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ON L_∞ UNIVERSALLY BAD SEQUENCES IN ERGODIC THEORY

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We prove an arithmetical condition (conditions (c) or (d) of the theorem) for a subsequence of the positive integers to be L_∞ universally bad. As a consequence, we prove (Corollary 1) that every L_∞ universally bad sequence is δ-sweeping out for some $\delta > 0$. This problem was posed by J. Rosenblatt [2, p. 231] and (as I was informed by R. Jones and J. Rosenblatt) A. Bellow and R. Jones also solve it by a different method in [1], as a corollary of their main result there. Our Corollary 2 shows that one can test L_∞ universal badness of sequences on the special dynamical system $([0, 1], B, \lambda, x \to 2x(\mathrm{mod}\ 1))$ consisting of Borel sets of [0, 1] with the Lebesgue measure, and transformation $x \to 2x(\mathrm{mod}\ 1)$.

By a *dynamical system* (Ω, A, μ, T) we mean a non-atomic probability space (Ω, A, μ) together with an ergodic measure preserving transformation T. A strictly increasing sequence $a_1 < a_2 < \cdots < a_n < \cdots$ of positive integers is called L_∞ *universally bad*, if for every dynamical system there is an $f \in L_\infty(\Omega)$ for which the pointwise ergodic theorem along the subsequence $\{a_i\}$ is not true, i.e., for which $\frac{1}{n}\sum_{i=1}^{n} f(T^{a_i}\omega)$ fails to converge in a set of ω of positive measure.

THEOREM. *The following statements are equivalent for a given strictly increasing sequence* $(a_1 < a_2 < \cdots)$ *of positive integers.*
(a) *There exists a dynamical system* (Ω, A, μ, T) *such that for every* $f \in L_\infty(\Omega)$,

$$\frac{1}{n}\sum_{i=1}^{n} f(T^{a_i}\omega)$$

is convergent for a.e. $\omega \in \Omega$; *i.e.,* $\{a_i\}$ *is not* L_∞ *universally bad.*
(b) *For every* $c > 0$ *there exists a dynamical system* (Ω, A, μ, T) *and an* $\varepsilon > 0$ *with the property that if* $G \in A$, γ *is the characteristic function of* G, *and*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \gamma(T^{a_i}\omega) \geq c$$

for a.e. $\omega \in \Omega$, *then* $\mu(G) \geq \varepsilon$.

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(c) *For every* $0 < c < 1$ *there is an* $\varepsilon > 0$ *such that if* n *is a positive integer,* $H \subseteq \{1, 2, \dots, n\}$, χ *is the characteristic function of* H, *and* $|H| < \varepsilon n$, *then*

$$\left|\left\{1 \le x \le n \colon \max_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \chi(x + a_i) \ge c\right\}\right| \le (1 - \varepsilon)n.$$

(d) *For every* $0 < c, c^* < 1$, *there is an* $\varepsilon > 0$ *such that if* n *is a positive integer,* $H \subseteq \{1, 2, \dots, n\}$, χ *is the characteristic function of* H, *and* $|H| < \varepsilon n$, *then*

$$\left|\left\{1 \le x \le n \colon \max_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \chi(x + a_i) \ge c\right\}\right| \le c^* n.$$

(e) *For every dynamical system* (Ω, A, μ, T), *with the notation*

$$M(g)(\omega) = \sup_{k=1,2,\dots} \left|\frac{1}{k} \sum_{i=1}^{k} g(T^{a_i}\omega)\right|,$$

if $g_i \in L_\infty(\Omega)$ *for* $i = 1, 2, \dots$, $\|g_i\|_\infty \le 1$, $\|g_i\|_1 \to 0$, *then* $\|M(g_i)\|_1 \to 0$.

Proof. (a) $\Rightarrow$ (b). Let (Ω, A, μ, T) be the dynamical system from (a), and $\varepsilon = c$. Since $\gamma \in L_\infty(\Omega)$, we have, by (a),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \gamma(T^{a_i}\omega) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \gamma(T^{a_i}\omega) \ge c$$

for a.e. $\omega \in \Omega$, so by Lebesgue's theorem,

$$\mu(G) = \lim_{n\to\infty} \int_\Omega \frac{1}{n} \sum_{i=1}^{n} \gamma(T^{a_i}\omega)\, d\mu = \int_\Omega \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \gamma(T^{a_i}\omega)\, d\mu \ge c,$$

which proves (b).

