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SOME WEAK TYPE ESTIMATES FOR CONE MULTIPLIERS

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ABSTRACT. We consider operators T^δ associated with a localized height of cone multipliers. It is shown that T^δ is of weak type (p, p) for the functions of the form $f(x, t) = g(|x|, t)$ if $p = 2n/(n+1+2\delta)$, $0 < \delta \le (n-1)/2$.

1. Introduction

For $\delta > 0$, define convolution operators T^δ by

$$\widehat{T^\delta f}(\xi', \xi_{n+1}) = \psi(\xi_{n+1}) \left(1 - \frac{|\xi'|^2}{\xi_{n+1}^2}\right)_+^\delta \hat{f}(\xi', \xi_{n+1}), \qquad (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$$

where $\psi \in C_0^\infty(1/2, 2)$ and $\hat{f}$ denotes the Fourier transform of a Schwartz function f on $\mathbb{R}^n \times \mathbb{R}$.

G. Mockenhaupt [4] proved that T^δ is bounded on $L^4(\mathbb{R}^3)$ if $\delta > 1/8$. J. Bourgain [1] developed a technique to prove this result for certain values of $\delta < 1/8$. See also Mockenhaupt, Seeger and Sogge [5] for the related results on the wave equation.

The main purpose of this paper is to prove a sharp weak type (p, p) inequality for the functions with radial symmetry of the form $f(x, t) = g(|x|, t)$. This result implies the weak type endpoint estimate for the Bochner-Riesz means on radial functions in $L^p(\mathbb{R}^n)$ where $p = 2n/(n+1+2\delta)$ and $0 < \delta \le (n-1)/2$, which was proved by Chanillo and Muckenhoupt [2] (see appendix).

We consider the actions of T^δ on L^p-functions f which are invariant under $SO(n) \times \{I\}$, that is, $f(x, t) = g(|x|, t)$ for g satisfying $(\int_{\mathbb{R}} \int_0^\infty |g(s, z)|^p s^{n-1} ds\, dz)^{1/p} = \|f\|_p < \infty$.

THEOREM 1. *Suppose $p = 2n/(n+1+2\delta)$ and $0 < \delta \le (n-1)/2$. Then the inequality*

$$|\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |T^\delta f(x, t)| > \alpha\}| \le C \left(\frac{\|f\|_{L^p(\mathbb{R}^{n+1})}}{\alpha}\right)^p$$

holds for all $SO(n) \times \{I\}$ invariant L^p-functions f and for all $\alpha > 0$. The constant C does not depend on α or f.

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With respect to the notation we shall denote by C positive constants that have different values in different lines.

2. The cone multiplier operator acting on L^p functions with spherical symmetries

Let $\varphi \in C_0^\infty(\mathbb{R})$ be supported in $(1/2, 2)$ such that $\sum_{k\geq 1} \varphi(2^k s) = 1$ for $0 < s < 1$. In order to show the kernel estimates, we need some properties of Bessel functions. These may be found in [7], [8]. J_μ is the Bessel function of order $\mu > -1/2$ defined by

$$J_\mu(t) = A_\mu t^\mu \int_{-1}^{1} e^{it\sigma}(1-\sigma^2)^{\mu-\frac{1}{2}}\, d\sigma \tag{2.1}$$

where $A_\mu = [2^\mu \Gamma(2\mu+1)\Gamma(\frac{1}{2})]^{-1}$.

$$|J_\mu(t)| \leq C\, t^\mu \quad \text{for} \quad t \geq 0. \tag{2.2}$$

Let $\mu = n/2 - 1$. For a function $g(x)$ on $(0, \infty)$, we define the Hankel transform $\widetilde{g}(y)$ of order μ by

$$\widetilde{g}(y) = \int_0^\infty g(x) J_\mu(yx)\,(yx)^{-\mu}\, x^{2\mu+1}\, dx, \tag{2.3}$$

where $J_\mu(x)$ is the Bessel function of order μ.

We shall need pointwise estimates for the kernels of

$$\begin{aligned} T_k^\delta f(x,t) \;=\; (2\pi)^{-(n+1)} &\iiiint e^{i<x-y,\xi'>+i(t-z)\xi_{n+1}}\, \varphi\left(2^k\left(1-\frac{|\xi'|^2}{\xi_{n+1}^2}\right)\right) \\ &\times \left(1-\frac{|\xi'|^2}{\xi_{n+1}^2}\right)_+^\delta \psi(\xi_{n+1})\, d\xi'\, d\xi_{n+1}\, f(y,z)\, dy\, dz. \end{aligned}$$

We let $|x| = r$ and define $T_k^\delta f(x,t) = U_k^\delta g(r,t)$. If $f(x,t) = g(|x|,t)$, then by (2.3),

$$U_k^\delta g(r,t) = (2\pi)^{-(n+1)} \int_{\mathbb{R}} \int_0^\infty H_k(r,t,s,z) g(s,z)\, ds\, dz$$

where

$$\begin{aligned} H_k(r,t,s,z) \;=\; s^{n/2} r^{-(n-2)/2} &\int_{\mathbb{R}} \int_0^1 J_{\frac{n-2}{2}}(\rho r \xi_{n+1}) J_{\frac{n-2}{2}}(\rho s \xi_{n+1}) \\ &\times \varphi(2^k(1-\rho^2))(1-\rho^2)^\delta \rho\, \psi(\xi_{n+1}) \xi_{n+1}^2 e^{i(t-z)\xi_{n+1}}\, d\rho\, d\xi_{n+1}. \end{aligned} \tag{2.4}$$

We denote $\sum_{k\ge 1} H_k(r,t,s,z)$ by $H(r,t,s,z)$.

