natural modules for* $\overline{L_0}$;

(a2) $2n_0 = n_1$;

(a3) Q_1 *is elementary abelian of order* 2^{4n_0}, *and* Q_1 *is an orthogonal module for* $\overline{L_1}$.

(b) *Assertion* (a) *holds with the rôles of* 0 *and* 1 *interchanged.*

Proof. Set $b = b_1 + 1$. Then $Z_b = Z(L_b)$, and (4.7) implies that b is in the same G-orbit as 0 or that $r = s - 1$ and that we are allowed to interchange the rôles of 0 and 1. Therefore we may assume without loss that b is in the same G-orbit as 0. This yields:

(1) $Z_0 = Z(L_0)$.

Now (3.2)(e) implies$Z_0 \leq Z_1$, otherwise Z_1 would be central in L_1 and $Z_0 \cap Z_1$ would be central in $<L_0, L_1>$. From (4.3) and (4.4) we get:

(2) $b = b_0 = r/2$ and $b_1 = r/2 - 1$, $Q = Z_b$, and Z_0 is elementary abelian of order 2^{n_0}.

Set $H = Z_1 \cap Q_b$. We first assume that $H \not\leq Q_{b+1}$. Since $Z(L_{b+1}) = 1$ (see (1) and (3.2)(e)), we have $R = [H, Z_{b+1}] \neq 1$. Let $a = [h, z]$ be a non-trivial element in R such that $h \in H$ and $z \in Z_{b+1}$. We may assume that z does not fix 0.

If $b_1 \geq 4$, then Z_1 fixes -1, and $(-1)^{s^{-1}}$ has distance two or four from 1. Therefore $s \geq 5$ and (2.6) imply that Z_1 fixes $(-1)^{s^{-1}}$, and we conclude that a stabilizes $\gamma' = (-1\ldots(b + b_1 + 1))$. But by (2), the length of γ' is greater than r, a contradiction. Together with (2) we have shown:

(3) $b_1 = 1$, $b_0 = 2$ and $r = 4$; or $b_1 = 3$, $b_0 = 4$ and $r = 8$.

Assume that $r = 8$. Then $b_1 = 3$, and with the same argument as above R stabilizes $(0\ldots8)$ of length r. This implies $R = Q = Z_4$, and $|R| = 2^{n_0}$. From (1), (1.3) and $Z(L_5) = 1$, we get $Z_5 = Z_4Z_6$. Now, conjugation with τ^{-2} yields $Z_1 = Z_0Z_2$. Hence (3) implies $Z_2 = H \leq Q_5$, a contradiction to the assumption $H \not\leq Q_{b+1}$.

Now assume $r = 4$. We want to show assertion (a). Since $s = 5$, we get

$$|L_0|_2 = 2^{2n_0}2^{2n_1}.$$

Additionally we have $Z_2Q_0 \in Syl_2(L_0)$ and $Z_2 \cap Q_0 = 1$. Therefore we get

$$Q_1 = Z_0 \times Z_2 \times (Q_0 \cap Q_2).$$

Assume that $\phi(Q_1) \neq 1$. Then $\phi(Q_0 \cap Q_2) \neq 1$, and $\phi(Q_0 \cap Q_2) \leq Q_{-1} \cap Q_3$, since $\overline{L_1}$ has elementary abelian Sylow 2-subgroups. Thus $\phi(Q_0 \cap Q_2)$ stabilizes $(-2\ldots4)$ of length 6, contradicting $r = 4$.

We have shown that Q_1 is elementary abelian of order $2^{2n_0}2^{n_1}$. Now (1.2) implies:

(4) $n_1 \leq 2n_0$.

Since Q_1 is abelian, Q_0/Z_0 is, by (1.3) and (4.1), a direct sum of k natural modules for $\overline{L_0}$, and (4) yields $k = 1$ or 2.

If $k = 1$, then (1.5) and $n_0 > 1$ imply that Q_0 is abelian. It follows that

$$Q_1 \cap Q_0 = Z(L_0)$$

by (1). This contradicts (4.1). Hence $k = 2$, and from (4) we get $n_1 = 2n_0$. In particular Q_1 is a module of order 2^{4n_0}. Thus $[Q_1, Q_0, Q_0] \neq 1$, (1.1) and (1.3) imply that Q_1 is an orthogonal module for $\overline{L_1}$.

From now on we assume that $H \leq Q_{b+1}$. Then H stabilizes $(-(b-2)\ldots(b+2))$ of length r. Hence (2), (1.3) and the operation of K on H imply:

(5) $H = Z_2$, and $Z_1 = Z_0Z_2$ is direct sum of natural modules for $\overline{L_1}$, in particular $n_1 \leq n_0$.

We have $K = K_0K_1$ (see (3.2)). On the other hand

$$Z_bQ_0 \in Syl_2(L_0) \quad \text{and} \quad [K_b, Z_b] = 1.$$

The structure of $\mathrm{Aut}(L_2(2^n))$ yields $[K_b, L_0] \leq Q_0$. This implies

$$K_b \cap K_0 = 1 \quad \text{and} \quad |K_bK_0| = |K_0|^2 \leq |K_1K_0| = |K|.$$

Hence (5) and (3.1) yield:

(6) $n_1 = n_0$, and Z_1 is a natural module for $\overline{L_1}$.

Assume $b_1 = 1$. Then (6) yields $[Z_1, Q_0] = Z_0$. Since Z_1 is not in Q_0 and K operates on Z_1, we get $[O^2(L_0), Q_0] = 1$ and $Z_1 = (Z_1 \cap O^2(L_0))Z_0$, which implies $[Z_1, Q_0] = 1$, a contradiction. Since b_1 is odd, we have shown:

(7) $b_1 \geq 3$.

Set $V_k = \langle Z_{k+1}^{L_k} \rangle$ for $k \in T$. Then (7), (2.6) and $s \geq 5$ yield

$$V_0 \leq Q_0 \cap Q_1,$$

and (6) implies $[V_0, Q_0] = Z_0$. In particular V_0 and V_{b-2} are abelian. The transitivity of L_0 on $\Delta(0)$ and (3.1) imply

$$Z_1^{L_0} = Z_1 \cup Z_{-1}^{Z_b},$$

since $Z_bQ_0 \in Syl_2(L_0)$. Set $R = [Z_{-1}, Z_b]$; then $V_0 = RZ_1Z_{-1}$. We get

$$R \leq V_0 \cap V_{b-2},$$

since Z_b is contained in V_{b-2}, and $[R, Z_b] = 1$, since V_{b-2} is abelian. Thus, by (1.3), $V_0/C_{V_0}(O^2(L_0))$ is a natural module for $\overline{L_0}$.

Assume that $R_0 = C_R(O^2(L_0)) \not\leq Z_0$. Since R_0 is contained in V_{b-2}, it fixes b. Pick

$$t \in N_{O^2(L_0)}(K_0) \setminus K_0.$$

By (4.2), R_0Z_0 stabilizes $(b^t \ldots b) = (-b \ldots b)$ of length r and $|R_0Z_0| = 2^{n_0}$. But now (2) yields $R_0 \leq Z_0$, a contradiction. We have shown:

(8) $V_0 = Z_1Z_{-1}$ and $|V_0| = 2^{3n_0}$.

V_0 stabilizes $(-(b-2)\ldots(b-2))$ and $R \neq 1$ stabilizes

$$\hat{\gamma} = (-(b-2)\ldots 2(b-2)).$$

The maximality of r and (2) yield $3(b-2) \leq r$ and:

(9) $r \leq 12$.

Assume $r = 12$. Then $\hat{\gamma}$ has length r, and $R = Z_2$. Since $Z_5 = Z_4Z_6$, we get $[Z_{-1}, Z_5] = Z_2$. Conjugation with τ yields:

(10) $[Z_j, Z_{j+6}] = Z_{j+3}$ for all $j \in T$ which are in the same G-orbit as 1.

Next we want to show that (10) holds for an arbitrary arc $\lambda = (\delta_{-3}\ldots\delta_3)$ of length 6 in Γ, where δ_{-3} is in the same G-orbit as 1. It suffices to show that λ is conjugate to a subarc of T. Applying (2.6) we may assume that

$$<\delta_{-2}\ldots\delta_3> = (0\ldots5).$$

But then Q fixes $(0\ldots5)$ and operates transitively on $\Delta(0)\setminus\{1\}$. Hence λ is conjugate to a subarc of T. We have shown:

(11) $[Z_{\delta_{-3}}, Z_{\delta_3}] = Z_{\delta_0}$ for all arcs $(\delta_{-3}\ldots\delta_0\ldots\delta_3)$ of length 6 in Γ, where δ_{-3} is in the same G-orbit as 1.

Pick $z\in Z_0$ and $z'\in Z_{10}$. Then z fixes 6, but not 7, and z' fixes 4 but not 3. Hence $(10^z\ldots6\ldots10)$ and $(0^{z'}\ldots4\ldots0)$ are arcs of length 8, and by (11),

$$[Z_9, Z_9^z] = Z_6 \quad\text{and}\quad [Z_1, Z_1^{z'}] = Z_4.$$

Since Z_1 and Z_9 are elementary abelian and contain Z_0 and Z_{10} respectively, the elements $(zz')^2$ and $(z'z)^2$ are involutions. But then

$$(zz')^2 = (z'z)^2 \in Z_4\cap Z_6,$$

and $Z_4\cap Z_6$ is a non-trivial subgroup stabilizing $(-2\ldots12)$, a contradiction to the maximality of r. We have shown (together with (2), (7) and (9)):

(12) $b_0 = 4$, $b_1 = 3$ and $r = 8$.

From (5), (6) and (8) we get $V_0 = Z_{-1}Z_1$ and $V_2 = Z_1Z_3 = Z_1Z_4$. Thus we have

$$V_2\cap Q_0 = Z_1 \le V_0 \quad\text{and}\quad [V_2, Q_0\cap Q_1] \le V_2\cap Q_0 \le V_0.$$

In particular, $[Q_1\cap Q_{-1}, <V_2, V_2^{\tau^{-2}}>] \le V_0$, and $Q_1\cap Q_{-1}$ is normal in G_0. Hence (4.1) and (1.3) imply that $Q_0/Q_1\cap Q_{-1}$ is a natural module for $\overline{L_0}$ (since $n_0 = n_1$) and

$$Q_1\cap Q_{-1} = C_{Q_0}(K_0)V_0.$$

Pick $t\in N_{O^2(L_0)}(K)\setminus K$. Then t normalizes K_0 and every subgroup of $C_{Q_0}(K_0)$ which contains Z_0, since $[C_{Q_0}(K_0), t] \le C_{Q_0}(K_0)\cap V_0 \le Z_0$.

Assume $|C_{Q_0}(K_0)| \ge 2^{2n_0}$. (4.2)(d) implies that $C_{Q_0}(K_0)\cap L_4$ stabilizes $(-4\ldots4)$ of length r. Hence $C_{Q_0}(K_0)\cap L_4 > Z_0$ would contradict (4.2)(a).

So we may assume that there exists $i\in\{2,3\}$ such that $L_i\cap C_{Q_0}(K_0) \not\le Q_i$. Then

$$(C_{Q_0}(K_0)\cap L_i)Q_i\in Syl_2(L_i),\quad L_i = C_{L_i}(K_0)Q_i \quad\text{and } Z_0 \le Q_i\cap C_{Q_0}(K_0).$$

If $i = 3$, then $C_G(K_0)_3$ and $C_G(K_0)_4$ operate transitively on $\Delta(3)$ and $\Delta(4)$ respectively, since $Z_0Q_4\in Syl_2(L_4)$. Hence (2.1) and $K_0 \ne 1$ imply $i = 2$.

Let x be an element in $N_{L_2}(Z_0)$. If $x\notin G_0$, then the arc joining 0 and 0^x has length $n \le 4$. Since $s \ge 5$, we may assume that $0^x\in T$. But then Z_0 stabilizes a subarc of length $r+n$ in T, a contradiction to the maximality of r.

So we have shown that $N_{L_2}(Z_0) \le G_0$. On the other hand $C_{Q_0}(K_0)\cap Q_2 = Z_0$, because otherwise either $C_{Q_0}(K_0)\cap L_4 > Z_0$ or $C_{Q_0}(K_0)\cap L_3 \not\le Q_3$, contradicting what we have already proved. Hence we get $C_{L_2}(K_0) \leq N_{L_2}(Z_0) \leq G_0$, a contradiction to $C_{L_2}(K_0)Q_2 = L_2$.

Now assume $|C_{Q_0}(K_0)| = 2^{n_0}$. Then $Q_1 \cap Q_{-1} = V_0$, and we get $|Q_0| = 2^{5n_0}$, and, by (1.3) and (1.4), $Q_0' = Z_0 = \phi(Q_0)$. In particular, $Q_0/Z_0 = W_1/Z_0 \times V_0/Z_0$, where W_1/Z_0 is a natural module for $\overline{L_0}$ and $W_1 \not\leq Q_1$. Since $Q_0' \leq Z_0$, we get that $Q_0 \cap Q_2$ is normal in G_1 and together with (4.1) and (1.3) that $Q_1/Q_0 \cap Q_2$ is a natural module for $\overline{L_1}$. Now (1.5) implies $Q_1' = Z_1$. On the other hand, by (12), $Z_{-1} \cap Q_2 = Z_0$, hence $[V_0, K_1] = V_0$. Pick

$$g \in L_1 \setminus G_0.$$

Then $<W_1, W_1^g>Q_1 = L_1$ normalizes $(W_1 \cap Q_1)(W_1^g \cap Q_1)/Z_1 = X$, and $W_1 \cap Q_1/Z_0$ has order 2^{n_0}. Hence X is a natural module for $\overline{L_1}$, and K_1 normalizes

$$(W_1 \cap Q_1)Z_1$$

and centralizes

$$Q_1/(W_1 \cap Q_1)(W_1^g \cap Q_1).$$

Thus we get

$$V_0 = [V_0, K_1] \leq (W_1 \cap Q_1)Z_1.$$

Now the order of V_0 implies $(W_1 \cap Q_1)Z_1 = V_0$ and $W_1 \cap V_0 \not\leq Z_0$, a contradiction.

5. A special case

(5.0) *Hypothesis and notation.* Hypothesis (4.0) holds with (4.0)(b) replaced by

(b') $n_0 > 1$ and $n_1 = 1$.

We use notation (3.3). In addition we define $\tilde{Z}_i = [Z_i, K]$ for $i \in T$. If $\tilde{Z}_i \neq 1$, we set

$$r_i = \max\{j - i / j \in T, j > i \text{ and } \tilde{Z}_i \leq G_j\}$$

and

$$\ell_i = \max\{i - j / j \in T, i > j \text{ and } \tilde{Z}_i \leq G_j\}$$

Clearly $b_i \leq r_i$ and $b_i \leq \ell_i$, and, by (2.10), any subarc of T of length greater than r has stabilizer of odd order. We will use this fact in this section without reference. Note that we no longer assume that $(0 \ldots r)$ is a maximal regular subarc of T. But the operation of τ yields that at least one of $(0 \ldots r)$ and $(1 \ldots (r+1))$ is maximal regular. Note also that $C_T(Z_i)$ for $i \in T$ may no longer be symmetric in i.

(5.1) *For $i \in T$ the following hold:*

(a) *$K \leq L_0$ and $[K, L_1] \leq Q_1$.*

(b) *$K \in S_{\tilde{\gamma},K}$ for $\tilde{\gamma} = (-1\ 0\ 1)$.*

(c) *$O^{2'}(N_G(K)_1)$ is isomorphic to a subgroup of $C_2 \times \Sigma_4$.*

(d) *If $Q_{i-1} \cap Q_{i+1}$ is normal in G_i, then $Q_i/Q_{i-1} \cap Q_{i+1}$ is elementary abelian of order 2^{2n_i} and $Q_i = (Q_{i-1} \cap Q_i)(Q_{i+1} \cap Q_i)$.*

(e) *If $[Z_i, K] = 1$, then $C_T(Z_i) = (i - b_i \ldots i + b_i)$.*

Proof. The hypothesis and (3.2)(b) yield

$$K = K_0 \quad \text{and} \quad [K, L_1] \leq Q_1.$$

Hence $N_G(K)_1$ operates transitively on $\Delta(1)$, and (3.1) implies $K \in S_{\tilde{\gamma},K}$ (for definition see Section 2). Thus we can apply (2.9). Any normal subgroup X of $O^{2'}(N_G(K)_1)$ which is also normal in $O^{2'}(N_G(K)_{-1})$ stabilizes $1^{N_G(K)}$ by (2.1). Since $\tau \in N_G(K)$, it follows that

$$X \leq G_T \cap O^{2'}(N_G(K)_1) = K \cap O^{2'}(N_G(K)_1) = 1.$$

Hence we can apply (1.10) and get (c).