(b) $\Rightarrow$ (c). The proof depends on the following lemma of Rohlin ([3], Theorem 8.1.):

Given any dynamical system (Ω, A, μ, T), $\varepsilon > 0$ and positive integer n, there is a measurable set $D \in A$ such that $D, T^{-1}D, \dots, T^{-(n-1)}D$ are pairwise disjoint, and $\mu(D \cup T^{-1}D \cup \dots \cup T^{-(n-1)}D) > 1 - \varepsilon$. □

Assume that for a given c assertion (c) is not true. Then for every $\varepsilon > 0$ there is a positive integer n, $H \subseteq \{1, 2, \dots, n\}$ such that $|H| < \varepsilon n$,

$$\left|\left\{1 \le x \le n \colon \max_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \chi(x + a_i) \ge c\right\}\right| > (1 - \varepsilon)n.$$

Apply Rohlin's lemma for this ε and n, and for any given dynamical system (Ω, A, μ, T); then we get a set $D \in A$, and let

$$G_\varepsilon = \bigcup_{h \in H} T^{-n+h} D.$$

Denote by γ_ε the characteristic function of G_ε; then for $1 \le x \le n$ and $\omega \in T^{-n+x} D$,

$$\sup_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \gamma_\varepsilon(T^{a_i}\omega) \ge \max_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \chi(x + a_i).$$

Hence, from the conditions, $\mu(G_\varepsilon) = |H|\mu(D) < \varepsilon n \mu(D) \le \varepsilon$. On the other hand,

$$\mu\left(\omega\colon \sup_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \gamma_\varepsilon(T^{a_i}\omega) \ge c\right) > (1-\varepsilon)n\mu(D) > (1-\varepsilon)^2 > 1 - 2\varepsilon.$$

Here we can neglect the small values of k, because the measure of the set $\{\omega\colon T^{a_i} \in G_\varepsilon$ for some $i \le 1/\sqrt{\varepsilon}\}$ is at most $\frac{1}{\sqrt{\varepsilon}}\mu(G_\varepsilon) < \sqrt{\varepsilon}$. So for G_ε we know that $\mu(G_\varepsilon) < \varepsilon$ and

$$\mu\left(\omega\colon \sup_{k > 1/\sqrt{\varepsilon}} \frac{1}{k} \sum_{i=1}^{k} \gamma_\varepsilon(T^{a_i}\omega) \ge c\right) > 1 - 2\varepsilon - \sqrt{\varepsilon}.$$

So if $G = \bigcup_{j=1}^{\infty} G_{\varepsilon/2^j}$, the characteristic function of G is γ, then $\mu(G) < \varepsilon$, and by the Borel-Cantelli lemma (since $\sum_{j=1}^{\infty} 2\frac{\varepsilon}{2^j} + \sqrt{\varepsilon/2^j} < \infty$) we have

$$\mu\left(\omega\colon \limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \gamma(T^{a_i}\omega) \ge c\right) = 1.$$

This proves that (b) is not true.

(c) $\Rightarrow$ (d). Implication (d) $\Rightarrow$ (c) would be trivial, but we have to prove now the converse. Assume that assertion (d) is not true with a given c and c^*. Then for every $\varepsilon > 0$ there is a positive integer n, $H \subseteq \{1, 2, \dots, n\}$, such that $|H| < \varepsilon n$, and with the notation

$$G = \left\{1 \le x \le n\colon \max_{k=1,2,\dots} \frac{1}{k} \sum_{i=1}^{k} \chi(x + a_i) \ge c\right\}$$

we have $|G|/n > c^*$. We would like to show that this can be satisfied with numbers arbitrarily close to 1 in place of the fixed $0 < c^* < 1$. Now choose an integer m with $1 \le m \le n$, and let $H_1 = H \cup (H + m)$, $G_1 = G \cup (G + m)$; these are subsets of $\{1, 2, \dots, n + m\}$. We want to maximize $\frac{|G_1|}{n+m}$. Assume that $\frac{|G_1|}{n+m} \le c_1^*$ for every $1 \le m \le n$ with some c_1^*. It follows that

$$|\{g \in G\colon g + m \in G\}| \ge 2|G| - c_1^*(n+m),$$

and summing with respect to m gives

$$2n|G| - c_1^* n^2 - c_1^* \frac{n(n+1)}{2} \leq \sum_{m=1}^{n} |\{g \in G\colon\ g+m \in G\}|.$$