LEMMA 1. *Suppose* $0 < r \le 1$, $0 < s \le 1$ *and* $t, z \in \mathbb{R}$. *Then for each k there is an estimate as follows: for every N*,

$$|H_k(r,t,s,z)| \le C_N\, s^{n-1}\, 2^{-k(\delta+1)} \min\{1, (2^{-k}|t-z|)^{-N}\}. \tag{2.5}$$

Thus,

$$|H(r,t,s,z)| \le C \frac{1}{(1+|t-z|)^{\delta+1}}\, s^{n-1}. \tag{2.6}$$

Proof. We integrate by parts N times with respect to ξ_{n+1} for every N in (2.4). So by (2.2), we obtain (2.5). Moreover, using (2.5) gives

$$s^{n-1}\left\{C \sum_{|t-z|\le 2^k} 2^{-k(\delta+1)} + C_N \sum_{|t-z|>2^k} 2^{-k(\delta+1-N)}\,|t-z|^{-N}\right\},$$

and thus (2.6) is established. □

We now consider the kernel estimates for the case $r > 1$. In order to estimate (2.4), we use dyadic decompositions of Bessel functions (2.1) following the article by Müller and Seeger [6].

LEMMA 2. *For* $r, s > 0$ *and* $t, z \in \mathbb{R}$ *there are estimates as follows: for fixed* $k \in \mathbb{Z}^+$,

$$\begin{aligned} &|H_k(r,t,s,z)| \\ &\quad \le C_{N,N_1}\left(\frac{s}{r}\right)^{(n-1)/2} 2^{-k(\delta+1)}(1+||t-z|-|r\pm s||)^{-N}\min\{1,(2^{-k}|r\pm s|)^{-N_1}\}. \end{aligned} \tag{2.7}$$

Moreover,

$$\begin{aligned} |H(r,t,s,z)| \le C\left(\frac{s}{r}\right)^{(n-1)/2} &\left\{\frac{1}{(1+|r+s|)^{\delta+1}}\frac{1}{(1+||t-z|-|r+s||)^N}\right. \\ &\left.+\frac{1}{(1+|r-s|)^{\delta+1}}\frac{1}{(1+||t-z|-|r-s||)^N}\right\}. \end{aligned} \tag{2.8}$$

Proof. Let $\eta \in C_0^\infty(\mathbb{R})$ be supported in $(-1/2, 2)$ and equal to 1 in $(-1/4, 1/4)$. Define $m = 0, 1, 2, \ldots$ and

$$\eta_{mk}(\sigma,\nu) = \begin{cases} \eta(2^{-k}\nu(1-\sigma^2)) & \text{if } m = 0 \\ \eta(2^{-k-m}\nu(1-\sigma^2)) - \eta(2^{-k-m+1}\nu(1-\sigma^2)) & \text{if } m > 0 \end{cases}$$

and set

$$J^m_{\mu,k}(\rho\nu) = A_\mu(\rho\nu)^\mu \int_{-1}^{1} e^{i(\rho\nu)\sigma}\,(1-\sigma^2)^{\mu-1/2}\eta_{mk}(\sigma,\nu)\,d\sigma.$$

Let $M > N + N_1 + (n-1)/2$ and set

$$\phi_{mk\nu}(\sigma) = \begin{cases} (1-\sigma^2)^{(n-3)/2}\,\eta_{mk}(\sigma,\nu) & \text{if } m = 0 \\ \left(\frac{1}{i\rho\nu}\right)^M \left(\frac{d}{d\sigma}\right)^M [\eta_{mk}(\sigma,\nu)(1-\sigma^2)^{(n-3)/2}] & \text{if } m > 0. \end{cases}$$

Then

$$J^m_{\frac{n-2}{2},k}(\rho\nu) = A_{\frac{n-2}{2}}(\rho\nu)^{\frac{n-2}{2}} \int_{-1}^{1} e^{i(\rho\nu)\sigma}\phi_{mk\nu}(\sigma)\,d\sigma \tag{2.9}$$

by an integration by parts if $m > 0$.

We may decompose the kernel (2.4) as

$$H_k = \sum_{m,d=0}^{\infty} H_k^{m,d}$$

where

$$\begin{aligned} H_k^{m,d}(r,t,s,z) \;=\; & s^{n/2}r^{-(n-2)/2}\int_{\mathbb{R}}\int_0^1 J^m_{\frac{n-2}{2},k}(\rho r\xi_{n+1})\,j^d_{\frac{n-2}{2},k}(\rho s\xi_{n+1}) \\ & \times \varphi(2^k(1-\rho^2))(1-\rho^2)^\delta \rho\psi(\xi_{n+1})\xi_{n+1}^2 e^{i(t-z)\xi_{n+1}}\,d\rho\,d\xi_{n+1}. \end{aligned}$$

Fix k and set $r\xi_{n+1} = u$ and $s\xi_{n+1} = v$. From (2.9), we have

$$\begin{aligned} H_k^{m,d}(r,t,s,z) \;=\; & C\,A^2_{\frac{n-2}{2}} s^{n-1}\int_{-1}^{1}\int_{-1}^{1} \phi_{mku}(\sigma_1)\phi_{dkv}(\sigma_2) \\ & \times \int_{\mathbb{R}}\int_0^1 \varphi(2^k(1-\rho^2))(1-\rho^2)^\delta\rho^{n-1}\,\psi(\xi_{n+1})\xi_{n+1}^n \\ & \times e^{i\{(t-z)+\rho r\sigma_1+\rho s\sigma_2\}\xi_{n+1}}\,d\rho\,d\xi_{n+1}\,d\sigma_1\,d\sigma_2. \end{aligned} \tag{2.10}$$