Assertion (d) follows as in (4.1).

Assume now that $[Z_i, K] = 1$ and without loss of generality that Z_i stabilizes $i + b_i$ but not $i - b_i$. Then there exists $i - b_i < h < i$ such that $Z_i \leq L_h$ but $Z_i \not\leq Q_h$. Hence we get

$$[L_h, K] \leq Q_h \quad \text{and} \quad [L_{i+b_i}, K] \leq Q_{i+b_i}.$$

If follows from (a) that h and $i + b_i$ are in the same G-orbit as 1, and

$$i - h \equiv b_i \quad (2);$$

in particular, $i - h \leq b_i - 2$.

Pick $\delta \in \{h, h-1\} \cap i^G$. Then $\delta + b_i > i$ and $[Z_\delta, Z_i] = 1$. If $\delta = h$, then $Z_h = Z(L_h)$ and hence also $Z_i = Z(L_i)$; in particular $[Z_{h-2}, Z_i] = 1$, since $b_i + h - 2 \geq i$. Thus we have found that $[Z_u, Z_i] = 1$ for $u = h - 1$ or $h - 2$. Then $d(u, u^x) = 2$ or 4 for $x \in Z_i \setminus G_u$. Since $s \geq 5$, this implies $Z_u = Z_{u+2}$ or $Z_u = Z_{u+4}$, and the operation of $\langle\tau\rangle$ yields $Z_u \leq G_T$, a contradiction.

(5.2) *One of the following holds.*

(a) *$b_0 = 1$, $b_1 = 2$, $r = 4$, $n_0 = 2$ and:*

(a1) *Q_0 is elementary abelian of order 2^4;*

(a2) *Q_0 is an orthogonal module for $\overline{L_0}$;*

(a3) *Q_1 is extra special of order 2^5;*

(a4) *Q_1/Z_1 is a direct sum of two natural modules for $\overline{L_1}$.*

(b) *$b_0 = 3$, $b_1 = 2$, $r = s - 1 = 6$, $n_0 = 3$ and:*

(b1) *Q_0 is extra special of order 2^9;*

(b2) *Z_1 is a natural module, $(Q_1/Z_1)/Z(L_1/Z_1)$ is a direct sum of three natural modules for $\overline{L_1}$, and Q_1/Z_1 is special.*

(c) *$b_0 = 3$, $b_1 = 2$, $s = 5$, $r = 6$, $n_0 = 2$ and:*

(c1) *Q_0 is extra special of order 2^5, and Q_0/Z_0 is a orthogonal module for $\overline{L_0}$;*

(c2) *Q_1 is special, Z_1 is a natural module for $\overline{L_1}$, and Q_1/Z_1 is a direct sum of two natural modules for $\overline{L_1}$;*

(c3) *$(1\ldots(r+1))$ is a maximal regular subarc of T.*

Proof. From (5.1)(a) and the operation of τ on T we get $K \le L_i$ for $i \in T$ and $i \equiv 0$ (2), and $[K, L_j] \le Q_j$ for $j \in T$ and $j \equiv 1$ (2).

Suppose first that $\tilde{Z}_0 \neq 1$. Then r_0 and $-\ell_0$ are in the same G-orbit as 0 (we write $r_0 \sim 0$ etc.), since otherwise $[\tilde{Z}_0, K]$ would be in Q_k, $k = r_0$ resp. $-\ell_0$, contradicting $[\tilde{Z}_0, K] = \tilde{Z}_0 \not\le Q_k$.

Set $b = r_0 - \ell_0$. If $\ell_0 < r_0$, we get $\tilde{Z}_{r_0} \not\le Q_b$ but $\tilde{Z}_b \le Q_{r_0}$. Hence

$$[\tilde{Z}_{r_0}, \tilde{Z}_b] = 1,$$

and $<\tilde{Z}_{r_0}, N_{L_b}(K)> Q_b = L_b$ centralizes $\tilde{Z}_b$, a contradiction since $K \le L_b$.

If $r_0 < \ell_0$ we apply the same argument with the rôles of r_0 and ℓ_0 interchanged. This shows:

(1) $r_0 = \ell_0$ and $r_0 \sim 0$.

We may choose the maximal regular subarc γ of T such that

$$\gamma = (0\ldots r) \text{ or } (1\ldots(r+1)).$$

Assume that $(0\ldots r)$ is a maximal regular subarc and $r_0 \le r/2 - 2$ or that $(1\ldots(r+1))$ is a maximal regular subarc and $r_0 \le r/2 - 1$. In both cases (2.6) yields $r \equiv 0$ (2), and Q centralizes $<Z_2, Z_{2r_0+2}>$. On the other hand

$$<Z_2, Z_{2r_0+2}> Q_{r_0+2} = L_{r_0+2},$$

and K normalizes $C_G(Q) \cap L_{r_0+2}$. Thus $K \le C_G(Q)$; in particular

$$\gamma = (1\ldots(r+1)),$$

and $(0\ldots r)$ is not regular.

Since $K \in S_{\tilde{\gamma},K}$ for $\tilde{\gamma} = (-1\ 0\ 1)$ (see (5.1)(b)), we can define $\tilde{\Gamma}$ with respect to $N_G(K)$ as in (2.9). From (5.1)(c) we get that maximal regular arcs in $\tilde{\Gamma}$ have length $\tilde{r} \le 4$, hence $r = 6$ or 8. If $r = 8$, then $r_0 = 2$ and $\tilde{r} = 4$, and Q is contained in $Z(N_G(K)_5)$. Hence $C_{L_5}(Q)$ and $C_{L_4}(Q)$ are transitive on $\Delta(5)$ and $\Delta(4)$ respectively, contradicting (2.1).

Thus we may assume $r = 6$ and $r_0 = 2$. If $b_0 = 1$, then Q centralizes $<Z_2, Z_4>$, and

$$<Z_2, Z_4> Q_3 = L_3.$$

Hence $C_{L_3}(Q)$ and $C_{L_4}(Q)$ are transitive on $\Delta(3)$ and $\Delta(4)$ respectively, contradicting (2.1). Thus $b_0 = 2$, and $1 \neq [Z_0, Z_2]$ stabilizes $(-2\ldots4)$ of length 6. Conjugation with τ yields $O_2(G_{(0\ldots6)}) \neq 1$, a contradiction. Hence we have shown (together with (2.6)):

(2)(a) $r_0 = r/2$, or

(b) $r_0 = r/2 - 1$, $(1\ldots(r+1))$ is not regular and $s < r$.

Set $\tilde{R} = [\tilde{Z}_0, \tilde{Z}_{r_0}]$. Since $<\tilde{Z}_0, N_G(K) \cap L_{r_0}> Q_{r_0} = L_{r_0}$, we have $\tilde{R} \neq 1$. Assume now that $\tilde{Z}_1 \neq 1$, too. By (5.1)(a), $\tilde{Z}_1$ is normal in L_1. Thus

$$\tilde{Z}_1 = (\tilde{Z}_0 \cap \tilde{Z}_1) \times (\tilde{Z}_2 \cap \tilde{Z}_1),$$

and $\tilde{Z}_1$ stabilizes $(-(r_0-2)\ldots r_0)$, which implies $r_1 \geq r_0 - 1 \leq \ell_1$. If $r_1 = r_0 - 1$, we get $[\tilde{Z}_1, \tilde{Z}_{r_0}] = \tilde{R} \neq 1$ contradicting $\tilde{Z}_{r_0} \leq Q_1$. With the same argument $\ell_1 > r_0 - 1$. Since r_0 is even and ℓ_1 and r_1 are odd, it follows that

$$r_1 \geq r_0 + 1 \leq \ell_1$$

and, by (2), $r_1 = \ell_1 = r_0 + 1$, $r_1 + \ell_1 = r$, and maximal regular subarcs in T are $<\tau>$-conjugates of $(0\ldots r)$. Hence $|\tilde{Z}_1| = |\tilde{Z}_0 \cap \tilde{Z}_1|^2 = 2^{n_0}$, which contradicts the operation of K on $\tilde{Z}_0$. We have shown:

(3) $\tilde{Z}_1 = 1$.

Assume $Z_1 \neq Z(L_1)$. By (1.11), $Z_i/Z(L_i)$ is a natural module for $\overline{L_i}$ $(i = 0, 1)$. But (3) yields $[Z(S), K] = 1$, contradicting the operation of K on $\tilde{Z}_0$. Together with (5.1)(c) we have shown:

(4) $Z_1 = Z(L_1)$ and $|Z_1| = 2$.

Assume $b_0 = r_0$ and, without loss of generality, $Z_0 \leq G_{r_0}$. Then

$$[Z_0, \tilde{Z}_{r_0}] \leq Z_0 \cap Z_{r_0},$$

and, by (1.3), $Z_0/Z(L_0)$ is a natural module for $\overline{L_0}$. Additionally, (4) and (3.2)(e) imply $Z(L_0) = 1$. Thus, by (1.3), $Z_0 = \tilde{Z}_0$, but $Z_1 \leq Z_0$ and $[Z_1, K] = 1$, a contradiction. We have shown:

(5) $b_0 < r_0$.

Assume $\tilde{R} \cap \tilde{Z}_0 \neq 1$. This yields $\tilde{R} \cap \tilde{Z}_0 \cap \tilde{Z}_{r_0} \neq 1$, since $\tilde{R} \leq Z_0 \cap Z_{r_0}$ and K normalizes $\tilde{R}$. Hence $\tilde{R} \cap \tilde{Z}_0 \cap \tilde{Z}_{r_0}$ stabilizes $(-r_0\ldots2r_0)$, and (2) and (5) imply $b_0 = 1$, $r_0 = 2$ and $r = 6$. Thus, by (5.1)(d),

$$Q_1 = (Z_0 \cap Q_1)(Z_2 \cap Q_1)(Q_2 \cap Q_0) \quad \text{and} \quad Z_0 \cap Z_2 = Z_1.$$

In particular, $\tilde{R} \leq Z_1$, and (4) contradicts $\tilde{R} \cap \tilde{Z}_0 \neq 1$.
We have shown:

(6) $\tilde{R} \cap \tilde{Z}_0 = 1$.

Assume $b_0 \geq 2$ and, as above without loss of generality, $Z_0 \leq G_{b_0}$. Then (5) yields

$$[Z_0, K] \le Q_{b_0}$$

and hence $b_0 \ge 3$.

If $Z_1 \le \tilde{R}$, then $Z_1 \le Z_0 \cap Z_{r_0}$ and $b_1 \ge (r_0 - 1) + b_0$. Thus by (3), (5.1)(e) and (2), $r \ge 2b_1 \ge 2(r_0 - 1) + 2b_0 \ge r - 4 + 2b_0$ and $b_0 \le 2$, a contradiction.

If $Z_1 \not\le \tilde{R}$, then by (5.1)(c), $C_{Z_0}(K) = Z_1\tilde{R}$, since $C_{Z_0}(K)$ is central in a Sylow 2-subgroup of $N_G(K)_1$, and $[Z_0, \tilde{Z}_{r_0}, \tilde{Z}_{r_0}] = 1$. Now (1.3) implies $Z_0 = Z(L_0)\tilde{Z}_0$. But (4) and (3.2)(e) yield $Z(L_0) = 1$ and $Z_1 \le \tilde{Z}_0$, a contradiction to $Z_1 \le C_G(K)$. Hence:

(7) $b_0 = 1$.

From (7) and (5.1)(d) we get

$$L_1 = \langle Z_0, Z_2 \rangle Q_1, \quad |Q_1/Q_0 \cap Q_2| = 2^{2n_0},$$

$$Q_1 = (Z_0 \cap Q_1)(Z_2 \cap Q_1)(Q_0 \cap Q_2) \quad \text{and} \quad Z_0 \cap Z_2 = Z_1.$$

In particular, $[Q_0 \cap Q_2, O^2(L_0)] = 1$; thus $Z_0 \cap Q_0 \cap Q_2$ is normal in L_1 and

$$Z_0 \cap Q_0 \cap Q_2 = Z_1.$$

This implies, together with (4), that $r_0 = 2$ and $|Z_0| = 2^{n_0}4$, and (1) and (1.2) yield the assertions (a1) and (a2) for Z_0. To prove assertion (a) it remains to show $Q_0 \cap Q_2 = Z_1$ and $r = 4$.

If $r = 4$, then $|L_0| = 4^3$ by (2.6) and $Q_0 \cap Q_2 = Z_1$. Hence it suffices to show $r = 4$.

Assume $r \ne 4$. Then (2) yields $r = 6$, $|L_0|_2 = 2^8$ and $|Q_0 \cap Q_2| = 8$. On the other hand we get $Q_0 = (Q_0 \cap Q_2)Z_0$ and $[Q_0, \tilde{Z}_2] \le Z_0$ which implies

$$[Q_0 \cap Q_2, K] \le Z_1,$$

since $K \le L_0$. Hence by (4) we have $Q_0 \cap Q_2 \le C_{Q_1}(K)$ and, by (5.1)(c), $|Q_0 \cap Q_2| \le 4$, a contradiction.

From now on we assume that $\tilde{Z}_0 = 1$. Then $Z_0 = Z(L_0)$, and (3.2)(e) yields $Z(L_1) = 1$. Hence $Z_0 \not\le Q_1$ or $Z_1 = Z_0 \times Z_2$. In the first case we get

$$Z(L_1) = Q_1 = 1 \quad \text{and} \quad |Q_0| = |L_0|_2 = 2,$$

a contradiction. In the second case we get $\tilde{Z}_1 = 1$, and (5.1)(c) implies:

(8) $\tilde{Z}_1 = 1$, $Z_1 = Z_0 \times Z_2$, $|Z_0| = 2$ and $b_0 \ge 2$.

Note that (5.1)(e) implies now that $C_T(Z_i)$ is symmetric in i for $i \in T$; in particular, $2b_i \le r$. Since $Z_1 = Z_0 \times Z_2$, we have $b_1 = b_0 - 1$, and since K centralizes Z_1, b_0 is in the same G-orbit as 1.

Set $R = [Z_1, Z_{b_0}]$. Then $R \ne 1$ and $R \le Z_1 \cap Z_{b_0}$. On the other hand,

$$Z_{b_0} = Z_{b_0-1}Z_{b_0+1},$$

which implies $R = Z_2 = Z_{b_0-1}$ and $b_0 = 3$, since $|G_T|$ is odd. We have shown:

(9) $b_0 = 3$ and $b_1 = 2$.

As $s \geq 5$, we know from (9) that Z_δ fixes exactly the vertices of distance less than 4 (resp. 3) from $\delta \in \Gamma$. Now choose T^* to be a line in Γ stabilized by K such that

$$T^* = (\ldots\delta_{-i}\ldots\delta_i\ldots) \text{ and } C_{T^*}(Q) = (\delta_0\ldots\delta_{r^*})$$

and r^* is maximal with this property. If $\delta_0 \sim 0$, then $[Q, Z_{\delta_0}] = 1$ and $z \in Z_{\delta_0}^{\#}$ fixes δ_3 but not δ_4. Hence we get another line stabilized by K:

$$T^{**} = (\ldots\delta_i\ldots\delta_3\,\delta_4^z\ldots\delta_i^z\ldots) \text{ and } c_{T^{**}}(Q) = (\delta_{r^*}\ldots\delta_3\ldots\delta_{r^*}^z).$$

The maximality of r^* implies $2(r^* - 3) \leq r^*$ and $r^* \leq 6$.