The right hand side can be estimated with the number of pairs in G, so it is smaller than $|G|^2/2$; hence

$$c_1^*\left(3+\frac{1}{n}\right) > 4\frac{|G|}{n} - \left(\frac{|G|}{n}\right)^2.$$

Since the function $x \to 4x - x^2$ is monotone increasing in [0, 1], one can choose $1 \leq m \leq n$ such that

$$\frac{|G_1|}{n+m} > \frac{4c^* - (c^*)^2}{3+(1/n)} = c^* \frac{4-c^*}{3+(1/n)}.$$

On the other hand, it is trivial that if χ_1 is the characteristic function of H_1, then for $x \in G_1$ we have $\max_{k=1,2,\ldots} \frac{1}{k}\sum_{i=1}^{k} \chi_1(x+a_i) \geq c$. Summing up this process: starting with a set H with the properties

$$H \subseteq \{1,2,\ldots,n\},\ |H| < \varepsilon n,\ \left|\left\{1 \leq x \leq n\colon \max_{k=1,2,\ldots} \frac{1}{k}\sum_{i=1}^{k} \chi(x+a_i) \geq c\right\}\right| > c^* n. \tag{1}$$

we can find a new set satisfying similar conditions with the same c, with larger n, with 2ε in place of ε, and with $c^* \frac{4-c^*}{3+(1/n)}$ in place of c^*. Fix an arbitrary $0 < q < 1$, and iterate this process. We want to achieve $c^* \geq q$.

Now q is fixed, and we can start with arbitrarily small ε. With the first set H $1 \leq |H| \leq \varepsilon n$, so for the first n we have $\frac{1}{n} \leq \varepsilon$, and since n increases, this will be always true. So we can assume for example $\frac{1}{n} < \frac{1-q}{2}$, and then, if $c^* < q$, then

$$c^* \frac{4-c^*}{3+(1/n)} \geq R_1 c^*$$

with a constant $R_1 > 1$ depending on q. This means that there is a constant R_2 (depending on q and on the original c^*, but these are fixed numbers) such that after R_2 steps we will have $c^* \geq q$, so we will have (1) for the actual H and n, with q in place of c^*, and with $2^{R_2}\varepsilon$ in place of the original ε. If the original ε is small enough, then $2^{R_2}\varepsilon$ is also small. This proves that assertion (c) is not true with this c.

(d) $\Rightarrow$ (e). Firstly, from (d) it follows that for every $c > 0$ there is an $\varepsilon > 0$ such that if n is a positive integer, $0 \leq \chi \leq 1$ is a function defined on the set $\{1,2,\ldots,n\}$, and $\sum_{x=1}^{n} \chi(x) < \varepsilon n$, then $\sum_{x=1}^{n} \max_{k=1,2,\ldots} \frac{1}{k}\sum_{i=1}^{k} \chi(x+a_i) < cn$. Indeed, if χ is a characteristic function, then this is obvious from (d), but for general χ we have for every $\delta > 0$ that $\chi \leq \delta + \chi_\delta$, where $\chi > \delta$.

If $0 \le g \le 1$ is a function measurable on Ω, n is a large positive integer, then

$$\int_\Omega g(\omega) = \int_\Omega \frac{1}{n} \sum_{x=1}^{n} g(T^x \omega).$$