If we integrate by parts with respect to ρ and ξ_{n+1} in (2.10), then by Fubini's theorem we obtain

$$\begin{aligned} |H_k^{m,d}(r,t,s,z)| \le\; & C\,2^{-k(n-1)}\,s^{n-1}\int_{1/2}^{2}\int_{-1}^{1}\int_{-1}^{1}\int_0^1 |\phi_{mku}(\sigma_1)|\;|\phi_{dkv}(\sigma_2)| \\ & \times (1+|u\sigma_1+v\sigma_2|)^{-N_1}(1+|t-z+\rho r\sigma_1+\rho s\sigma_2|)^{-N} \\ & \times \left|\left(\frac{\partial}{\partial\rho}\right)^{N_1}\varphi(2^k(1-\rho^2))(1-\rho^2)^\delta\right| \\ & \times \left|\left(\frac{\partial}{\partial\xi_{n+1}}\right)^{N}\psi(\xi_{n+1})\xi_{n+1}^n\right| d\rho\, d\sigma_1\, d\sigma_2\, d\xi_{n+1}. \end{aligned} \tag{2.11}$$

Next note the size estimate

$$|\phi_{mku}(\sigma)| \le C\,2^{-mM}\,(2^{m+k}u^{-1})^{(n-3)/2}. \tag{2.12}$$

Moreover, ϕ_{mku} vanishes unless either $1-\sigma^2 \approx 2^{m+k}u^{-1}$ for $m>0$, or $1-\sigma^2 \le 2^k u^{-1}$ for $m=0$. Hence if σ is in the support of ϕ_{mku} then either $|u-u\sigma| \le 2^{m+k}$ or $|u+u\sigma| \le 2^{m+k}$. Then from (2.12), the integrand of (2.11) is bounded by

$$\begin{aligned} &C\,2^{-k\delta}|\phi_{mku}(\sigma_1)\phi_{dkv}(\sigma_2)|\frac{1}{(1+2^{-k}|u\sigma_1+v\sigma_2|)^{N_1}}\frac{1}{(1+t-z+\rho r\sigma_1+\rho s\sigma_2)^N}\\ &\le C\,2^{k\{(n-3)-\delta\}}\,(uv)^{-(n-3)/2}\,2^{(m+d)((n-3)/2+N+N_1-M)}\,\xi_{n+1}^{-N}\\ &\quad\times\left\{\frac{1}{(1+2^{-k}|u+v|)^{N_1}}\frac{1}{(1+||t-z|-\rho|r+s||)^N}\right.\\ &\qquad\left.+\frac{1}{(1+2^{-k}|u-v|)^{N_1}}\frac{1}{(1+||t-z|-\rho|r-s||)^N}\right\}. \end{aligned}$$

If we integrate over the support of $\varphi(2^k(1-\rho^2))\otimes\phi_{mku}\otimes\phi_{dkv}\otimes\psi$ in (2.11) for $m\ge 0$ and $d\ge 0$, we obtain an additional factor of $C\,2^{m+d+k}(rs)^{-1}$. Since $M>N_1+N+(n-1)/2$, we may sum over m and d and get (2.7). Thus using (2.7) gives

$$\left(\frac{s}{r}\right)^{(n-1)/2}(1+\|t-z|-|r\pm s\|)^{-N}\left\{C\sum_{|r\pm s|\le 2^k}2^{-k(\delta+1)}+C_{N_1}\sum_{|r\pm s|>2^k}2^{-k(\delta+1-N_1)}|r\pm s|^{-N_1}\right\},$$

and (2.8) is established. $\square$

For the next lemma, we use (2.2) for $J_{\frac{n-2}{2}}(\rho s\xi_{n+1})$ in (2.4) and then apply the same arguments used for Lemma 2.

LEMMA 3. *For $r>1$, $0<s\le 1$ and $t,z\in\mathbb{R}$ there are estimates as follows: for fixed $k\in\mathbb{Z}^+$,*

$$|H_k(r,t,s,z)| \le C\,s^{n-1}\,2^{-k(\delta+1)}\,r^{-(n-1)/2}\,(1+||t-z|-r|)^{-N}\min\{1,(2^{-k}r)^{-N_1}\}.$$

Moreover,

$$|H(r,t,s,z)| \le C\,s^{n-1}\,(1+r)^{-(n+1+2\delta)/2}\left\{\frac{1}{(1+|t-z+r|)^N}+\frac{1}{(1+|t-z-r|)^N}\right\}.$$

3. Weak type (p,p) estimates

In this section we shall show that the multiplier operator T^δ is of weak type (p,p) acting on $L^p(\mathbb{R}^n\times\mathbb{R})$ with $f(\cdot,t)$ spherically symmetric for all t in $\mathbb{R}$ where $p=2n/(n+1+2\delta)$, $0<\delta\le(n-1)/2$ and prove Theorem 1.