If $\delta_0 \sim 1$, then $C_{Z_{\delta_0}}(Q) = Z_{\delta_1}$, and $z \in Z_{\delta_1}^{\#}1$ fixes δ_4 but not δ_5. Arguing as above we get

$$T^{**} = (\ldots\delta_i\ldots\delta_4\,\delta_5^z\ldots\delta_i^z\ldots) \text{ and } c_{T^{**}}(Q) = (\delta_{r^*}\ldots\delta_4\ldots\delta_{r^*}^z)$$

and $r^* \leq 8$. Hence in both cases we get $r \leq r^* \leq 8$.

We define $V_0 = \langle Z_1^{L_0}\rangle$ and $V_2 = V_0^\tau$. Then $V_0' = Z_0$ and $V_2' = Z_2$ by (8) and (9), and $Q_0 \cap Q_2$ is normal in L_1. Hence (5.1)(d) implies

$$Q_1 = (Q_1 \cap V_0)(Q_1 \cap V_2)(Q_0 \cap Q_2) \quad \text{and} \quad L_1 = \langle V_0, V_2\rangle(Q_0 \cap Q_2).$$

Thus

$$Q_0 \cap Q_2 = D \times Z_1,$$

where $D = C_{Q_1}(d)$ and d is an element of order 3 in $\langle V_0, V_2\rangle$. Moreover

$$\phi(Q_0 \cap Q_2) = \phi(D) = 1,$$

since D has trivial intersection with Z_1. We have shown:

(10) $r \leq 8$, $Q_0 = DV_0$ is extra special and $Q_1/D \times Z_1$ is a direct sum of natural modules for $\overline{L_1}$.

If $r = 8$ then $r = r^*$, and we have shown above that $(0\ldots8)$ can not be regular, hence $KQ = G_{(1\ldots9)}$ and $[K, Q] = 1$. On the other hand

$$Q \leq Q_4 \cap Q_6 = D^{\tau^2} \times Z_5,$$

and we get $[K, Q, Q_5] = 1$ and $[Q_5, Q, K] \leq [Z_5, K] = 1$. Thus the 3-subgroup-lemma yields $Q \leq Z_5$, which contradicts (8) and (9).

We have shown $r \leq 6$. Since $(-3\ldots3)$ is stabilized by Z_0, we get after conjugation with τ^2:

(11) $r = 6$, and $(1\ldots7)$ is maximal regular subarc of T.

Assume first that $(0\ldots6)$ is also a maximal regular subarc of T. Then (2.6) implies $r = s - 1$, and we are in a similar situation as in (4.6) after steps (4) and (5). With the same argument as there we get assertion (b).

Assume now that $(0\ldots6)$ is not regular. Then (2.6) implies $s = 5$ and $|L_0|_2 = 2^{2n_0}8$. Thus we are in a similar situation as in the proof of (4.5) after step (9) (with the roles of 0 and 1 interchanged). In (4.5) we used (1.4) and Hypothesis (4.0)(a) to get a contradiction. Since in our situation now $n_1 = 1$, we get no contradiction but with the same argument as in (4.4) that Q_1 is special and that Q_1/Z_1 is direct sum of natural modules. Since $|Q_0/Z_0| = 2^{n_0}4$, we get $n_0 = 2$ from (1.2), and assertion (c) follows with (1.1) and (1.5).

6. The case $|G_T| \equiv 0\ (2)$

(6.0) *Hypothesis and notation.* Hypothesis (3.0) and notation (3.3) hold in this section. Additionally we choose $0 \in \mathcal{O}$ and assume:

(a) $|G_T| \equiv 0\ (2)$.

(b) $s \equiv 1\ (2)$ and $s \geq 5$.

(c) $Z_0 \neq 1 \neq Z_1$.

(d) $\max\{n_0, n_1\} > 1$.

(e) $(0 \ldots r)$ is a maximal regular subarc of T.

Note that maximal regular subarcs of T have length $s - 1$ or are in $(\mathcal{O}, 2m)$ (see (2.6)).

(6.1) *For $Q = O_2(G_T)$ and $\gamma = (012)$ the following hold:*

(a) $Q \neq 1$ *and* $G_T = QK$.

(b) $Q \in S_{\gamma,K}$.

Proof. For the definition of $S_{\gamma,K}$ see (2.9). By (3.2)(c), $G_{(01)}$ is 2-closed, hence (a) holds.

Set $M = N_G(Q)$. There is a finite subarc $\tilde{\gamma}$ in T of maximal length such that $G_\gamma \neq G_T$ (see (2.10)). $\tilde{\gamma}$ is a maximal regular subarc of T, and Q is a normal subgroup of G_γ; thus $G_{\tilde{\gamma}} = M_{\tilde{\gamma}}$. We may assume that $\tilde{\gamma} = (0 \ldots 2m)$ and $2m \geq s - 1$ (see (2.6)). Hence $o^{\tau^m} = 2m$ and $\langle M_{\tilde{\gamma}}, M_{\tilde{\gamma}}^{\tau^m} \rangle$ is transitive on $\Delta(2m)$. Conjugation with τ implies that M_0 and M_2 are transitive on $\Delta(0)$ and $\Delta(2)$ respectively.

Next we shall prove that there is an element $x \in M_1$ such that $0^x = 2$. Assertion (3.1)(b) implies that it suffices to show $N_{M_1}(K) \not\subseteq M_0 \cup M_2$. Pick

$$x' \in N_{M_0}(K) \quad \text{and} \quad x'' \in N_{M_2}(K)$$

such that $(-1)^{x'} = 1$ and $3^{x''} = 1$. Then

$$0^{\tau^{-1}x'} = (-2)^{x'} \neq 0, \quad 2^{\tau x''} = 4^{x''} \neq 2 \quad \text{and} \quad 1^{\tau^{-1}x'} = 1^{\tau x''} = 1.$$

Since $\langle \tau^{-1}x', \tau x'' \rangle \leq N_{M_1}(K)$, we have $N_{M_1}(K) \not\subseteq M_0 \cup M_2$.

To prove assertion (b) it remains to show that M_1 normalizes $\{0, 2\}$. Assume not; then (3.1) implies that M_1 is transitive on $\Delta(1)$. Hence, by (2.1), M is edge-transitive on Γ and $Q = 1$, a contradiction to (a).

Notation. $Q = O_2(G_T)$, $\gamma = (0\ 1\ 2)$, $M = N_G(Q)$. For $X, Y \in S_{\gamma,K}$ we define $X << Y$, if $N_G(X)_0 \leq N_G(Y)_0$. Let $S^*_{\gamma,K}$ be the set of $<<$-maximal elements in $S_{\gamma,K}$.

(6.2) *Suppose that $X \in S_{\gamma,K}$ and $\tilde{M} = N_G(X)$. Then the following hold:*

(a) *$\tilde{M}_1$ normalizes $\{0,2\}$ and $\tilde{M}_1 \not\leq \tilde{M}_0$.*

(b) *$Q_1 \cap \tilde{M}_0 \in Syl_2(\tilde{M}_0) \cap Syl_2(\tilde{M}_2)$.*

Suppose that $X \in S^*_{\gamma,K}$; then no non-trivial characteristic subgroup of $Q_1 \cap \tilde{M}_0$ is normal in $\tilde{M}_0$.

Proof. Assertion (a) follows from the definition of $S_{\gamma,K}$, and (b) is a consequence of (a), (3.1) and (3.2).

Assume that $X \in S^*_{\gamma,K}$ and that $C \neq 1$ is a characteristic subgroup of $Q_1 \cap \tilde{M}_0$, which is normal in $\tilde{M}_0$. From (a) and (b), it follows that C is also normal in $\tilde{M}_1$ and $\tilde{M}_2$. Hence $C \in S_{\gamma,K}$ and $\tilde{M}_0 \leq N_G(C)_0$. The maximality of X implies $\tilde{M}_0 = N_G(C)_0$. Thus $Q_1 \cap \tilde{M}_0 \in Syl_2(G_0)$, and (b) implies that G_1 is 2-closed, a contradiction to the hypothesis.

(6.3) *Suppose that $X \in S^*_{\gamma,K}$ and $\tilde{M} = N_G(X)$. Define $\tilde{\Gamma}$ with respect to $\tilde{M}$ as in (2.9), and let Δ be the connected component of $\tilde{\Gamma}$ containing 0. Then the following hold:*

(a) *$\tilde{M}_\Delta \leq Q_0K$.*

(b) *$\tilde{M}/\tilde{M}_\Delta$ is vertex-transitive on Δ, and 0 has the same valency in Δ as in Γ.*

(c) *$|\tilde{M}_0|_2 = 2^{kn_0}|\tilde{M}_\Delta|$, $k = 1,2,3$ or 4.*

(d) *$O_2(\tilde{M}_0)$ is elementary abelian.*

(e) *If $k \leq 2$, then Sylow 2-subgroups of $\tilde{M}_0$ are elementary abelian.*

(f) *If $k > 2$, then $O_2(\tilde{M}_0)/Z(O^{2'}(\tilde{M}_0))$ is a natural module for $O^{2'}(\tilde{M}_0/O_2(\tilde{M}_0))$.*

(g) *Maximal regular arcs in Δ have length k.*

Proof. Since $\tilde{M}_\Delta$ fixes $\Delta(0)$ pointwise, we get (a) from (3.2).

Set $T = Q_1 \cap \tilde{M}_0$, $W = \tilde{M}/\tilde{M}_\Delta$ and $B = T\tilde{M}_\Delta/\tilde{M}_\Delta$. Then (6.2)(b) implies

$$B \in Syl_2(W_0) \cap Syl_2(W_2),$$

and from (2.9) we get assertion (b). Now (2.1) yields that no non-trivial normal subgroup of $O^{2'}(W_i)$ is normal in $O^{2'}(W_j)$ for $\{i,j\} = \{0,2\}$. Thus we can apply (1.10) and get:

(1) B is elementary abelian of order 2^{kn_0}, $k \leq 2$, or

(2) $O_2(W_i)$ is elementary abelian of order 2^{2n_0} or 2^{3n_0}, and

$O_2(W_i)/Z(O^{2'}(W_i))$ is a natural module for $O^{2'}(W_i/O_2(W_i))$.

It is now easy to verify (c) and (g), and (e) and (f) follow, if we have proved (d). Hence it remains to prove (d).

Set $Y = O^2(O^{2'}(\tilde{M}_0))$; then $\tilde{M}_0 = YKT$. If $[Y, O_2(\tilde{M}_0)] = 1$, then $\phi(T)$ is characteristic in T and normal in $\tilde{M}_0$, and by (6.2)(c), $\phi(T) = 1$. Thus we may assume $V = [Y, O_2(\tilde{M}_0)] \neq 1$ and $Z_1 \leq O_2(\tilde{M}_0)$, and again by (6.2)(c) we can apply (1.6). Since (2.1) implies $[Z_1, Y] \neq 1$, we get $V = [Z_1, Y]$ and $V \leq Z(O_2(\tilde{M}_0))$.

If $T = Q_1$, then, by (1.7), there exists a non-trivial subgroup A in Q_1 which is normal in $O^2(G_1)$ and $\tilde{M}_0$. Since $\tilde{M}_0$ is transitive on $\Delta(0)$ and $O^2(G_1)$ on $\Delta(1)$, (2.1) contradicts $A \neq 1$. Hence $T < Q_1$, and we can choose $t' \in N_{Q_1}(T) \setminus T$ such that $t'^2 \in T$. From (6.2)(a) we have $t \in N_{M_1}(K) \setminus \tilde{M}_0$ such that $t^2 \in T$. Thus, in addition, we may choose t' such that $[t, t'] \in T$. Note that $\langle t', K\rangle$ normalizes $O_2(\tilde{M}_0)$, since $\langle t', K\rangle \leq G_0$ and $O_2(\tilde{M}_0) = Q_0 \cap T$.

First assume that $[O^{2'}(\tilde{M}_\Delta), Y] \neq 1$. Then (1.6) yields

$$V = [O^{2'}(\tilde{M}_\Delta), Y] \leq O^{2'}(\tilde{M}_\Delta).$$

Set $R = \langle (VV^t)^{\langle t', K\rangle}\rangle$. As shown above, $R \leq O_2(\tilde{M}_0)$ and $[R, Y] \leq V \leq R$. Hence R is normal in $\tilde{M}_0$. On the other hand $\langle t, t', K\rangle$ normalizes R, so $R \in S_{\gamma,K}$ and $t' \in N_G(R)_0 \setminus \tilde{M}_0$. This contradicts the maximality of X. Thus we have shown:

(3) $[O^{2'}(\tilde{M}_\Delta), Y] = 1$.

Now assume that $H \neq \phi(O_2(\tilde{M}_0)) \neq 1$. Then (2) and (3) imply $H \leq \tilde{M}_\Delta$ and $[H, Y] = 1$. Since t' normalizes $O_2(\tilde{M}_0)$, it also normalizes H. Thus HH^t is normalized by $\langle t, t', \tilde{M}_0, K\rangle$, and $HH^t \in S_{\gamma,K}$. Again, $t' \in N_G(HH^t)_0 \setminus \tilde{M}_0$, contradicting the maximality of X.

(6.4) *There exists $\tilde{s} \in \{4,5\}$ such that the following hold:*

(a) $|M_0|_2 = 2^{(\tilde{s}-1)n_0}|Q|$.

(b) *$O_2(M_0)$ is elementary abelian.*

(c) *$O_2(M_0)/Z(O^{2'}(M_0))$ is a natural module for $O^{2'}(M_0/O_2(M_0))$.*

(d) *Maximal regular subarcs of T have length $2\tilde{s} - 2$.*

(e) $s \leq 2\tilde{s} - 3$.

Proof. (6.1)(b) yields $Q \in S_{\gamma,K}$. Choose $X \in S^*_{\gamma,K}$ such that $Q << X$. Set $\tilde{M} = N_G(X)$. Then, by definition, $M_0 \leq \tilde{M}_0$, and an application of (6.3)(d) yields $M_0 = \tilde{M}_0$ and, without loss, $Q = X$, since $Q \leq O_2(\tilde{M}_0)$. Thus we may apply (6.3) to M_0. Let k and Δ be as in (6.3). Define $\tilde{s} = k + 1$. Then $\tilde{s} = 2, 3, 4$ or 5, and maximal regular arcs in Δ have length k.

Let $\tilde{\gamma}$ be a maximal regular subarc of T of length r. Then we may assume $\hat{\gamma} \in (\mathcal{O}, r)$ and $r \equiv 0$ (2) (see (2.6)) and, by (6.0)(e), $\hat{\gamma} = (0 \ldots r)$. The restriction of $\tilde{\gamma}$ to Δ is again a maximal regular arc, since Q is normal in $G_{\tilde{\gamma}}$. Hence $r = 2k = 2\tilde{s} - 2$. It remains to show (e), since then $s \geq 5$ implies $\tilde{s} = 4$ or 5.

Assume that $s = 2\tilde{s} - 1$. Then $\gamma_1 = (1 \ldots (2\tilde{s} - 1))$ is also a maximal regular subarc of T, and Q is normal in G_{γ_1}. Pick $\tau^* \in <\tau>$ such that

$$\gamma_1^{\tau^*} = (-(2\tilde{s} - 3) \ldots 1).$$

Then $<G_{\gamma_1}, G_{\gamma_1}^{\tau^*}>$ is a subgroup of M_1, and (3.1) implies that $<G_{\gamma_1}, G_{\gamma_1}^{\tau^*}>$ is transitive on $\Delta(1)$. This contradicts (6.2)(a).

(6.5) $Q \cap Z_i = 1$ *for* $i \in T$.

Proof. It suffices to show $Z_0 \cap Q = Z_1 \cap Q = 1$. Assume that $R = Z_i \cap Q \neq 1$ for some $i \in \{0, 1\}$. Then (6.4) yields $[R, O^{2'}(M_0)] = 1$.