Hence with the notation $\Omega_1 = \{\omega\colon \frac{1}{n} \sum_{x=1}^{n} g(T^x\omega) \ge \varepsilon\}$ we have $\mu(\Omega_1) \le \frac{1}{\varepsilon} \int_\Omega g(\omega)$. Let h be a fixed positive integer, $\Omega_2 = \Omega - \Omega_1$; then applying the remark above for $\chi(x) = g(T^x\omega)$, we get

$$\sum_{x=1}^{n-a_h} \max_{k=1,2,\dots,h} \frac{1}{k} \sum_{i=1}^{k} g(T^{x+a_i}\omega) < cn$$

for $\omega \in \Omega_2$. Hence

$$\begin{aligned} \int_\Omega \max_{k=1,2,\dots,h} \frac{1}{k} \sum_{i=1}^{k} g(T^{a_i}\omega) &= \frac{1}{n-a_h} \int_\Omega \sum_{x=1}^{n-a_h} \max_{k=1,2,\dots,h} \frac{1}{k} \sum_{i=1}^{k} g(T^{x+a_i}\omega) \\ &\le \frac{1}{\varepsilon} \int_\Omega g(\omega) + c\frac{n}{n-a_h}, \end{aligned}$$

estimating separately on the sets Ω_1 and Ω_2. If we first let $n \to \infty$, and then $h \to \infty$, we obtain $\|M(g)\|_1 \le c + \frac{1}{\varepsilon}\|g\|_1$. Since c can be arbitrarily small, this proves (e) (using positive and negative parts for general g).

(e) $\Rightarrow$ (a). It is easily seen from (e) (which is a weakened version of the usual maximal inequality) that if $f \in L_\infty(\Omega)$, $f_1, f_2, \dots, f_i, \dots \in L_\infty(\Omega)$, $\|f\|_\infty \le C$, $\|f_i\|_\infty \le C$, $\|f - f_i\|_1 \to 0$, C is a constant, and the pointwise ergodic theorem along the subsequence $\{a_i\}$ is true for all f_i, then it is also true for f. So it is enough to find a dynamical system in which we know the pointwise ergodic theorem along the subsequence $\{a_i\}$ for an (in this sense) dense set of functions.

Consider the dynamical system $([0,1], B, \lambda, x \to 2x (\mathrm{mod}\, 1))$. Here the set of trigonometric polynomials is dense in the above sense, so it is enough to know the theorem for the functions $x \to e^{2\pi i n x}$ (where $n \ne 0$ is an integer), and for this it is enough to show that for any $\varepsilon > 0$ the function series $h_r(x) = \frac{1}{(1+\varepsilon)^r} \sum_{j \le (1+\varepsilon)^r} e^{2\pi i (2^{a_j} n)x}$ is convergent for a.e. x. But this is true, because by Parseval's formula

$$\int_0^1 \sum_{r=1}^{\infty} |h_r(x)|^2\, dx \le \sum_{r=1}^{\infty} \frac{1}{(1+\varepsilon)^r} < \infty;$$

hence $\sum_{r=1}^{\infty} |h_r(x)|^2$ is finite a.e., so $h_r(x) \to 0$ a.e., and this proves (a). The proof of the theorem is now complete.

Following Rosenblatt [2], we say that the sequence $\{a_i\}$ is *δ-sweeping out* (for a given $\delta > 0$), if for every dynamical system (Ω, A, μ, T) and every $\varepsilon > 0$ there is a $G \in A$ with characteristic function γ such that $\mu(G) < \varepsilon$, and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \gamma(T^{a_i}\omega) \ge \delta$$

for a.e. $\omega \in \Omega$. Now we see that condition (b) of the theorem is exactly the statement that $\{a_i\}$ is not δ-sweeping out for any $\delta > 0$. So equivalence (a) $\Rightarrow$ (b) gives the following.

COROLLARY 1. *The sequence $\{a_i\}$ is L_∞ universally bad if and only if it is δ-sweeping out for some $\delta > 0$.*

From the proof of the theorem it is clear that if condition (a) is satisfied, then we can choose the special system $([0, 1], B, \lambda, x \to 2x(\text{mod } 1))$ there, the pointwise ergodic theorem along $\{a_i\}$ will be true for $f \in L_\infty$, and the a.e. limit will be $\int_0^1 f$. So we have:

COROLLARY 2. *The sequence $\{a_i\}$ is not L_∞ universally bad if and only if the pointwise ergodic theorem along $\{a_i\}$ is true in $([0, 1], B, \lambda, x \to 2x(\text{mod } 1))$ for L_∞ functions, i.e., if and only if for every bounded measurable function f on the real line, periodic with respect to 1 one has for a.e. x,*

$$\frac{1}{n}\sum_{i=1}^{n} f(2^{a_i}x) \to \int_0^1 f.$$

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