Now since $U^\delta g = \sum_{k\geq 1} U_k^\delta g$, by summing over k in (2.4), we see that

$$U^\delta g(r,t) = (2\pi)^{-(n+1)} \int_{\mathbb{R}} \int_0^\infty H(r,t,s,z) g(s,z)\, ds\, dz$$

where

$$\begin{aligned} H(r,t,s,z) = {} & s^{n/2} r^{-(n-2)/2} \int_{\mathbb{R}} \int_0^1 \rho(1-\rho^2)^\delta J_{\frac{n-2}{2}}(\rho r \xi_{n+1}) J_{\frac{n-2}{2}}(\rho s \xi_{n+1}) \\ & \times \psi(\xi_{n+1})\, \xi_{n+1}^2\, e^{i(t-z)\xi_{n+1}}\, d\rho\, d\xi_{n+1}. \end{aligned}$$

We will refer the following elementary estimates in the proof of Propositions 1 and 2.

LEMMA 4. *Suppose $r, s > 0$ and $t, z \in \mathbb{R}$. Suppose that*

$$|U^\delta g(r,t)| \leq C \int_{\mathbb{R}} \int_0^\infty \left(\frac{s}{r}\right)^{(n-1)/2} \frac{1}{(1+|r \pm s|)^{\delta+1}} |g(s,z)|\, \chi_{D(r,t)}\, ds\, dz$$

where $D(r,t) = \{(s,z)\colon |r \pm s| > 1,\ |t-z| > 1,\ ||t-z| - |r\pm s|| \leq 1\}$. Then the right-hand side of the inequality above is bounded by the sum of the two terms

$$C \int_{\mathbb{R}} \int_0^\infty \frac{1}{|t-z|^{\delta+1}} |g(s,z)|\, \chi_{\{|r\pm s|\leq 1, |t-z|>1\}}\, ds\, dz \tag{3.1}$$

and

$$C \int_{\mathbb{R}} \int_0^\infty \left(\frac{s}{r}\right)^{(n-1)/2} \frac{1}{|r\pm s|^{\delta+1}} |g(s,z)|\, \chi_{\{|t-z|\leq 1, |r\pm s|>1\}}\, ds\, dz. \tag{3.2}$$

Proof. It suffices to show that

$$C \int_{\mathbb{R}} \int_0^\infty \left(\frac{s}{r}\right)^{(n-1)/2} \frac{1}{(1+|r-s|)^{\delta+1}} |g(s,z)| \chi_{D_1(r,t)}\, ds\, dz \tag{3.3}$$

is bounded by (3.1) where $D_1(r,t) = \{(s,z)\colon |r-s| > 1,\ |t-z| > 1,\ |(r-s) - (t-z)| \leq 1\}$. By the transformation described by $u = (r-s) - (t-z)$ and $v = t - z$, (3.3) is bounded by

$$C \iint_{\{|u|\leq 1, |v|>1\}} \left(\frac{r-u-v}{r}\right)^{(n-1)/2} \frac{1}{|u+v|^{\delta+1}} |g(r-u-v, t-v)|\, du\, dv. \tag{3.4}$$

Then since $|u| \leq 1$ and $|v| > 1$ and by substituting $r' = r - v$, the integral (3.4) is bounded by

$$C \iint_{\{|u|\le 1,|v|>1\}} \left(\frac{r'-u}{r'+v}\right)^{(n-1)/2} \frac{1}{|v|^{\delta+1}} |g(r'-u,t-v)|\,du\,dv.$$

Moreover, since $|\frac{r'-u}{r'+v}|$ for some constant C, we obtain

$$C \iint_{\{|u|\le 1,|v|>1\}} \frac{1}{|v|^{\delta+1}} |g(r'-u,t-v)|\,du\,dv.$$

For the case $|r-s|>1$, $|t-z|>1$ and $|(r-s)+(t-z)|\le 1$, we replace $v=t-z$ with $v=r-s$ and apply similar arguments to get (3.2). $\square$

PROPOSITION 1. *Suppose $f(x,t)=g(|x|,t)$ and $f\in L^p(\mathbb{R}^n\times\mathbb{R})$. Suppose $p=2n/(n+1+2\delta)$, $0<\delta\le(n-1)/2$ and $n\ge 2$. Then*

$$|\{(x,t)\in\mathbb{R}^n\times\mathbb{R} : |T^\delta f(x,t)|\,\chi_{\{0<|x|\le 1\}}>\alpha/2\}|\le C\alpha^{-p}\|f\|_p^p \tag{3.5}$$

for all $\alpha>0$. The constant C does not depend on α or f.

Proof. Recall that if $f(x,t)=g(|x|,t)$ and $T^\delta f(x,t)=U^\delta g(r,t)$,

$$U^\delta g(r,t)=(2\pi)^{-(n+1)}\int_{\mathbb{R}}\int_0^\infty H(r,t,s,z)g(s,z)\,ds\,dz.$$

We now consider the case $p>1$, that is, $0<\delta<(n-1)/2$.