If $i = 1$, then $R \in S_{\gamma,K}$ and $Q_1 \leq N_G(R)_1$, and (6.1)(b) implies

$$Q_1 \in Syl_2(N_G(R)_0).$$

If $i = 0$, then $R \leq Z(L_0)$, and (6.2)(b) implies $R \leq Z_1$. Thus we may assume $i = 1$, $R \in S_{\gamma,K}$ and $Q_1 \in Syl_2(N_G(R)_0)$. But now (6.2) implies that $R \in S^*_{\gamma,K}$ and that no non-trivial characteristic subgroup of Q_1 is normal in $N_G(R)_0$. Hence (6.4)(c) and (1.7) yield a contradiction to (2.1), as in (6.3).

Note that (6.4), (6.5) and (2.10) imply that b_i (for $i \in T$) is an integer.

(6.6) *Suppose that there exists* $i \in T$ *such that* $Q_{i-1} \cap Q_{i+1}$ *is normal in* G_i. *Then* $Q_i = [Q_i, Q_{i-1}][Q_i, Q_{i+1}](Q_{i-1} \cap Q_{i+1})$, *and* $Q_i/Q_{i-1} \cap Q_{i+1}$ *is elementary abelian of order* $2^{2n_{i-1}}$.

The proof is the same as in (4.1).

(6.7) $b_0 > 2$.

Proof. In the following we apply (6.4) without reference. Suppose that $b_0 = 2$. We get $[O^{2'}(M_0), O_2(M_0)] \leq Z_0$, and $Z_0/Z_0 \cap Z(O^{2'}(M_0))$ is a natural module. In particular Z_1Z_0 is normal in M_0 and thus also normal in G_0.

First assume that $C_{L_1}(Z_1) = Q_1$. Then $Q_0 \cap Q_1 = C_{Q_0}(Z_1Z_0)$, and $Q_0 \cap Q_1$ is normal in G_0. Hence $Q_0 \cap Q_1 = Q_1 \cap Q_{-1}$, and $(-1\ 0\ 1\ 2)$ is left singular. This contradicts (2.6) and $s \geq 5$.

Assume now $C_{L_1}(Z_1) \neq Q_1$. Then $Z_1 = Z(L_1) \leq Z_0$, and (3.2)(e) implies $Z(L_0) = 1$. Hence by (1.3):

(1) $b_1 = 3$, $[S, Z_0] = Z_1$, and Z_0 is a natural module for $\overline{L_0}$.

Set $V = <Z_0^{G_1}>$, $A = V \cap Q_0$ and $B = V^{\tau^{-1}} \cap Q_0$, then $[V, Q_1] = Z_1$ and $S = VQ_0$ (since $b_0 = 2$). In particular we get $[Q_1 \cap Q_{-1}, <V, V^{\tau^{-1}}>] = Z_0$, and $Q_1 \cap Q_{-1}$ is normal in G_0. Together with (6.6) and (1.3) we have shown:

(2) (a) $[Q_1, V] = Z_1$,

(b) $Q_0 = AB(Q_1 \cap Q_{-1})$,

(c) $Q_0/Q_1 \cap Q_{-1}$ is direct sum of natural modules for $\overline{L_0}$,

(d) $|Q_0/Q_1 \cap Q_{-1}| = 2^{2n_1}$ and $n_1 \geq n_0$.

Suppose that $n_0 = 1$ and pick $q \in Q^{\#}$. Then (1) and (2)(a) imply $|Z_0| = 4$, $|Z_1| = 2$ and $[q, V] \leq Z_1$. Hence:

(3) $|V/C_V(q)| \leq 2$.

Set $X = C_G(q)$, and note that $BQ_1 = S \in Syl_2(L_1)$. Since $O^{2'}(M_0) \leq X$, X_0 is transitive on $\Delta(0)$. Thus, by (2.1), X_1 is not transitive on $\Delta(1)$. There exists $y \in X_0$ with $1^y = -1$ and $A^y = B$, hence, by (3), $|B/C_B(q)| \leq 2$. Now, (2) implies

$$|C_B(q)Q_1/Q_1| \geq 2^{n_1-1}.$$

Thus $\overline{X_0 \cap X_1}$ and $\overline{X_2 \cap Z_1}$ generate a subgroup of $\overline{L_1}$ with Sylow 2-subgroups of order at least 2^{n_1-1}. Since $<X_0 \cap X_1, X_2 \cap X_1>$ is not transitive on $\Delta(1)$, from [6, II § 8] we have $n_1 = 2$ and $|C_B(q)Q_1/Q_1| = 2$ is the only possibility.

Let N be a normal subgroup of G_1 such that $Z_1 \leq N \leq V$ and N/Z_1 is a minimal normal subgroup of G_1/Z_1. We want to show $N = V$, so assume $N \neq V$. From (2)(a) and (b) we have $[N \cap Q_{-1}, Q_0] \leq N \cap Z_0 = Z_1$ and hence

$$[(N \cap Q_{-1})Z_0, Q_0] \leq Z_1.$$

But $(N \cap Q_{-1})Z_0$ is normal in $<V, V^{\tau^{-1}}>$ and $\overline{<V, V^{\tau^{-1}}>} = \overline{L_0}$. Thus (1) implies $[N \cap Q_{-1}, Q_0] = 1$ and $N \cap Q_{-1} = Z_1$. Now the order of N/Z_1 is at most 2^3 ((2)(d)), and (1.2) yields $|N/Z_1| = 2$ and $[N, K] = 1$. On the other hand, $N \cap Q_0 \not\leq Q_{-1}$ and $K = K_1 = K_{-1}$, since $N \cap Q_{-1} = Z_1$ and $n_0 = 1$. This contradicts (3.1)(b) and (c). We have shown:

(4) V/Z_1 is an irreducible module for $\overline{L_1}$.

Since the orthogonal and the natural module are the only irreducible $GF(2)$-modules for $L_2(4)$ (see (1.12)), we get $|V| = 2^5$. We conclude that $V \cap Q_{-1} = Z_0$ and, by (6.5), $Q \cap V = 1$.

On the other hand $[Q, V^{\tau^{-1}}] \leq Z_{-1} \leq Z_0$ and $[Q, V^{\tau}] \leq Z_3 \leq Z_2$ ((2)(a)), and it follows that $[Q, <B, A^{\tau}>] \leq V$. Since K normalizes $<B, A^{\tau}>$ and $\overline{<B, A^{\tau}>} = \overline{L}_1$, we have $K \leq <B, A^{\tau}>$ and $[Q, K] \leq Q \cap V = 1$. But now $K \leq X_1$, and (3.1)(f) implies that X_1 is transitive on $\Delta(1)$, a contradiction. We have shown:

(5) $n_0 > 1$.

Choose $t \in N_{M_1}(K) \setminus M_0$ with $t^2 \in Q_1$. Note that by (3.2)(b), (1.3) and (1), $K = K_1 \times K_0$, since $Z_1 = Z(L_1)$. If $[K_0, t] = 1$, then the structure of $\mathrm{Aut}(L_2(2^{n_1}))$ implies $[K_0, L_1] \leq Q_1$, in particular $[K_0, B] \leq Q_1 \cap Q_{-1}$. This contradicts (2)(c) and (1.3). Hence $[K_0, t] \neq 1$ and $R = K_0K_0^t \cap K_1 \neq 1$. Note that R centralizes Q.

Since (2)(a) yields $[Q, A] \leq Z_1$, with the 3-subgroup-lemma we get $[A, R, Q] = 1$. Thus $[A, R] \leq O_2(M_0) \leq Q_{-1}$, and it follows that $[L_{-1}, R] \leq Q_{-1}$, since $AQ_{-1} \in Syl_2(L_{-1})$. On the other hand $Z_1Q_{-2} \in Syl_2(L_{-2})$, since $b_1 = 3$, and $[L_{-2}, R] \leq Q_{-2}$. Therefore $C_{L_i}(R)$ is transitive on $\Delta(i)$ for $i = -1, -2$, and (2.1) implies $R = 1$, a contradiction.

(6.8) *There exists no pair* (G, Γ) *which satisfies* (6.0).

Proof. Let (G, Γ) be a counterexample, and let $\tilde{s}$ be the integer defined in (6.4). If $Z_0 \not\leq Z(O^{2'}(M_0))$, then (6.4) implies $b_0 = 2$ which contradicts (6.7). Hence:

(1) $Z_0 \leq Z(O^{2'}(M_0))$.

Now (6.4) and (6.5) yield:

(2) (a) $\tilde{s} = 5$, $s = 5$ or 7, and maximal regular subarcs of T have length 8.

(b) $Z_0 = Z(L_0)$, $b_0 = 4$ and $|Z_0| = 2^{n_0}$.

In addition (6.2)(b) implies $Z_1 \leq Z(S \cap M_0)$ and $Z_1 = Z_0 \times Z_2$. Thus with (1.2) and (1.3):

(3) $b_1 = 3$, $|Z_1| = 2^{2n_0}$, and Z_1 is direct sum of natural modules for $\overline{L_1}$, in particular $n_0 \geq n_1$.

Set $V = \langle Z_1^{L_0} \rangle$ and $V_2 = V^\tau$. Then (6.4) implies

$$V \leq O_2(M_0) \text{ and } Z_1Z_{-1} \leq V \leq Z_1Z_{-1}Q.$$

According to (6.1)(b) and (6.2) there exists $t \in N_{M_1}(K) \setminus M_0$. Since K_0 centralizes $Z(O^{2'}(M_0))$ and K_0^t centralizes $Z(O^{2'}(M_2)) = Z(O^{2'}(M_0^t))$ and

$$Z(O^{2'}(M_0)) \cap Z(O^{2'}(M_2)) = Q,$$

by (1.3) and (6.4)(c) we have $K_0 \cap K_0^t = 1$. Since (3.2)(b) and (c) and (3) imply $K = K_0K_1$ and $|K| \leq |K_0|^2$, we derive:

(4) $K = K_0 \times K_0^t$ and $n_0 = n_1$.

In particular we have $Q \leq C_{Q_0}(K)$ and $Q = C_{Q_0}(K)$ by (3.4).
Hence

$$\tilde{V} = [O_2(M_0), K_4] \leq Z_1Z_{-1},$$

and (2)(b) implies $Z_4O_2(M_0) \in Syl_2(M_0)$. Now the structure of $\mathrm{Aut}(L_2(2^{n_0}))$ yields

$$[K_4, M_0] \leq O_2(M_0).$$

Thus $\tilde{V}$ is normal in M_0 and $[O_2(M_0), M_0] \leq \tilde{V}$. It follows that $Z_1\tilde{V} = V = Z_1Z_{-1}$. Conjugation with τ yields:

(5) $V = Z_1Z_{-1}$ and $V_2 = Z_1Z_3$.

We have $L_0 = \langle Z_{-4}, Z_4\rangle Q_0$, since $b_0 = 4$, and get the following commutator relations:

$$[Q_1 \cap Q_{-1}, Z_4] = [Z_{-1}(Q_2 \cap Q_{-1}), Z_4] = [Z_{-1}, Z_3]Z_2,$$

since $b_1 = 3$ and $[V_2, Q_2] = Z_2$, and

$$[Q_1 \cap Q_0, Z_4] \le V_2 \cap Q_0 = Z_1$$

by (5).
Thus we have $[Q_1 \cap Q_{-1}, Z_4] \le Q_1 \cap Q_{-1}$, and $Q_1 \cap Q_{-1}$ is normal in L_0. From (6.6), (5), the second commutator relation and (1.3), $Q_0/Q_1 \cap Q_{-1}$ is a natural module for $\overline{L_0}$.

Next we show $Q_1 \cap Q_{-1} = QV$. As shown above, $[\langle Z_4, Z_{-4}\rangle, Q_1 \cap Q_{-1}] \le V$, hence

$$Q_1 \cap Q_{-1} = C_{Q_0}(K_0)V,$$

since K_0 operates fixed-point-freely on $Q_0/Q_1 \cap Q_{-1}$ (note that $K_0 \ne 1$ by hypothesis and (4)).

Set $D = C_{Q_0}(K_0)$. Assume $D \not\le Q_2$. Then the structure of $\mathrm{Aut}(L_2(2^n))$ yields

$$[K_0, L_2] \le Q_2 \quad \text{and} \quad L_2 = C_{L_2}(K_0)Q_2$$

which implies $[K_0, V_2] \le Z_2$, since $Z_1 = Z_0Z_2$ and $[Z_0, K_0] = 1$. But then

$$[K_0, L_4] \le Z_2 \le Q_0 \quad \text{and} \quad [K_0, L_0] \le Q_0$$

a contradiction.

Now assume $D \not\le Q_3$; then $[K_0, L_3] \le Q_3$ and $L_3 = C_{L_3}(K_0)Q_3$. On the other hand $b_0 = 4$ and $Z_0Q_4 \in Syl_2(L_4)$, hence $[K_0, L_4] \le Q_4$ and $L_4 = C_{L_4}(K_0)Q_4$. Thus $C_G(K_0)_i$ is transitive on $\Delta(i)$ for $i = 3, 4$. Now (2.1) yields $K_0 = 1$, a contradiction.

We have shown that $D \le Q_3$ and therefore $D \le L_4$. Since $b_0 = 4$, we get

$$D = Z_0(D \cap Q_4) \quad \text{and} \quad Z_0 \cap Q_4 = 1.$$

If $D \cap Q_4 \ne Q$, then $N_{D\cap Q_4}(Q) > Q$ and $N_{D \cap Q_4}(Q) \not\le QZ_0$, but

$$N_{D\cap Q_4}(Q) \le O_2(M_0)$$

and QZ_0 is the centralizer of K_0 in $O_2(M_0)$ (see (6.4) and (3)). This contradiction shows $D \cap Q_4 = Q$ and $D = QZ_0$, in particular $Q_1 \cap Q_{-1} = QV$.

Now we apply (1.5) and (6.4) to Q_0/V and $\overline{L_0}$ and get that Q_0/V is elementary abelian, in particular $[Q_0, Q] \le V$. On the other hand

$$[Q, Q_1] = [Q, Z_4(Q_1 \cap Q_0)] = [Q, Z_{-2}(Q_1 \cap Q_2)] = [Q, Q_1 \cap Q_0] =$$
$$[Q, Q_1 \cap Q_2] \le V \cap V_2 = Z_1 \quad \text{(see (5))}.$$

Now let K^* be the subgroup of maximal order in K such that $[K^*, L_1] \le Q_1$. From (4) we get $|K^*| = |K_0| \ne 1$. This yields

$$[L_1, K^*, Q] \le [Q_1, Q] \le Z_1 \quad \text{and} \quad [K^*, Q, L_1] = 1,$$

hence, with the 3-subgroup-lemma, $[Q, L_1, K^*] \le Z_1$; in particular

$[Q, Q_2, K^*] \le Z_1$. Since $[Q, Q_2]$ is a module for M_2, by (6.5) either $[Q, Q_2] \le Z_2$ or $[Q, Q_2]Z_2 = V_2$. In the first case, $[Q, Q_2, K_2] = 1$ and $[K_2, Q, Q_2] = 1$ and hence, as above, $[Q_2, K_2, Q] = 1$. Conjugation with τ^{-1} yields $[Q_0, K_0, Q] = 1$. But, as we have seen, $Q_0 = [Q_0, K_0](Q_1 \cap Q_{-1})$ and $Q_1 \cap Q_{-1} = QV$; thus $[Q_0, Q] = 1$ which contradicts (6.5).

Assume $[Q, Q_2]Z_2 = V_2$. Then $[V_2, K^*] \le Z_1$ and $[Z_4, K^*] \le Z_1 \le Q_0$, and we get

$$[K^*, L_0] \le Q_0 \quad \text{and} \quad L_0 = C_{L_0}(K^*)Q_0.$$

But now $C_G(K^*)_i$ is transitive on $\Delta(i)$ for $i = 0, 1$, and (2.1) yields $K^* = 1$, a contradiction.

7. Some small cases

(7.0) *Hypothesis and notation.* Hypothesis (3.0) and notation (3.3) hold in this section. In addition we assume that $(0 \dots s)$ is right singular. Note that by (3.5), $O_2(G_{(0\dots s)}) \le Q_s$.