We write

$$\begin{aligned}|U^\delta g(r,t)| &\le C\left(\int_{\mathbb{R}}\int_0^1+\int_{\mathbb{R}}\int_1^\infty |H(r,t,s,z)||g(s,z)|\,ds\,dz\right)\\ &:= C\,(I+II).\end{aligned}$$

For the integral I, by Lemma 1 when $0<r\le 1$ and $0<s\le 1$,

$$|H(r,t,s,z)|\le C\,s^{n-1}\frac{1}{(1+|t-z|)^{\delta+1}}.$$

We thus have

$$\begin{aligned} I &\le C\Bigg(\int_{\mathbb{R}}\int_0^1 |g(s,z)|s^{n-1}\,\chi_{\{|t-z|\le 1\}}\,ds\,dz\\ &\qquad+\int_{\mathbb{R}}\int_0^1 |g(s,z)|s^{n-1}|t-z|^{-(\delta+1)}\chi_{\{|t-z|>1\}}\,ds\,dz\Bigg)\\ &:= C\,(I_a+I_b).\end{aligned}$$

We now consider the integral I and let q be the conjugate component, $1/p+1/q=1$. For the integral I_a, since $r, s \le 1$ and $|t-z| \le 1$, we have

$$\begin{aligned} I_a &\le C\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p} \\ &\quad\times\left(\int_{\mathbb{R}}\int_0^1 s^{[(n-1)-\frac{(n-1)}{p}]q}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/q} \\ &\le C\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}. \end{aligned}$$

Since $r \le 1$,

$$I_a \le C\, r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}.$$

For the integral I_b, since $|t-z|>1$, we have

$$\begin{aligned} I_b &\le C\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\,ds\,dz\right)^{1/p} \\ &\quad\times\left(\int_{|z|\le\frac{1}{2}|t|}+\int_{\frac{1}{2}|t|<|z|<2|t|}+\int_{|z|\ge 2|t|}\int_0^1 s^{[(n-1)-\frac{(n-1)}{p}]q}|t-z|^{-(\delta+1)q}\,ds\,dz\right)^{1/q} \\ &\le C\, r^{-(n+1+2\delta)/2}(1+|t|)^{-(\frac{1}{p}+\delta)}\,\|f\|_p. \end{aligned}$$

Consequently for $r \le 1$,

$$\begin{aligned} I &\le C\, r^{-(n+1+2\delta)/2}(1+|t|)^{-(\frac{1}{p}+\delta)}\,\|f\|_p \\ &\quad+C\, r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}. \end{aligned} \tag{3.6}$$

For the integral II, since $0<r\le 1$ and $s>1$, by Lemma 2,

$$|H(r,t,s,z)| \le C\left(\frac{s}{r}\right)^{(n-1)/2}\frac{1}{(1+|r\pm s|)^{\delta+1}}\,\frac{1}{(1+||t-z|-|r\pm s||)^N}.$$

Let $E_1=\{(s,z)\colon |r\pm s|>1, |t-z|\le 1, ||t-z|-|r\pm s||>1\}$, $E_2=\{(s,z)\colon |r\pm s|>1, |t-z|>1, ||t-z|-|r\pm s||>1\}$, and $E_3=\{(s,z)\colon |r\pm s|>1, |t-z|>1, ||t-z|-|r\pm s||\le 1\}$. Then

$$\begin{aligned} II &\le C\left(\iint_{E_1}+\iint_{E_2}+\iint_{E_3}|g(s,z)||H(r,t,s,z)|\,ds\,dz\right) \\ &:= C\,(II_a+II_b+II_c). \end{aligned}$$

Now consider II and let q be the conjugate exponent, $1/p + 1/q = 1$. For the integral II_a, since $|r \pm s| \approx s$ and $||t-z| - |r \pm s|| \approx s$, we have

$$\begin{aligned} II_a &\le C \int_{\mathbb{R}} \int_1^\infty \left(\frac{s}{r}\right)^{(n-1)/2} s^{-(\delta+1+N)} \chi_{\{|t-z|\le 1\}} |g(s,z)|\, ds\, dz \\ &\le C r^{-(n-1)/2} \left(\int_{\mathbb{R}} \int_1^\infty |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p} \\ &\quad \times \left(\int_{\mathbb{R}} \int_1^\infty s^{[\frac{(n-1)}{2} - (\delta+1+N) - \frac{(n-1)}{p}]q} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/q} \\ &\le C r^{-(n-1)/2} \left(\int_{\mathbb{R}} \int_0^\infty |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}. \end{aligned}$$

Since $r \le 1$,

$$II_a \le C r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_0^\infty |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}.$$

For the integral II_b, since $||t-z| - |r \pm s|| > 1$ and $|t-z| > 1$ and $|r \pm s| \approx s$, we have

$$\begin{aligned} II_b &\le C \int_{\mathbb{R}} \int_1^\infty \left(\frac{s}{r}\right)^{(n-1)/2} s^{-(\delta+1)} |t-z|^{-N} |g(s,z)| \chi_{\{|t-z|>1\}}\, ds\, dz \\ &\le C r^{-(n-1)/2} \left(\int_{\mathbb{R}} \int_1^\infty |g(s,z)|^p s^{n-1}\, ds\, dz \right)^{1/p} \\ &\quad \times \left(\int_{|z|\le \frac{1}{2}|t|} + \int_{\frac{1}{2}|t|<|z|<2|t|} \right. \\ &\qquad \left. + \int_{|z|\ge 2|t|} \int_1^\infty s^{[\frac{(n-1)}{2} - (\delta+1) - \frac{(n-1)}{p}]q} |t-z|^{-Nq}\, ds\, dz \right)^{1/q} \\ &\le C r^{-(n-1)/2} (1+|t|)^{-(\frac{1}{p}+N-1)} \|f\|_p. \end{aligned}$$

Since $r \le 1$,

$$II_b \le C r^{-(n+1+2\delta)/2} (1+|t|)^{-(\frac{1}{p}+N-1)} \|f\|_p.$$

Finally, for the integral II_c we apply Lemma 4. So

$$\begin{aligned} II_c &\le C \int_{\mathbb{R}} \int_1^\infty \frac{1}{|t-z|^{\delta+1}} |g(s,z)| \chi_{\{|r\pm s|\le 1, |t-z|>1\}}\, ds\, dz \\ &\quad + C \int_{\mathbb{R}} \int_1^\infty \left(\frac{s}{r}\right)^{(n-1)/2} \frac{1}{|r \pm s|^{\delta+1}} |g(s,z)| \chi_{\{|t-z|\le 1, |r\pm s|>1\}}\, ds\, dz. \end{aligned} \tag{3.7}$$

The first expression in (3.7) is bounded by

$$C\left(\int_{\mathbb{R}}\int_1^\infty |g(s,z)|^p \chi_{\{|r\pm s|\le 1\}}\,ds\,dz\right)^{1/p}$$
$$\times\left(\left(\int_{|z|\le\frac{1}{2}|t|}+\int_{\frac{1}{2}|t|<|z|<2|t|}+\int_{|z|\ge 2|t|}\right)\int_1^\infty \chi_{\{|r\pm s|\le 1\}}|t-z|^{-(\delta+1)q}\,ds\,dz\right)^{1/q}$$
$$\le C(1+|t|)^{-(\frac{1}{p}+\delta)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p \chi_{\{|r\pm s|\le 1\}}\,ds\,dz\right)^{1/p}.$$

For the second integral in (3.7), since $|r\pm s|\approx s$ and $|t-z|\le 1$, by the similar computations used for the integral II_a, it is bounded by

$$C\,r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}.$$

Hence

$$\begin{aligned}(3.8)\qquad II \le\ & C\,r^{-(n+1+2\delta)/2}(1+|t|)^{-(\frac{1}{p}+N-1)}\|f\|_p\\ &+C\,r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}\\ &+C(1+|t|)^{-(\frac{1}{p}+\delta)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p \chi_{\{|r\pm s|\le 1\}}\,ds\,dz\right)^{1/p}.\end{aligned}$$

Collecting (3.6) and (3.8), it follows that for $p>1$,

$$\begin{aligned}|U^\delta g(r,t)|\chi_{\{0<r\le 1\}} \le\ & C\,r^{-(n+1+2\delta)/2}(1+|t|)^{-(\frac{1}{p}+N-1)}\|f\|_p\\ &+C\,r^{-(n+1+2\delta)/2}(1+|t|)^{-(\frac{1}{p}+\delta)}\|f\|_p\\ &+C\,r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}\\ &+C(1+|t|)^{-(\frac{1}{p}+\delta)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p \chi_{\{|r\pm s|\le 1\}}\,ds\,dz\right)^{1/p}.\end{aligned}$$

We now consider the case $p=1$, that is $\delta=(n-1)/2$. The only expressions we treat differently for this case are the integral II_a and II_b. Thus,

$$\begin{aligned}II_a \le\ & C\,r^{-(n-1)/2}\int_{\mathbb{R}}\int_1^\infty s^{-1-N}\chi_{\{|t-z|\le 1\}}|g(s,z)|\,ds\,dz\\ \le\ & C\,r^{-n}\int_{\mathbb{R}}\int_0^\infty |g(s,z)|s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz.\end{aligned}$$

Similarly,

$$\begin{aligned} II_b &\le C\, r^{-(n-1)/2} \int_{\mathbb{R}} \int_1^\infty s^{-1} |t-z|^{-N}\, |g(s,z)| \chi_{\{|t-z|>1\}}\, ds\, dz \\ &\le C\, r^{-n}\, (1+|t|)^{-N} \int_{\mathbb{R}} \int_1^\infty |g(s,z)| s^{n-1}\, ds\, dz \\ &\le C\, r^{-n}\, (1+|t|)^{-N}\, \|f\|_1. \end{aligned}$$

We may now proceed as in the case $p > 1$. This completes the proof of Proposition 1. □

PROPOSITION 2. *Suppose* $f(x,t) = g(|x|,t)$ *and* $f \in L^p(\mathbb{R}^n \times \mathbb{R})$. *Suppose* $p = 2n/(n+1+2\delta)$, $0 < \delta \le (n-1)/2$ *and* $n \ge 2$. *Then*

$$|\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |T^\delta f(x,t)|\, \chi_{\{|x|>1\}} > \alpha/2\}| \le C\, \alpha^{-p} \|f\|_p^p \tag{3.9}$$

for all $\alpha > 0$. *The constant* C *does not depend on* α *or* f.

Proof. We first consider the case $p > 1$, that is $0 < \delta < (n-1)/2$.
We now write

$$\begin{aligned} |U^\delta g(r,t)| &\le C \left(\int_{\mathbb{R}} \int_0^1 + \int_{\mathbb{R}} \int_1^\infty |g(s,z)||H(r,t,s,z)|\, ds\, dz \right) \\ &:= C\,(III + IV) \end{aligned}$$

and claim that

$$\begin{aligned} &|U^\delta g(r,t)| \chi_{\{r>1\}} \\ &\quad \le C\, r^{-(n+1+2\delta)/2} (1+|t|)^{-(\frac{1}{p}+N-1)} \|f\|_p \\ &\quad\quad + C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_0^\infty |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p} \\ &\quad\quad + C\, (1+|t|)^{-(\frac{1}{p}+\delta)} \left(\int_{\mathbb{R}} \int_0^\infty |g(s,z)|^p \chi_{\{|r\pm s|\le 1\}}\, ds\, dz \right)^{1/p} \\ &\quad\quad + C\, (1+|t|)^{-(\frac{1}{p}+N-1)} \left(\int_{\mathbb{R}} \int_0^\infty |g(s,z)|^p \chi_{\{|r-s|\le 1\}}\, ds\, dz \right)^{1/p} \\ &\quad\quad + C \left(\int_{\mathbb{R}} \int_0^\infty |g(s,z)|^p \chi_{\{|r-s|\le 1, |t-z|\le 1, r/2<s<2r\}}\, ds\, dz \right)^{1/p}. \end{aligned} \tag{3.10}$$