(7.1) *$s \ge 3$, or $G_0 \simeq G_1 \simeq L_2(2^{n_0})$ and $n_0 > 1$.*

Proof. Assume $s \le 2$. Let S be a Sylow 2-subgroup of $L_0 \cap L_1$. If $s = 1$, then $S = Q_1$, and L_1 is 2-closed, a contradiction.

If $s = 2$, then $Q_1 \le Q_2$, and (3.2)(e) yields $Q_1 < Q_2$ or $Q_1 = Q_2 = 1$. In the first case the operation of K implies $Q_2 \in Syl_2(L_2)$ and L_2 is 2-closed, a contradiction. In the second case (after conjugation with τ^{-1}) we find that $L_0 \simeq L_1 \simeq L_2(2^{n_0})$, and S has order 2^{n_0}. The operation of $K = K_0K_1$ and (3.2) yield $K = K_0 = K_1$, $n_0 = n_1 > 1$, $L_0 = G_0$ and $L_1 = G_1$.

(7.2) *Suppose that $s = 3$. Then one of the following holds.*

(a) *$n_0 = 1$, $n_1 > 1$ and:*

(a1) *$O^2(L_0) \simeq C_3$;*

(a2) *Q_1 is elementary abelian and $C_{L_1}(Q_1) = Q_1$;*

(a3) *There exist arcs of length s with stabilizers of even order.*

(b) *$n_0 > 1$, $n_1 > 1$ and:*

(b1) *$O^2(L_0) \simeq L_2(2^{n_0})$ and $O^2(L_1) \simeq L_2(2^{n_1})$;*

(b2) *Sylow 2-subgroups of G_0 are elementary abelian of order $2^{n_0+n_1}$.*

Proof. Set $R = Q_1 \cap Q_2$, then R is in Q_3. Since $Q_1 \cap Q_2$ and $Q_2 \cap Q_3$ are L_2-conjugates, we get $R = Q_2 \cap Q_3$, and R is normal in $\langle Q_1, Q_3\rangle Q_2 = L_2$; in particular

$$L_2/R \simeq L_2(2^{n_0}) \times Q_2/R \quad \text{and} \quad Q_1 \in Syl_2(\langle Q_1, Q_3\rangle).$$

If $C_{<Q_1,Q_3>}(R) \leq R$, we apply (1.7) and get a contradiction to (3.2)(e).

Thus we may assume $C_{<Q_1,Q_3>}(R) \not\leq R$. From (1.9) we get:

(1) $O^2(L_1) \simeq L_2(2^{n_1})'$ and $O^2(L_0) \simeq L_2(2^{n_0})'$, and Sylow 2-subgroups of G_0 are elementary abelian of order 2^{n_0} or $2^{n_0+n_1}$; or

(2) $n_0 = 1$. $O^2(L_0) \simeq C_3$ and $C_{L_1}(Q_1) \leq Q_1$, and Q_1 is elementary abelian.

In case (1) we get $|G_0|_2 = 2^{n_0+n_1}$ since $s = 3$, and (3.2)(b) yields assertion (b).

In case (2), again, (3.2)(b) implies $n_1 > 1$ and assertion (a).

(7.3) *Suppose that* $Z_0 = 1$ *or* $Z_1 = 1$. *Then* $s = 2$ *and* $G_0 \simeq G_1 \simeq L_2(2^{n_0})$, $n_0 > 1$.

Proof. If $Z_i = 1$ for some $i \in \{0,1\}$, then $Q_i = 1$ and $|L_i|_2 = 2^{n_i}$. This implies $s = 2$, and the assertion follows from (7.1).

(7.4) *Suppopse that s is even and* $s > 2$. *Then* $s = 4$, *and* Q_i *is elementary abelian and a natural module for* $\overline{L_i}$ $(i = 0,1)$.

Proof. Let $\gamma = (0\ldots s)$ be a subarc of length s in T, and set $Q = O_2(G_\gamma)$. From (2.6)(a) we get that all arcs of length greater than $s-1$ are singular. Hence (3.5) and (2.10) imply $Q = O_2(G_T)$.

Assume $Q \neq 1$. Then there exists $\delta \in \Gamma$ of minimal distance from 0 such that $Q \not\leq G_\delta$. Let $\tilde{\gamma} = (\delta_0\ldots\delta_n)$, $\delta_0 = 0$ and $\delta_n = \delta$, be the arc joining 0 and δ. The minimality of n yields $Q \leq G_{\delta_i}$ for $i < n$. Now define $\hat{\gamma}$ to be the arc

$$(\delta_{n-s-1}\ldots\delta_{n-1})$$

if $n-1 \geq s$, and

$$(s-(n-1)\ldots\delta_0\ldots\delta_{n-1}),$$

if $n-1 < s$, such that $\hat{\gamma}$ has length s. Then $\hat{\gamma}$ is a subarc of T^g for some G-conjugate of the K-track (T,τ,K) (see (2.6)). In particular $<K^g, Q> \leq G_{\hat{\gamma}}$, and (3.5) and (2.10) imply $Q \leq G_\delta$, a contradiction.

We have shown that G_γ is a $2'$-group. Now (2.7) implies $s = 4$.

Pick $S \in Syl_2(L_0 \cap L_1)$. The transitivity of $L_0 \cap L_1 = N_{L_0}(S)$ on the arcs

$$(0\ 1\ \delta_2\ \delta_3\ \delta_4) \quad \text{and} \quad (1\ 0\ \delta_{-1}\ \delta_{-2}\ \delta_{-4})$$

(see (2.6)) yields

$$|S| = 2^{2n_1+n_1} = 2^{2n_0+n_1}.$$

This implies $n_1 = n_0$ and $|S| = 2^{3n_0}$; in particular $|Q_0| = |Q_1| = 2^{2n_0}$.

Assume first that $C_{L_i}(Q) \leq Q_i$ for $i = 0,1$. Then we apply (1.11) and get either the assertion or $Z_j = Z(L_j)$ for some $j \in \{0,1\}$. In the second case $|Q_j/Z_j| < 2^{2n_0}$, and (1.2) yields a contradiction.

We may assume now without loss that $C_{L_0}(Q_0) \not\leq Q_0$. Applying (1.9) we get $n_0 = 1$ and $L_1 \simeq \Sigma_4$. But now (3.2) implies

$$S = (S \cap O^2(L_0))(S \cap O^2(L_1)) = Q_1,$$

a contradiction.

8. The stabilizer of $\Delta(\alpha)$

(8.0) *Hypothesis and notation.* In this section we assume Hypothesis B and use notation (3.3) as far as it suits this hypothesis. In addition,

$$X_{\Delta(\delta)} = \bigcap_{\rho \in \Delta(\delta)} X_{\delta\rho}$$

for $\delta \in \Gamma$ and $X \leq G$.

(8.1) *Suppose that* Γ *is a tree. Then* $G_{\Delta(\delta)}$ *is solvable and* $O(G_{\Delta(\delta)}) = 1$ *for* $\delta \in \Gamma$*, and one of the following holds.*

(a) *There exists an edge-transitive normal subgroup E of G such that:*

(a1) $O^{2'}(E_\delta)/O_2(E_\delta) \cong L_2(2^{n_\delta})$, *or* $n_\delta = 1$ *and* $O_2(E_\delta) \in Syl_2(E_\delta)$;

(a2) *no proper normal subgroup of E is edge-transitive on* Γ;

(a3) $C_{G_\alpha}(Q_\alpha) \leq Q_\alpha$ *if and only if* $C_{G_\beta}(Q_\beta) \leq Q_\beta$.

(b) $s = 3$, and $\{G_\alpha, G_\beta\}$ is parabolic of type

$\mathrm{Aut}(L_2(2^{n_\beta})) \int \mathrm{Aut}(L_2(2^{n_\alpha}))$ or $\mathrm{Aut}(L_2(2^{n_\alpha})) \int \mathrm{Aut}(L_2(2^{n_\beta}))$.

(c) *(possibly after changing notion)* $n_\beta = 1$ *and* $s = 3$, Q_α *is elementary abelian,*

$$G_\alpha/Q_\alpha \simeq H \leq \mathrm{Aut}(L_2(2^{n_\alpha})),$$

Q_α *is isomorphic to a submodule of the natural permutation GF(2)-module for* $G_\alpha/Q_\alpha, G_\beta = G_{\alpha\beta}W$, $W \simeq \Sigma_3$, *and W is normal in* G_β.

Proof. The first property is obvious:

(1) $G_\delta/G_{\Delta(\delta)}$ is isomorphic to a subgroup of $\mathrm{Aut}(L_2(2^{n_\delta}))$ which contains $L_2(2^{n_\delta})'$, $\delta \in \Gamma$.

Since $O(G_{\Delta(\alpha)})$ is normal in $G_{\alpha\beta}$, we get $[O(G_{\Delta(\alpha)}), G_{\alpha\beta}] \leq O(G_{\Delta(\alpha)}) \leq O(G_{\alpha\beta})$. Hence (1) and the structure of $\mathrm{Aut}(L_2(2^{n_\beta}))$ yield $O(G_{\Delta(\alpha)}) \leq O(G_{\Delta(\beta)})$. The same argument applied to $O(G_{\Delta(\beta)})$ shows $O(G_{\Delta(\alpha)}) = O(G_{\Delta(\beta)})$. Hence $O(G_{\Delta(\alpha)})$ is normal in $\langle G_\alpha, G_\beta \rangle = G$. We get:

(2) $O(G_{\Delta(\alpha)}) = O(G_{\Delta(\beta)}) = 1$.

Let H_δ be the largest perfect normal subgroup in $G_{\Delta(\delta)}$. Again the structure of $\mathrm{Aut}(L_2(2^{n_\delta}))$ yields $H_\alpha = H_\beta$ and:

(3) $H_\alpha = H_\beta = 1$, in particular $G_{\Delta(\delta)}$ is solvable for $\delta \in \Gamma$.

If $Q_\alpha \leq G_{\Delta(\beta)}$ and $Q_\beta \leq G_{\Delta(\alpha)}$, then the above argument shows $Q_\alpha = Q_\beta = 1$, and (2) and (3) imply $G_{\Delta(\alpha)} = G_{\Delta(\beta)} = 1$, and (a) holds.

Thus we may assume, without loss, $Q_\alpha \not\leq G_{\Delta(\beta)}$. Since Q_α is normal in $Q_{\alpha\beta}$, we get:

(4) $[Q_\alpha, G_{\Delta(\beta)}] \leq Q_\alpha \cap G_{\Delta(\beta)} \leq Q_\beta$.

Set $W_\beta = \langle Q_\alpha^{G_\beta} \rangle Q_\beta$. Then (4) implies that every chief factor of W_β which is in $W_\beta \cap G_{\Delta(\beta)}$ but not in Q_β is central. Hence, [6, V 25.7] and the structure of $\mathrm{Aut}(L_2(2^{n_\beta}))$ yield $W_\beta \cap G_{\Delta(\beta)} = Q_\beta$ and $W_\beta/Q_\beta \simeq L_2(2^{n_\beta})$.

Assume that $Q_\beta \not\leq G_{\Delta(\alpha)}$. Then we define $W_\alpha = \langle Q_\beta^{G_\alpha} \rangle Q_\alpha$ and, as above, get

$$W_\alpha \cap G_{\Delta(\alpha)} = Q_\alpha \quad \text{and} \quad W_\alpha/Q_\alpha \simeq L_2(2^{n_\alpha}).$$

Set

$$E = \langle O^2(W_\alpha)(O^2(W_\beta) \cap G_\alpha),\ O^2(W_\beta)(O^2(W_\alpha) \cap G_\beta) \rangle$$

and $T_\delta = C_{G_\delta}(Q_\delta)$ for $\delta = \alpha, \beta$. Then (2.3) and (2.4) imply that (a1) and (a2) hold in E, and (3) and (4) yield $T_\delta \cap G_{\Delta(\delta)} = Z(Q_\delta)$. Hence $T_\delta \not\leq Q_\delta$ if and only if $C_{W_\delta}(Q_\delta) \not\leq Q_\delta$.

Thus either case (a) holds for E, or we have one of the following:

(I) $Q_\beta \not\leq G_{\Delta(\alpha)}$, $C_{W_\beta}(Q_\beta) \not\leq Q_\beta$ and $C_{W_\alpha}(Q_\alpha) \leq Q_\alpha$,

(II) $Q_\beta \not\leq G_{\Delta(\alpha)}$, $C_{W_\alpha}(Q_\alpha) \not\leq Q_\alpha$ and $C_{W_\beta}(Q_\beta) \leq Q_\beta$,

(III) $Q_\beta \leq G_{\Delta(\alpha)}$.

Since (I) and (II) only differ in notation, we may assume without loss of generality that we are in case (I) or (III).

Assume (III). This implies $Q_\beta \leq Q_\alpha$ and $Q_\alpha \in Syl_2(W_\beta)$. By (2.1), $\langle W_\beta, O^2(G_\alpha)Q_\alpha \rangle$ is edge-transitive on Γ. Thus no non-trivial subgroup of W_β is normalized by $O^2(G_\alpha)Q_\alpha$. Hence (1.7) implies $C_{W_\beta}(Q_\beta) \not\leq Q_\beta$. So we have shown in both cases (I) and (III):

(5) $C_{W_\beta}(Q_\beta) \not\leq Q_\beta$.

Then $W_\beta = Q_\beta C_{W_\beta}(Q_\beta)$, and $\phi(Q_\alpha)$ is normal in the edge-transitive subgroup $\langle W_\beta, G_\alpha \rangle$. This implies:

(6) $W_\beta = Q_\beta W_\beta^*$, $W_\beta^* \simeq L_2(2^{n_\beta})$, and Q_α is elementary abelian.

Set $R_\beta = \bigcap_{\beta \neq \beta' \in \Delta(\alpha)} G_{\Delta(\beta')}$. The subgroup $\bigcap_{\beta' \in \Delta(\alpha)} G_{\Delta(\beta')}$ is normal in $\langle G_\alpha, W_\beta \rangle$. Hence, as above,

$$\bigcap_{\beta' \in \Delta(\alpha)} G_{\Delta(\beta')} = 1 = R_\beta \cap G_{\Delta(\beta)}.$$

For subsets $\{\beta_1 \ldots \beta_k\}$ in $\Delta(\alpha)$ we define

$$Y_k = \bigcap_{i=1}^{k} G_{\Delta(\beta_i)} \quad \text{and} \quad \tilde{Y}_k = \prod_{i=1}^{k} R_{\beta_i}.$$

Assume first that $R_\beta = 1$. Since $[Q_\alpha K_\beta] \leq R_\beta$ it follows that $K_\beta = 1$ and $2^{n_\beta} = 2$. We know that $Z(Q_{\beta_i}Q_\alpha) = \langle a_i \rangle$ is cyclic of order 2, since, by (3.2)(e), $Z(Q_{\beta_i}Q_\alpha) \cap Q_{\beta_j} = 1$ for $i \neq j$.

If $\Pi_{i=1}^k a_i = 1$, then $a_k = \Pi_{i=1}^{k-1} a_i \in Q_{\beta_k}$ and thus $k - 1 \equiv 0$ (2). On the other hand, if $k < 2^{n_\alpha} + 1$, then $a_1, \ldots, a_k \in Q_\alpha \setminus Q_{\beta_{k+1}}$ yields $k \equiv 0$ (2), a contradiction. This shows that Q_α is isomorphic to the non-trivial submodule of a natural permutation $GF(2)$-module for G_α / Q_α.

Now assume that $R_\beta \neq 1$, and let k be maximal such that

$$\tilde{Y}_k = \mathop{\mathsf{X}}_{i=1}^{k} R_{\beta_i} \quad \text{and} \quad \tilde{Y}_k \cap Y_k = 1,$$

and assume that there exists $\beta_{k+1} \in \Delta(\alpha) \setminus \{\beta_1, \ldots, \beta_k\}$. Then $R_{\beta_{k+1}} \leq Y_k$ and hence $\tilde{Y}_{k+1} = \mathsf{X}_{i=1}^{k+1} R_{\beta_i}$. By the maximality of k there exists

$$1 \neq ry \in \tilde{Y}_{k+1} \cap Y_{k+1}$$

for $r \in R_{\beta_{k+1}}^{\#}$ and $y \in \tilde{Y}_k$. Then

$$y \in G_{\Delta(\beta_{k+1})} \quad \text{and} \quad r \in Y_{k+1} y^{-1} \subseteq G_{\Delta(\beta_{k+1})}$$

which contradicts $R_{\beta_{k+1}} \cap G_{\Delta(\beta_{k+1})} = 1$.