Next we split the range of the integration as follows: Let $G_1 = \{(s,z)\colon |r\pm s| > 1, |t-z| \le 1, ||t-z| - |r\pm s|| > 1\}$, $G_2 = \{(s,z)\colon |r\pm s| > 1, |t-z| > 1,$

$||t-z|-|r\pm s|| > 1\}$, $G_3 = \{(s,z)\colon |r\pm s| > 1, |t-z| > 1, ||t-z|-|r\pm s|| \le 1\}$, $G_4 = \{(s,z)\colon |r-s| \le 1, |t-z| > 1, ||t-z|-|r-s|| > 1\}$, and $G_5 = \{(s,z)\colon |r-s| \le 1, |t-z| \le 1, ||t-z|-|r-s|| \le 1, r/2 < s < 2r\}$.

Consider III and let q be the conjugate exponent, $1/p + 1/q = 1$. By Lemma 3 when $r > 1$ and $0 < s \le 1$,

$$|H(r,t,s,z)| \le C\, r^{-(n+1+2\delta)/2}\, s^{n-1}\, \frac{1}{(1+||t-z|-r|)^N}.$$

$$\begin{aligned} III &\le C\left(\iint_{G_1} + \iint_{G_2} + \iint_{G_3} |g(s,z)||H(r,t,s,z)|\, ds\, dz\right) \\ &:= C\,(III_a + III_b + III_c). \end{aligned}$$

For the integral III_a, since $||t-z|-r| \approx r$, we have

$$\begin{aligned} III_a &\le C\, r^{-(n+1+2\delta)/2} \int_{\mathbb{R}}\int_0^1 r^{-N} |g(s,z)| s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \\ &\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}}\int_0^1 |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz\right)^{1/p} \\ &\qquad \times \left(\int_{\mathbb{R}}\int_0^1 s^{[(n-1)-\frac{(n-1)}{p}]q} \chi_{\{|t-z|\le 1\}}\, ds\, dz\right)^{1/q} \\ &\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz\right)^{1/p}. \end{aligned}$$

Now consider III_b. Since $||t-z|-r| > 1$ and $|t-z| > 1$,

$$\begin{aligned} III_b &\le C\, r^{-(n+1+2\delta)/2} \int_{\mathbb{R}}\int_0^1 |t-z|^{-N} |g(s,z)| s^{n-1} \chi_{\{|t-z|>1\}}\, ds\, dz \\ &\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}}\int_0^1 |g(s,z)|^p s^{n-1}\, ds\, dz\right)^{1/p} \\ &\qquad \times \left(\left(\int_{|z|\le\frac{1}{2}|t|} + \int_{\frac{1}{2}|t|<|z|<2|t|} + \int_{|z|\ge 2|t|}\right)\int_0^1 s^{[(n-1)-\frac{(n-1)}{p}]q} |t-z|^{-Nq}\, ds\, dz\right)^{1/q} \\ &\le C\, r^{-(n+1+2\delta)/2} (1+|t|)^{-(\frac{1}{p}+N-1)}\, \|f\|_p. \end{aligned}$$

Finally, consider the integral III_c. By Lemma 4,

$$\begin{aligned} III_c &\le C \int_{\mathbb{R}}\int_0^1 \frac{1}{|t-z|^{\delta+1}} |g(s,z)| \chi_{\{|r\pm s|\le 1, |t-z|>1\}}\, ds\, dz \\ &\qquad + C\, r^{-(n+1+2\delta)/2} \int_{\mathbb{R}}\int_0^1 |g(s,z)| s^{n-1} \chi_{\{|t-z|\le 1, |r\pm s|>1\}}\, ds\, dz. \end{aligned} \tag{3.11}$$

The first row integral on the right-hand side of (3.11) is bounded by

$$C\,(1+|t|)^{-(\frac{1}{p}+\delta)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r\pm s|\le 1\}}\,ds\,dz\right)^{1/p}.$$

By the similar computations used for the integral III_a, the second row integral of (3.11) is bounded by

$$C\,r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}.$$

Thus,

$$\begin{aligned}(3.12)\quad III \;\le\;& C\,r^{-(n+1+2\delta)/2}\,(1+|t|)^{-(\frac{1}{p}+N-1)}\,\|f\|_p\\ &+C\,r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\chi_{\{|t-z|\le 1\}}\,ds\,dz\right)^{1/p}\\ &+C\,(1+|t|)^{-(\frac{1}{p}+\delta)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r\pm s|\le 1\}}\,ds\,dz\right)^{1/p}.\end{aligned}$$

For the integral IV, since $r, s > 1$ by Lemma 2,

$$|H(r,t,s,z)|\le C\left(\frac{s}{r}\right)^{(n-1)/2}\frac{1}{(1+|r\pm s|)^{\delta+1}}\,\frac{1}{(1+||t-z|-|r\pm s||)^N}.$$

Thus,

$$\begin{aligned}IV \;\le\;& C\left(\int_{\mathbb{R}}\int_1^{r/2}+\int_{\mathbb{R}}\int_{r/2}^{2r}+\int_{\mathbb{R}}\int_{2r}^{\infty}|g(s,z)||H(r,t,s,z)|\,ds\,dz\right)\\ :=\;& C\,(IV_a+IV_b+IV_c).\end{aligned}$$