We have shown that there exists a normal subgroup $W = \mathsf{X}_{\beta' \in \Delta(\alpha)} R_{\beta'}$ in G_α, and R_β is a subgroup of $\mathrm{Aut}(W_\beta^*)$ containing the normalizer of a Sylow 2-subgroup of W_β^*. In particular, $(R_\beta \cap Q_\alpha) W_\beta^{*\prime} \simeq L_2(2^{n_\beta})$, and $(R_\beta \cap Q_\alpha) W_\beta^{*\prime}$ is normal in G_β, since $G_{\alpha\beta}$ normalizes R_β. According to (6) we may choose $W_\beta = (R_\beta \cap Q_\alpha) W_\beta^{*\prime}$.

There exists an involution $t \in W_\beta$ with $\alpha^t = \alpha'$ for $\alpha \neq \alpha' \in \Delta(\beta)$. Set $X = G_\alpha \cap G_\beta \cap G_{\alpha'}$. Then $[X, t] \leq W_\beta \cap G_\alpha \cap G_{\alpha'} = 1$. Hence a subgroup X_0 in X is transitive on $\Delta(\alpha) \setminus \{\beta\}$, if and only if it is also transitive on $\Delta(\alpha') \setminus \{\beta\}$. This shows that $s \geq 3$ and that there exists no regular arc $(\alpha\beta\alpha'\beta')$ of length 3, and since $Q_\alpha \not\leq G_{\Delta(\beta)}$ we get $s = 3$.

Assume that $n_\beta > 1$. Then $C_{G_\alpha}(W) = 1$ and $G_\alpha \leq \mathrm{Aut}(W)$ (here and in the following we interpret the natural monomorphism into the automorphism group as inclusion). Set

$$W_0 = \mathop{\times}_{\beta' \in \Delta(\alpha)} \mathrm{Aut}(R_{\beta'}).$$

As $G_{\Delta(\alpha)}$ fixes every $\beta' \in \Delta(\alpha)$ and $\mathrm{Aut}(W) = \mathrm{Aut}(R_\beta) \wr \Sigma_{|\Delta(\alpha)|}$, we get

$$G_{\Delta(\alpha)} = W_0 \cap G_\alpha \leq G_\alpha \leq \mathrm{Aut}(R_\beta) \wr \Sigma_{|\Delta(\alpha)|}.$$

On the other hand $G_\alpha W_0 / W_0 \simeq H \leq \mathrm{Aut}(L_2(2^{n_\alpha}))$, and G_α operates in its natural permutation representation on $\{R_{\beta'} / \beta' \in \Delta(\alpha)\}$. But then $G_\alpha W_0$ is conjugate in $\mathrm{Aut}(W)$ to $\mathrm{Aut}(R_\beta) \wr H$. Hence we may assume

$$R_\beta \wr L_2(2^{n_\alpha})' \leq G_\alpha \leq \mathrm{Aut}(R_\beta) \wr \mathrm{Aut}(L_2(2^{n_\alpha})).$$

It is easy to see with Schur's lemma that $\mathrm{Aut}(R_\beta)$ is a subgroup of $\mathrm{Aut}(L_2(2^{n_\beta}))$, hence

$$G_\alpha \leq \mathrm{Aut}(R_\beta) \wr \mathrm{Aut}(L_2(2^{n_\alpha})) \leq \mathrm{Aut}(L_2(2^{n_\beta})) \wr \mathrm{Aut}(L_2(2^{n_\alpha})).$$

With the same argument we get

$$G_\beta \le \operatorname{Aut}((\underset{\beta\neq\beta\in\Delta(\alpha)}{\times} R_{\beta'}) \times L_2(2^{n_\beta}))$$
$$\le \operatorname{Aut}(\underset{\beta'\in\Delta(\alpha)}{\times} L_2(2^{n_{\beta'}}))$$
$$\le \operatorname{Aut}(L_2(2^{n_\beta})) \wr \Sigma_{|\Delta(\alpha)|}.$$

Set

$$\tilde{W}_0 = \underset{\beta'\in\Delta(\alpha)}{\times} \operatorname{Aut}(L_2(2^{n_{\beta'}})).$$

Then $G_{\Delta(\alpha)}W_\beta \le \tilde{W}_0$, and $G_\beta/G_\beta \cap \tilde{W}_0$ is isomorphic to a subgroup of the normalizer of a Sylow 2-subgroup in $\operatorname{Aut}(L_2(2^{n_\alpha}))$. In particular the permutation representation of $G_\beta/G_\beta \cap \tilde{W}_0$ on $\{R_{\beta'}/\beta \neq \beta' \in \Delta(\alpha)\}$ is unique, and G_β is in $\operatorname{Aut}(X_{\beta'\in\Delta(\alpha)}L_2(2^{n_{\beta'}}))$ conjugate to a subgroup of

$$\operatorname{Aut}(L_2(2^{n_\alpha})) \wr \operatorname{Aut}(L_2(2^{n_\alpha})).$$

This shows assertion (b), if $n_\beta > 1$.

Assume $n_\beta = 1$. Then W is elementary abelian of order $2^{2^{n_\alpha+1}}$, and G_β is no longer a subgroup of $\operatorname{Aut}(W)$. But now $O^2(G_{\Delta(\alpha)})$ is normal in $<G_\alpha, G_\beta> = G$. Hence $G_{\Delta(\alpha)} = Q_\alpha = W$, and assertion (c) follows.

9. Finite graphs

(9.0) *Hypothesis and notation.* In this section we assume Hypothesis (3.0) and use notation (3.3). In addition:

(1) $\max\{n_0, n_1\} > 1$,

(2) $s \ge 3$,

(3) arcs of length s have stabilizers of odd order in G.

It follows from (3) and (3.1)(e) that there are involutions

$$t_0 \in N_{L_0}(K) \setminus L_1 \quad \text{and} \quad t_1 \in N_{L_1}(K) \setminus L_0.$$

Hence we may assume $\tau = t_0 t_1$ (see (2.8)); then $\tau^{t_i} = \tau^{-1}$ and $k^{t_0} = -k$ and $k^{t_1} = 2 - k$ for $k \in T$. Furthermore

$$\operatorname{Aut}^0(\Gamma) = <x \in \operatorname{Aut}(\Gamma) \,/\, 0^x \in 0^G>,$$

$X = N_{\operatorname{Aut}^0(\Gamma)}(G)$, $\mathscr{X} = \{T^g/g \in X\}$ and

$$\mathscr{X}_{2(s-1)} = \{\gamma/\gamma \text{ arc of length } 2(s-1) \text{ and } \gamma \subseteq T^g \in \mathscr{X}\};$$

$\gamma(\delta_1, \delta_2)$ denotes the unique arc starting at δ_1 which joins the two vertices δ_1 and δ_2.

(9.1) *Suppose that γ is an arc of length s. Then γ is contained in a unique element of $\mathcal{X}$.*

Proof. Since γ is conjugate to $(0\ldots s)$ or $(1\ldots s+1)$ (see (2.6)), γ is contained in some element of $\mathcal{X}$.

Now assume that γ is a counterexample. Then $\gamma \subseteq T \cap T^g$ for some $g \in X$ and $T \neq T^g$, and without loss of generality we may assume

$$T \cap T^g = (0\ldots w), \quad w \geq s.$$

In particular $G_{(0\ldots w)} = K = K^g$, since G_γ has odd order. Thus

$$(0\ldots w) \quad \text{and} \quad (0^g\ldots w^g)$$

are both subarcs of T^g.

First suppose that $w \equiv 1\ (2)$. Then $\Delta(0)$ or $\Delta(w)$ contains more than three elements which contradicts $K = K^g$ and (3.1)(b). Hence $w \equiv 0\ (2)$, and there exists $\tau^* \in \langle \tau^g \rangle$ such that

$$0^{g\tau^*} = 0 \quad \text{and} \quad w^{g\tau^*} = w$$

or

$$0^{g\tau^*} = w \quad \text{and} \quad w^{g\tau^*} = 0.$$

In the first case $g\tau^* \in G_{(0\ldots w)} = K^g$, and $g\tau^*$ and τ^* normalize T^g. It follows that $\langle g \rangle$ normalizes T^g, contradicting $T \neq T^g$.

In the second case there exists a reflection t' on T^g such that

$$g\tau^* t' \in G_{(0\ldots w)}.$$

Thus as above, t', τ^* and g normalize T^g, a contradiction.

(9.2) *Let $X = O^{2'}(\langle G_{(0\ldots s-1)}, G_{(s-1\ldots 2(s-1))}\rangle)$. Then:*

(a) $X/X \cap Q_{s-1} \simeq L_2(2^{n_{s-1}})$.

(b) *K normalizes X.*

(c) *$X \cap Q_{s-1}$ is a natural module for $X/X \cap Q_{s-1}$, or $X \cap Q_{s-1} = 1$.*

Proof. We define

$$T_1 = O_2(G_{(0\ldots s-1)}), \quad T_2 = O_2(G_{(s-1\ldots 2(s-1))}),$$

$K^* = C_K(T_1)$ and $R = \langle T_1, T_2 \rangle \cap Q_{s-1}$. Since K operates on T_1 and T_2 and arcs of length s have stabilizers of odd order, we get together with [6, I 14.4]:

(1) T_i is elementary abelian of order $2^{n_{s-1}}$ and $T_i \cap Q_{s-1} = 1$, $i = 1,2$.

(2) $T_iQ_{s-1} \in Syl_2(L_{s-1})$, $i = 1,2$, and $\langle T_1, T_2 \rangle / R \simeq L_2(2^{n_{s-1}})$.

(3) K^* centralizes $\langle T_1, T_2 \rangle$.

(4) There exists a complement $X \simeq L_2(2^{n_{s-1}})$ in $\langle T_1, T_2 \rangle$ which contains K_{s-1}.

Hence it suffices to show that R is a natural module or $R = 1$.

If $s = 3$, we apply (7.2) and get $<T_1, T_2> \leq O^2(L_{s-1})$ and $R = 1$, since $[T_i, K] = T_i$ for $i = 1,2$.

If $s \equiv 0$ (2), we apply (7.3) and get that $R = 1$ or $R = Q_{s-1}$ is a natural module.

Hence we may assume $s \geq 5$ and $s \equiv 1$ (2), in particular $\mu = (s-1)/2$ is an integer and a vertex in T.

Suppose first that $K^* \neq 1$. If $C_{Q_{s-1}}(K^*) \not\leq Q_{s-2}$, then the operation of K on $C_{Q_{s-1}}(K^*)$ yields $Q_{s-2}C_{Q_{s-1}}(K^*) \in Syl_2(L_{s-2})$ and $[L_{s-2}, K^*] \leq Q_{s-2}$. Together with (2) and (3) this contradicts (2.1).

We have shown:

(5) $C_{Q_{s-1}}(K^*) \leq Q_{s-2}$.

Since $<T_1, T_2>$ operates transitively on $\Delta(s-1)$, we get

$$R \leq C_{Q_{s-1}}(K^*) \leq \bigcap_{\rho \in \Delta(s-1)} Q_\rho = H.$$

Now, an application of (4.6), (4.8) and (5.2) yields one of the following cases:

(i) $H = 1$.

(ii) $H = Z_{s-1} = Z(L_{s-1})$ and $H \leq G_\mu$.

(iii) $s = 7$, $H = T_3 Z_{s-1}$, where $T_3 = O_2(G_{(\mu\ldots s+\mu-1)})$, and Z_{s-1} is a natural module for $\overline{L}_{s-1}$.

In case (i) we get $R = 1$. In case (ii), $R \leq Z(<T_1, T_2>)$, and (4) and the operation of K imply $R = 1$.

Assume now case (iii). With the help of (4.8) and (5.2) it is easy to check that $[T_1, K_\mu] = 1$ and hence $K_\mu \leq K^*$. On the other hand, $\mu = 3$ and $s - 1 = 6$, and (3.2) implies $K_\mu = K^*$. Since T_3 stabilizes the maximal regular arc $(\mu\ldots s+\mu-1)$, we get $T_3 \cap Q_\mu = 1$ and $C_{T_3}(K_\mu) = 1$ or $K_\mu = 1 = K^*$. So $C_{T_3}(K_\mu) = 1$ and $R = 1$ or $R = Z_{s-1}$, and the assertion holds.

Suppose now that $K^* = 1$. Then we are in case (5.2)(a) or (b) and $K = K_{s-1}$. If $s = 5$, then $C_{L_3}(K) = Z_3 \times <Z_1, Z_5>$ and $|Z_3| = 2$ and $<Z_1, Z_5> \simeq \Sigma_3$. If $s = 7$, then $C_{L_5}(K) = <Z_7, Z_3> \simeq \Sigma_4$. Let d be an element of order 3 in $C_{L_{s-2}}(K)$, and let Ω be the set of all elementary abelian subgroups F in Q_{s-2} such that $F \cap Q_{s-3} \cap Q_{s-1} = 1$, $|F| = 2^{n_{s-1}}$ and $[K, F] = F$. If (5.2)(a) holds, it is easy to check that $\Omega = \{T_1, T_1^d, T_1^{d^{-1}}\}$. We want to show the same, if (5.2)(b) holds.

Define $\overline{Q}_{s-2} = Q_{s-2}/Q_{s-3} \cap Q_{s-1}$ and $\overline{\Omega} = \{\overline{F}/F \in \Omega\}$. Clearly $|\overline{\Omega}| \geq 3$, since $\overline{T}_1$, $\overline{T}_1^d$ and $\overline{T}_1^{d^{-1}}$ are contained in $\overline{\Omega}$. Assume $|\overline{\Omega}| > 3$, then the operation of d implies $|\overline{\Omega}| \geq 6$, and there are at least 42 images of involutions and at most 21 images of 4-elements in $\overline{Q}_{s-2}$. We now take a factor group $\tilde{Q}$ of Q_{s-2} which is a non-abelian extension of $\overline{Q}_{s-2}$ of order 2^7. All such possible extensions contain more than 21 4-elements. Hence we have shown:

(6) $|\overline{\Omega}| = 3$.

Now let $T_3 = O_2(G_{(2\ldots 8)})$. Then $Q_4 \cap Q_6 = T_3Z_5$, and there exists a reflection t on T in L_4 which inverts the elements of K and interchanges T_1 and T_3. Since $|K| > 3$, there are only two K-modules of order 2^3 in $T_1T_3Z_5/Z_5$, namely T_1Z_5/Z_5 and T_3Z_5/Z_5. On the other hand

$$Z_5 = Z_6Z_6^d \leq C_{Q_5}(K) \quad \text{and} \quad \Omega \cap T_1Z_5 = \{T_1\};$$

thus we have shown for $s = 5$ and 7, $\Omega = \{T_1, T_1^d, T_1^{d^{-1}}\}$. One of these three elements in Ω, say T_1^d, is contained in Q_{s-1} and since $d \in <Z_s, Z_{s-4}>$, there exists $z \in Z_s$ such that $T_1^{d^{-1}} = T_1^z$. Hence we have shown:

(7) T_1 and T_1^z are the only complements for Q_{s-1} in T_1Q_{s-1} which are normalized by K.

Now reflecting T with $t_0^{\tau^3}$ yields:

(8) T_2 and $T_2^{\tilde{z}}$ are the only complements for Q_{s-1} in T_2Q_{s-1} ($\tilde{z} \in Z_{s-1}$) which are normalized by K.

If we now take Y as described in (4), we can fine $x \in <Z_{s-4}, Z_s>$ such that $Y^x = <T_1, T_2>$.

(9.3) *Suppose that* $\alpha_1, \alpha_2, \alpha_3 \in \Gamma$, $\gamma(\alpha_1, \alpha_3) \in \mathscr{X}_{2(s-1)}$ *and*

$$d(\alpha_2, \alpha_3) = 2(s-1).$$

Then $\gamma(\alpha_2, \alpha_3) \in \mathscr{X}_{2(s-1)}$.

Proof. We use the following notation: $\gamma_i = \gamma(\alpha_j, \alpha_k)$ for $\{i, j, k\} = \{1, 2, 3\}$,

$$\gamma_1 \cap \gamma_2 \cap \gamma_3 = \{\lambda\},\ T_i = O_2(G_{\gamma(\alpha_i, \lambda)}),$$

$L = <T_1, T_2>$, t_λ is a reflection on γ_3 contained in $O_2(G_{(\lambda\,\delta_1\ldots\delta_{s-1})})$ for some arc $(\lambda\delta_1\ldots\delta_{s-1})$ of length $s-1$.

By (9.2), $L/O_2(L) \simeq L_2(2^{n_\lambda})$, and $O_2(L)$ is a natural module or $O_2(L) = 1$. It is easy to check that $T_1^\nu \cap T_2 \neq 1$ ($\nu \in L$) implies $T_1^\nu = T_2$.

There exists $t \in T_1$ which interchanges the two vertices in $\Delta(\lambda) \cap \gamma_1$. Hence γ_2 and γ_3^t have an arc of length s in common. It follows from (9.1) that $\gamma_2 = \gamma_3^t$. The structure of $L_2(2^{n_\lambda})$ yields the existence of $t' \in T_2$ such that $<t, t'>Q_\lambda/Q_\lambda \simeq \Sigma_3$, and the structure of L implies $<t, t'> \simeq \Sigma_3$. Note that the relation $t^{t't} = t'$ holds.

Set $\nu = t'^t$, then $t' \in T_1^\nu \cap T_2$ and $T_1^\nu = T_2$. On the other hand $T_1^{t_\lambda} = T_2$, thus νt_λ normalizes T_1 and T_2. From the structure of L and L_λ we conclude that $[L, \nu t_\lambda] = 1$. By (7.4), this is only possible if $s \equiv 1$ (2). Hence νt_λ stabilizes the arc $(\lambda_1\ldots\lambda\ldots\lambda_2)$ of length $s-1$ where λ_i is the midpoint in $\gamma(\alpha_i, \lambda)$. So νt_λ has order 1 or 2, and ν and t_λ commute. Therefore $\gamma(\lambda, \alpha_3)$ and $(\lambda\ldots\delta_{s-1})$ have an arc $(\lambda\ldots\lambda_3)$ of length $(s-1)/2$ in common. Since both ν and t_λ fix two vertices in $\Delta(\lambda_3)$, we get $\nu, t_\lambda \in Q_{\lambda_3}$, and ν and t_λ fix the elements in $\Delta(\lambda_3) \cap \gamma_2$. Thus νt_λ stabilizes $\tilde{\gamma} = (\lambda_1\ldots\lambda\ldots\lambda_3\mu)$ where $\mu \in \Delta(\lambda_3) \cap \gamma_2$ and $\mu \notin (\lambda\ldots\lambda_3)$. Since $\tilde{\gamma}$ has length s, it follows that $\nu t_\lambda = 1$ and $\nu = t_\lambda$. Hence $\alpha_1^\nu = \alpha_1^{t_\lambda} = \alpha_2$ and

$\alpha_3^{t\lambda} = \alpha_3^{\nu} = \alpha_3$, since $\nu = t'^t \in T_2^t = T_3$, and we have shown $\gamma_2^{t\lambda} = \gamma_1 \in K_{2(s-1)}$.

(9.4) *There exists an equivalence relation* $\approx$ *on* Γ *such that:*

(a) $\tilde{\Gamma} = \Gamma/\approx$ *is an* $(s-1)$*-gon (where two equivalence classes are adjacent, iff they contain some pair of adjacent vertices).*

(b) X *operates on* $\tilde{\Gamma}$.

(c) X_0 *and* X_1 *operate faithfully on* $\tilde{\Gamma}$.

Moreover, for $\tilde{X} = X^{\tilde{\Gamma}}$, *one of the following holds:*

(1) $s = 3$, $\tilde{G} \simeq L_2(2^{n_0}) \times L_2(2^{n_1})$, $\tilde{X} \lesssim \mathrm{Aut}(L_2(2^{n_0}) \times L_2(2^{n_1}))$, *and* $\{X_0, X_1\}$ *is parabolic of type* $L_2(2^{n_0}) \times L_2(2^{n_1})$.

(2) $s = 4$, $\tilde{G} \simeq L_3(2^{n_0})$, $\tilde{X} \lesssim \mathrm{Aut}(L_3(2^{n_0}))$ *and* $\{X_0, X_1\}$ *is parabolic of type* $L_3(2^{n_0})$.

(3) $s = 5$, $\tilde{G} \simeq Sp_4(2^{n_0})$ *or* $U_4(2^{n_0})$, $\tilde{X} \lesssim \mathrm{Aut}(Sp_4)2^{n_0}))$ *(resp.* $Aut(U_4(2^{n_0})))$, *and* $\{X_0, X_1\}$ *is parabolic of type* $Sp_4(2^{n_0})$ *(resp.* $U_4(2^{n_0})$*).*

(4) $s = 7$, $\tilde{G} \simeq G_2(2^{n_0})$ or ${}^3D_4(2^{n_0})$, $\tilde{X} \lesssim \mathrm{Aut}(G_2(2^{n_0}))$ *(resp.* $\mathrm{Aut}\,({}^3D_4(2^{n_0}))$*), and* $\{X_0, X_1\}$ *is parabolic of type* $G_2(2^{n_0})$ *(resp.* ${}^3D_4(2^{n_0})$*).*

Proof. For $\delta \in \Gamma$ we define:

$$\Gamma_\delta = \{\lambda \in \Gamma / \gamma(\delta, \lambda) \in \mathscr{K}_{2(s-1)}\} \cup \{\delta\}.$$

Note that $\gamma(\delta,\lambda) \in \mathscr{K}_{2(s-1)}$ implies $\gamma(\lambda,\delta) \in \mathscr{K}_{2(s-1)}$, since the elements in $\mathscr{K}$ allow reflections. X operates on the graph $\hat{\Gamma}$ with vertex set $\{\Gamma_\delta / \delta \in \Gamma\}$, where two vertices Γ_δ and $\Gamma_{\delta'}$ are adjacent iff $\delta \neq \delta'$ and $\{\delta, \delta'\} \subseteq \Gamma_\delta \cap \Gamma_{\delta'}$. Now we define an equivalence relation $\approx$ on Γ:

$\delta \approx \delta'$ for $\delta, \delta' \in \Gamma$ iff Γ_δ is in $\hat{\Gamma}$ in the same connected component as $\Gamma_{\delta'}$.

Set $\tilde{\Gamma} = \Gamma/\approx$ and denote by $\tilde{\delta}$ the equivalence class of $\delta \in \Gamma$. Two vertices $\tilde{\alpha}, \tilde{\beta}$ are adjacent iff there exist $\alpha' \in \tilde{\alpha}$ and $\beta' \in \tilde{\beta}$ such that $\beta' \in \Delta(\alpha')$.

It is easy to see that X operates on $\tilde{\Gamma}$. We want to show first that $\tilde{\Gamma}$ is nontrivial:

(1) If δ has distance less than $2(s-1)$ from 0 (resp. 1), then $\tilde{\delta} \neq \tilde{0}$ (resp. $\tilde{\delta} \neq \tilde{1}$) or $\delta = 0$ (resp. $\delta = 1$).

Let $\delta \neq 0$ be of distance less than $2(s-1)$ from 0. Assume that $\delta \in \tilde{0}$. Then there exist elements $\delta_0, \delta_1, \ldots, \delta_n$ such that $\delta_0 = 0$ and $\delta_n = \delta$ and Γ_{δ_i} is adjacent to $\Gamma_{\delta_{i+1}}$ in $\hat{\Gamma}$ for $i = 0, \ldots, n-1$, which means

$$\gamma(\delta_i, \delta_{i+1}) \in \mathscr{K}_{2(s-1)}.$$

Let n be minimal with these properties.

There exists δ_k, $0 < k < n$, such that $d(\delta_0, \delta_k)$ is maximal. Set

$$\gamma_1 = \gamma(\delta_k, \delta_{k+1}) \cap \gamma(\delta_k, \delta_{k-1}) = (\delta_k \dots \lambda),$$
$$\gamma_2 = (\lambda \dots \delta_{k+1}) \quad \text{and} \quad \gamma_3 = (\lambda \dots \delta_{k-1}).$$

Since γ_1 is contained in at least two different elements of $\mathscr{X}$, it has length less than s. On the other hand $d(\delta_0, \lambda) + |\gamma_i| \leq d(\delta_0, \delta_k)$ for $i = 2,3$. Hence the length of γ_i is $s - 1$ for $i = 1,2,3$, and we can apply (9.3) to get

$$\delta_{k-1}, \delta_{k+1} \in \Gamma_{\delta_{k-1}} \cap \Gamma_{\delta_{k+1}}.$$

But now $\delta_0, \dots, \delta_{k-1}, \delta_{k+1}, \dots, \delta_n$ have the same properties as $\delta_0, \dots, \tilde{\delta}_n$, contradicting the minimality of n.

The same argument holds for 1 in place of 0.

(2) Suppose that $\tilde{\delta}$ and $\tilde{\lambda}$ are adjacent in $\tilde{\Gamma}$. Then for every $\delta \in \tilde{\delta}$ there exists $\lambda \in \tilde{\lambda}$ such that $\delta \in \Delta(\lambda)$.

By definition, there exist $\delta_0 \in \tilde{\delta}$ and $\lambda_0 \in \tilde{\lambda}$ such that $\delta_0 \in \Delta(\lambda_0)$. Assume that $\delta \in \tilde{\delta}$ and $\gamma(\delta_0, \delta) \in \mathscr{X}_{2(s-1)}$. It suffices to show (2) for all such vertices δ.

Let λ^* be the vertex of distance $s - 1$ from δ_0 and δ in $\gamma(\delta_0, \delta)$. Then $\gamma(\lambda_0, \lambda^*)$ has length s, and (9.1) implies that there is a unique element T^* in $\mathscr{X}$ containing $\gamma(\lambda_0, \lambda^*)$. Pick $\lambda_1 \in T^*$ of distance $2(s-1)$ from λ_0 and $2(s-1) - 1$ from δ_0 and $\delta_1 \in T^* \cap \Delta(\lambda_1)$ of distance $2(s-1)$ from δ_0. Note that $\tilde{\delta}_0 = \tilde{\delta}_1$ and $\tilde{\lambda}_0 = \tilde{\lambda}_1$.

If $\delta \in T^*$, then $\delta = \delta_1$ and $d(\delta, \lambda_1) = 1$. So assume $\delta \notin T^*$. Then we can apply (9.3), and get $\gamma(\delta_1, \delta) \in T^{**} \in \mathscr{X}$.

Hence there exists $\lambda_2 \in \Delta(\delta) \cap T^{**}$ of distance $2(s-1)$ from λ_1, and since $\lambda_1 \in T^{**}$ it follows that $\lambda_2 \in \tilde{\lambda}_1 = \tilde{\lambda}$.

(3) For $\delta, \lambda \in \Gamma$ the following hold:

(a) $d(\tilde{\delta}, \tilde{\lambda}) = \min\{d(\delta', \lambda') \mid \delta' \in \tilde{\delta}, \lambda' \in \tilde{\lambda}\}$.

(b) $|\Delta(\tilde{\delta})| = |\Delta(\delta)|$.

(c) $\tilde{\Gamma}$ is a generalized $(s-1)$-gon; in particular $\tilde{\Gamma}$ is finite.

Parts (a) and (b) are easy consequences of (2). By (1), $\tilde{\Gamma}$ has diameter $s - 1$, and the classes of vertices in T form a circuit of length $2(s-1)$. Again by (1), $2(s-1)$ is the girth of $\tilde{\Gamma}$.

Set $\tilde{X} = X^{\tilde{\Gamma}}$. In the following we use $\sim$ convention for subgroups and subsets of $\tilde{X}$ and $\tilde{\Gamma}$.

(4) Any arc of length s in $\tilde{\Gamma}$ is contained in a unique element of $\tilde{\mathscr{X}}$.

Since the elements of $\tilde{\mathscr{X}}$ are circuits of length $2(s-1)$, this follows immediately from (2.6) and (3)(c).

(5) X_0 and X_1 operate faithfully on $\tilde{\Gamma}$.

Suppose that $x \in X_0^{\#}$ fixes every $\tilde{\delta}$ in $\tilde{\Gamma}$. Then we can choose δ such that x

fixes δ' for $\gamma(0,\delta) = (0\ldots\delta'\ \delta)$ but not δ. Hence $d(\delta,\delta^x) = 2$ and $\delta^x \in \tilde{\delta}$ which contradicts (1). The same argument shows that X_1 operates faithfully on $\tilde{\Gamma}$.

(6) Suppose that $s = 4$. Then assertion (9.4)(2) holds.

If $s = 4$, then $\tilde{\Gamma}$ is a generalized 3-gon. It follows that $\tilde{\Gamma}$ is the incidence graph of a projective plane $\mathscr{P}$ of order q_0. Hence $\tilde{X}$ operates as a group of collineations on $\mathscr{P}$, and the elements in $Z_i^\#(i \in T)$ induce elations on $\mathscr{P}$. Since $\tilde{G}$ is transitive on the points and lines of $\mathscr{P}$, the assertion follows from [13, 13.11].

From now on we assume $s \neq 4$ and refer to Sections 4 and 5, where the structure of L_0 and L_1 is described, and (6.8) and (7.2) as (*). Set $\mu = (s-1)/2$, $W = \langle t_0, t_1\rangle$ and $q_i = 2^{n_i}$.

(7) $K_i = K_{i+2\mu}$ for $i \in T$.

We apply (*). Then $s = 3$, 5 or 7, and in all but one case there exists a subgroup D_i such that $[D_i, K_i] = 1$,

$$C_T(D_i) = (i-\mu,\ldots,i+\mu) \quad \text{and} \quad D_iQ_{i\pm\mu} \in Syl_2(L_{i\pm\mu}).$$

In the remaining case ((4.8)(a), resp. (5.2)(a)) we have $K_i = K_i^- \times K_i^+$ with $|K_i| = q^2-1$, $|K_i^-| = q-1$ and $|K_i^+| = q+1$, and get $[D_i, K_i^-] = 1$, where D_i has all the other above properties. In addition,

$$|K| \mid (q^2-1)(q-1),$$

and K_i^+ is the unique subgroup in K of order $q+1$. Hence $K_i^+ = K_{i+2\mu}^+$, and it is easy to apply the following argument to K_i^- instead of K_i to get $K_i = K_{i+2\mu}$.

Thus we assume $[D_i, K_i] = 1$. This implies $[K_i, \overline{L_{i+\mu}}] = 1$ and with the same argument $[K_{i+2\mu}, \overline{L_{i+\mu}}] = 1$. If $K_i = C_K(\overline{L_{i+\mu}})$ and $K_{i+2\mu} = C_K(\overline{L_{i+\mu}})$, then $K_i = K_{i+2\mu}$. Hence we may assume $K_i \neq C_K(\overline{L_{i+\mu}})$. Since $K = K_iK_{i+1}$ (by (3.2)), it follows that $i+\mu \in i^G$ and $q_i < q_{i+1}$. Hence we are in case (4.8)(a) (resp. (5.2)(a)), $|K| = (q_i-1)^2(q_i+1)$ and $|C_K(\overline{L_{i+\mu}})| = q_i^2-1$. But then there is a unique subgroup of order $q_i - 1$ in $C_K(\overline{L_{i+\mu}})$ and again $K_i = K_{i+2\mu}$.

(8) $\tau^{2\mu} \in X_{\tilde{\Gamma}}$ and $\tilde{W} \simeq D_{4\mu}$.

Since W is an infinite dihedral group and $\tau^k \notin X_{\tilde{\Gamma}}$ for $0 < k \leq 2\mu - 1$ by (1), it suffices to show $\tau^{2\mu} \in X_{\tilde{\Gamma}}$.

We define $t_{2i} = t_0^{\tau^i}$ and $t_{2i+1} = t_1^{\tau^i}$ for $i \in \mathbf{Z}$. Note that t_i inverts the elements in K_i and $\tau^{2\mu} = t_0t_{2\mu} = t_1t_{2\mu+1}$. From (7) we know that $t_0t_{2\mu}$ centralizes K_0 and that $t_1t_{2\mu+1}$ centralizes K_1. Hence $\tau^{2\mu}$ centralizes K.

Set $A = \langle\tau^{2\mu}\rangle$, and suppose that $\tilde{A} \neq 1$. If we are in cases (4.8)(a) (resp. (5.2)(a))—we shall call this the U_4-case—we choose notation such that $q_0 = q_1^2$. The elements in $\tilde{A} \cap \tilde{K}$ are inverted by $\tilde{t}_0$ and $\tilde{t}_1$, thus

$$\tilde{A} \cap \tilde{K} \leq \tilde{K}_0 \cap \tilde{K}_1$$

and, by (*), $\tilde{K}_0 \cap \tilde{K}_1 = 1$. Hence we get a direct product $\tilde{A} \times \tilde{K}_i$ $(i = 0,1)$, and since $\tilde{K}_i$ operates transitively on $\Delta(\tilde{i}) \setminus \tilde{T}$ (see (3)(b)), there exists

$$x = ak \in \tilde{A} \times \tilde{K}_i, \ \langle a\rangle = \tilde{A} \text{ and } k \in \tilde{K}_i,$$

which fixes every element in $\Delta(\tilde{i})$ and in $\tilde{T}$. Thus x also fixes

$$\Delta(\tilde{i})^{\tilde{\tau^\mu}} = \Delta(\widetilde{i + 2\mu})$$

by (4). Hence x and $x^{\tilde{\tau^\mu}} = ak^{\tilde{\tau^\mu}}$ are in $C_{\tilde{X}}(\Delta(\widetilde{i + 2\mu}))$. Now (7) implies

$$k^{-1}k^{\tilde{\tau^\mu}} \in C_{\tilde{X}}(\Delta(\widetilde{i + 2\mu})) \cap \tilde{K}_{i+2\mu} = 1,$$

and $k = k^{\tilde{\tau^\mu}}$. It follows that $k^{\tilde{t}_{i+\mu}} = k^{\tilde{t}_i} = k^{-1}$, since $\tau^\mu = t_i t_{i+\mu}$. If we are not in the U_4-case or if $i = 1$, then by (*), $\tilde{K}_i \cap \tilde{K}_{i+\mu} = 1$. On the other hand,

$$k^{-2} = [k, \tilde{t}_{i+\mu}] \in \tilde{K}_i \cap \tilde{K}_{i+\mu};$$

thus we have $k = 1$.

If $i = 0$ and we are in the U_4-case, it follows that $\tilde{K}_0 \cap \tilde{K}_2 = \tilde{K}_0^+$, where $\tilde{K}_0^+$ is the unique subgroup of order $q_1 + 1$ in $\tilde{K}$, and $k \in \tilde{K}_0^+$.

The operation of $\tilde{\tau}$ on $\tilde{T}$ implies that we have to treat the following two cases:

(i) $\tilde{A} \le C_{\tilde{X}}(\Delta(\tilde{i}))$ for all $\tilde{i} \in \tilde{T}$,

(ii) the U_4-case holds, and $\tilde{A} \le C_{\tilde{X}}(\Delta(\tilde{i}))$ for all odd $\tilde{i} \in \tilde{T}$.

Assume (ii). Then $k \in \tilde{K}_0^+$, and k fixes every element in $\Delta(\tilde{i})$ for $i \equiv 1$ (2). Hence x fixes every element in $\Delta(\tilde{i})$, $i \equiv 1$ (2), $\Delta(\tilde{0})$ and $\Delta(\tilde{4})$. Pick

$$\tilde{\delta}_3 \in \Delta(\tilde{3}) \setminus \tilde{T} \quad \text{and} \quad \tilde{\delta}_5 \in \Delta(\tilde{5}) \setminus \tilde{T}.$$

For $i = 3, 5$ and $\tilde{\varrho} \in \Delta(\tilde{0})$, $\gamma(\tilde{\varrho}, \tilde{\delta}_i)$ denotes the arc

$$(\tilde{\varrho}\ \tilde{0}\ \tilde{1} \ldots \tilde{3}\ \tilde{\delta}_3) \text{ (resp. } (\tilde{\varrho}\ \tilde{0}\ \tilde{1} \ldots \tilde{5}\ \tilde{\delta}_5)).$$

By (4), $\gamma(\tilde{\varrho}, \tilde{\delta}_i)$ is contained in a unique element $\tilde{T}(\tilde{\varrho}, \tilde{\delta}_i)$ of $\tilde{\mathscr{X}}$, and x fixes all of these $\tilde{T}(\tilde{\varrho}, \tilde{\delta}_i)$. Hence again by (4), x fixes every element in $\Delta(\tilde{\delta}_5)$, $\Delta(\tilde{5})$, $\Delta(\tilde{4})$, $\Delta(\tilde{3})$, $\Delta(\tilde{\delta}_3)$, and $(\tilde{\delta}_3\ \tilde{3}\ \tilde{4}\ \tilde{5}\ \tilde{\delta}_5)$ is a G-conjugate of $(\tilde{0} \ldots \tilde{4})$.

Thus, in both cases (i) and (ii) it suffices to prove (**) to get a contradiction:

(**) Let x be an element in $\tilde{X}$ which fixes the elements in

$$\Delta(\tilde{0}), \ldots, \Delta(\widetilde{s - 1}).$$

Then $x = 1$.

By (4), x stabilizes every vertex in $\tilde{T}$. Pick $\tilde{k} \in \tilde{T}$, $s \le k \le 2s - 3$, and

$$\tilde{\delta} \in \Delta(\tilde{k}) \setminus \{\widetilde{k - 1}\}.$$

Then $\gamma = (\tilde{\delta}\ \tilde{k} \ldots \widetilde{k - (s - 1)})$ is an arc of length s contained in a unique element $\tilde{T}^g$ of $\tilde{\mathscr{X}}$. Since x stabilizes

$$(\tilde{k} \ldots \widetilde{k - (s - 1)})$$

and the vertices in

$$\Delta(\widetilde{k - (s - 1)}) \cap \tilde{T}^g,$$

it follows from (4) that x stabilizes $\tilde{T}^g$ and hence $\tilde{\delta}$. Thus we have shown that x fixes the elements in $\Delta(\tilde{i})$ for $\tilde{i} \in \tilde{T}$.

Now let $\tilde{\delta}$ be any vertex in $\tilde{\Gamma}$, and choose $\tilde{k} \in \tilde{T}$ such that $d(\tilde{\delta}, \tilde{k})$ is minimal. By induction we may assume that x fixes every vertex in $\tilde{\Gamma}$ which has distance less than $d(\tilde{\delta}, \tilde{k})$ from some vertex in $\tilde{T}$. Let $(\tilde{\delta}_0, \dots, \tilde{\delta}_n)$, $\tilde{\delta}_0 = \tilde{k}$, $\tilde{\delta}_n = \tilde{\delta}$, be the arc joining $\tilde{k}$ and $\tilde{\delta}$. Then $n \le s - 1$ ((3)(c)) and

$$(\widetilde{k + (s - n + 1)} \dots \tilde{k} \dots \tilde{\delta}_{n-1})$$

is an arc of length s contained in some $\tilde{T}^g \in \tilde{\mathscr{X}}$. As above x stabilizes $\tilde{T}^g$ and therefore $\tilde{\delta}$.

(9) *Set* $N = \tilde{W}\tilde{K}$ *and* $B = \widetilde{G_{01}}$. *Then* (B, N) *is a BN-pair of* $\tilde{G}$.

For the definition of a BN-pair see [11]. It suffices to show:

(***) $\tilde{t}_i B w \subseteq BwB \cup B\tilde{t}_i wB$ for $i = 0, 1$ and $w \in \tilde{W}$.

Every $w \in W$ can be written as $\tilde{t}_i \tilde{\tau}^m$ or $\tilde{\tau}^m$ for some $0 \le m \le s - 1$. We shall show (***) for $i = 0$ and $w = \tilde{t}_1 \tilde{\tau}^m$. The other cases follow with the same argument. For $x \in G_{01}$ we get

$$(0\ 1)^{t_0 x t_1 \tau^m} = (2\ 1^{t_0 x t_1})^{\tau^m} = (2 + 2m\ 1^{t_0 x t_1 \tau^m}).$$

Pick $2k(s - 1) + 1 \in T$ such that $d(2 + 2m, 2k(s - 1) + 1)$ is minimal. Then

$$d(2 + 2m, 2k(s - 1) + 1) \le s - 1$$

and there exists $y \in G_{2k(s-1)} \cap G_{2k(s-1)+1}$ such that

$$(0\ 1)^{t_0 x t_1 \tau^m y} \subseteq T \quad \text{and} \quad (0\ 1)^{t_0 x t_1 \tau^m y \tau^{-m-1}} = (01)$$

or

$$(0\ 1)^{t_0 x t_1 \tau^m y \tau^{-m-1} t_0} = (01).$$

Hence $t_0 x t_1 \tau^m y \tau^{-m-1} \in G_{01}$ or $t_0 x t_1 \tau^m y \tau^{-m-1} t_0 \in G_{01}$, and from

$$\widetilde{G_{2k(s-1)}} \cap \widetilde{G_{2k(s-1)+1}} = B$$

we get

$$\tilde{t}_0 \tilde{x} \tilde{t}_1 \tilde{\tau}^m \in B\tilde{t}_0 \tilde{\tau}^{m+1} B \cup B\tilde{\tau}^{m+1} B = BwB \cup Bt_0 wB.$$

Note that $B = S\tilde{K}$ for $S \in Syl_2(B)$ by (3.2)(c). Hence we can apply (9) and [11] to get the assertion.

10. Proofs of Theorems 1 and 2 and Corollary 1

Proof of Theorem 2. Let G be a counterexample. Suppose first that Γ is a tree and that G is not vertex-transitive. We apply (8.1) and conclude that (8.1)(a) holds for some normal subgroup E in G.

Assume that Hypothesis (3.0) holds in E. Then it follows from Sections 4, 5, 6, 7 and 9 that E is no counterexample. Since $G \le \mathrm{Aut}(E)$ and G is a counterexample, the singularity s_E of E cannot be the singularity of G. Hence there exists an arc $\gamma = (\lambda \ldots \delta)$ of length s_E which is regular under the operation of G. By (2.6) we may assume additionally $\gamma \subseteq T$ for some K-track (T, τ, K) defined in (3.3) with respect to E. Again by the above mentioned sections we get $|E_\gamma|_2 = 1$ or 2 and $s_E \equiv 1\ (2)$ or $n_\alpha = n_\beta > 1$. Thus without loss of generality we may assume $n_\lambda > 1$, and the choice of K assures that K does not fix every vertex in $\Delta(\lambda)$. But $[K, G_\gamma] \leq E_\gamma$, and the structure of G_λ and the transitivity of G_γ on $\Delta(\lambda) \setminus \gamma$ imply $2^{n_\lambda} \mid |[K, G_\gamma]|$, a contradiction.

Now assume that Hypothesis (3.0) does not hold in E. By (8.1)(a) we may assume that $n_\alpha = 1$ and E_α is 2-closed. Pick $S_0 \in Syl_2(E_{\alpha\beta})$. Then S_0 is normal in E_α and $[S_0, E_{\alpha\beta}] \leq S_0$. The structure of $\mathrm{Aut}(L_2(2^{n_\beta}))$ and (2.1) imply $E_\beta / O_2(E_\beta) \simeq L_2(2^{n_\beta})$ and $E = \langle O^2(E_\alpha), E_\beta \rangle$. Hence no non-trivial characteristic subgroup of S_0 is normal in E_β. From (1.7) we get

$$C_{E_\beta}(O_2(E_\beta)) \not\leq O_2(E_\beta)$$

and thus, by (8.1), $C_{E_\alpha}(O_2(E_\alpha)) \not\leq O_2(E_\alpha)$. Again, (2.1) implies

$$E = \langle C_{E_\beta}(O_2(E_\beta)), C_{E_\alpha}(O_2(E_\alpha)) \rangle .$$

Therefore $O_2(E_\beta) \cap O_2(E_\alpha) = 1$, $E_\alpha \simeq O_2(E_\alpha) \times A_3$, $|O_2(E_\alpha)| = 2^{n_\beta}$ and $E_\beta \simeq L_2(2^{n_\beta})$.It is now easy to check that $s = 3$ and $\{G_\alpha, G_\beta\}$ is parabolic of type $L_2(2^{n_\beta}) \times L_2(2)'$, and G is not a counterexample.

Now assume that Γ is not a tree, and let G^* be the amalgamated product of G_α and G_β with respect to $G_\alpha \cap G_\beta$. We identify G_α and G_β with the corresponding subgroups in G^*. There exists a normal subgroup N in G^* such that $G^*/N \simeq G$. Let φ be the natural homomorphism from G^* to G.

G^* operates by right multiplication on the graph Γ^* with vertex set

$$\{G_\alpha x / x \in G^*\} \cup \{G_\beta x / x \in G^*\}$$

where two vertices are adjacent iff they have non-empty intersection.

According to [4, (2.4) and (2.5)], G^* and Γ^* fulfill Hypothesis B, Γ^* is a tree, and the vertex stabilizers are conjugate to G_α or G_β. What we have already proved shows that G^* is not a counterexample to Theorem 2.

Let $\approx$ be the equivalence relation on Γ^* induced by N (i.e., $\delta' \approx \delta$ for $\delta', \delta \in \Gamma^*$ iff $\delta' \in \delta^N$) and define δ'^N to be adjacent to δ^N iff there exist $\delta_1 \in \delta^N$ and $\delta_2 \in \delta'^N$ such that $\delta_1 \in \Delta(\delta_2)$. As the vertices of Γ^* are the cosets of G_α and G_β, the vertices in $\Gamma^*/\approx$ are the cosets of $G_\alpha N$ and $G_\beta N$. If G is not vertex-transitive on Γ,

$$(G_\delta N x)\psi = \delta^{x\varphi}, \quad x \in G \text{ and } \delta \in \{\alpha, \beta\},$$

defines an isomorphism from $\Gamma^*/\approx$ to Γ. This isomorphism is compatible with φ. Hence G operates in the same way on Γ as on $\Gamma^*/\approx$, and G is no counterexample.

Now assume that G is vertex-transitive. Then $n_\alpha = n_\beta > 1$, and G_α is conjugate to G_β in G. From the structure of G^* we see that $\{G_\alpha, G_\beta\}$ is parabolic of type $L_2(2^{n_\alpha}) \times L_2(2^{n_\alpha})$, $L_3(2^{n_\alpha})$ or $Sp_4(2^{n_\alpha})$. It is now easy to check that $s = 3, 4$ or 5 respectively. This shows that G is not a counterexample.

Proof of Theorem 1. Let G^* be the amalgamated product of M_1 and M_2 with respect to $M_1 \cap M_2$. We define the graph Γ^* as in the proof of Theorem 2. As we have shown there, Hypothesis B holds in G^* with respect to Γ^*, and vertex-stabilizers in G^* are conjugate to M_1 or M_2. Hence Theorem 2 implies Theorem 1.

Proof of Corollary 1. Let G be a counterexample. Then either (c) or (d) in Theorem 1 holds.

Assume case (d). Then $|O_2(M_1)| = 2^{2^{n_1}+1}$ and $n_1 > 1$. Now an easy application of [3, Corollary 4] and the Main Theorem in [3] shows $G = M_1O(G)$.

Now assume case (c). We choose notation such that $n_1 > 1$. Since maximal elementary abelian subgroups of $O_2(M_1)$ have order 2^3, it is easy to see that M_1 has sectional 2-rank 4 and that $O_2(M_1)$ is weakly closed in a Sylow 2-subgroup S of M_1. Hence S is a Sylow 2-subgroup of G, and G has sectional 2-rank 4. Now [12] implies that $\{M_1, M_2\}$ is parabolic of type J_2.

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