Then,

$$\begin{aligned}IV_a+IV_c \;\le\;& \left(\iint_{G_1}+\iint_{G_2}+\iint_{G_3}|g(s,z)||H(r,t,s,z)|\,ds\,dz\right)\\ :=\;& (\mathcal{P}(r,t)+\mathcal{Q}(r,t)+\mathcal{R}(r,t)).\end{aligned}$$

For the integral $\mathcal{P}(r,t)$, we first assume $s < r/2$ and $s > 1$. Then since $|r\pm s|\approx r$ and

$||t-z|-|r\pm s|| \approx r$, with $1/p+1/q = 1$ for $p = 2n/(n+1+2\delta)$, $0 < \delta < (n-1)/2$,

$$C \int_{\mathbb{R}} \int_1^{r/2} \left(\frac{s}{r}\right)^{(n-1)/2} r^{-(\delta+1+N)} \chi_{\{|t-z|\le 1\}} |g(s,z)|\, ds\, dz$$
$$\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_1^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}$$
$$\times \left(\int_{\mathbb{R}} \int_1^{\infty} s^{[\frac{(n-1)}{2} - \frac{(n-1)}{p}]q} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/q}$$
$$\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_1^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}.$$

If $s > 2r$ and $r > 1$, with $1/p+1/q = 1$, then since $|r\pm s| \approx s$ and $||t-z|-|r\pm s|| \approx s$,

$$C \int_{\mathbb{R}} \int_{2r}^{\infty} \left(\frac{s}{r}\right)^{(n-1)/2} s^{-(\delta+1+N)} \chi_{\{|t-z|\le 1\}} |g(s,z)|\, ds\, dz$$
$$\le C\, r^{-(n-1)/2} \left(\int_{\mathbb{R}} \int_{2r}^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}$$
$$\times \left(\int_{\mathbb{R}} \int_{2r}^{\infty} s^{[\frac{(n-1)}{2} - (\delta+1+N) - \frac{(n-1)}{p}]q} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/q}$$
$$\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_{2r}^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}.$$

Consequently,

$$(3.13)\quad \mathcal{P}(r,t) \le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_0^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz \right)^{1/p}.$$

Likewise, for the integral $\mathcal{Q}(r,t)$, we also consider the cases $s < r/2$ and $s > 2r$. If $s < r/2$, then with $1/p + 1/q = 1$ we have

$$C \int_{\mathbb{R}} \int_1^{r/2} \left(\frac{s}{r}\right)^{(n-1)/2} r^{-(\delta+1)} |t-z|^{-N} |g(s,z)| \chi_{\{|t-z|>1\}}\, ds\, dz$$
$$\le C\, r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_1^{\infty} |g(s,z)|^p s^{n-1}\, ds\, dz \right)^{1/p}$$
$$\times \left(\int_{|z|\le \frac{1}{2}|t|} + \int_{\frac{1}{2}|t|<|z|<2|t|} + \int_{|z|\ge 2|t|} \int_1^{\infty} s^{[\frac{(n-1)}{2} - \frac{(n-1)}{p}]q} |t-z|^{-Nq}\, ds\, dz \right)^{1/q}$$
$$\le C\, r^{-(n+1+2\delta)/2} (1+|t|)^{-(\frac{1}{p}+N-1)} \|f\|_p.$$

If $s > 2r$, then with $1/p + 1/q = 1$,

$$C \int_{\mathbb{R}} \int_{2r}^{\infty} \left(\frac{s}{r}\right)^{(n-1)/2} s^{-(\delta+1)} |t-z|^{-N} |g(s,z)| \chi_{\{|t-z|>1\}}\, ds\, dz$$

$$\leq C r^{-(n-1)/2} \left(\int_{\mathbb{R}} \int_{2r}^{\infty} |g(s,z)|^p s^{n-1}\, ds\, dz \right)^{1/p}$$

$$\times \left(\left(\int_{|z| \leq \frac{1}{2}|t|} + \int_{\frac{1}{2}|t|<|z|<2|t|} + \int_{|z|\geq 2|t|} \right) \int_{2r}^{\infty} s^{[\frac{(n-1)}{2} - (\delta+1) - \frac{(n-1)}{p}]q} |t-z|^{-Nq}\, ds\, dz \right)^{1/q}$$

$$\leq C r^{-(n+1+2\delta)/2} (1+|t|)^{-(\frac{1}{p}+N-1)} \|f\|_p.$$

Accordingly,

$$\mathcal{Q}(r,t) \leq C r^{-(n+1+2\delta)/2} (1+|t|)^{-(\frac{1}{p}+N-1)} \|f\|_p. \tag{3.14}$$

We now pass to the integral $\mathcal{R}(r,t)$ with $1/p + 1/q = 1$. By Lemma 4 we have

$$\begin{aligned} \mathcal{R}(r,t) \leq\ & C \int_{\mathbb{R}} \int_1^{\infty} \frac{1}{|t-z|^{\delta+1}} |g(s,z)| \chi_{\{|r\pm s|\leq 1, |t-z|>1\}}\, ds\, dz \\ & + C \int_{\mathbb{R}} \int_1^{\infty} \left(\frac{s}{r}\right)^{(n-1)/2} \frac{1}{|r\pm s|^{\delta+1}} |g(s,z)| \chi_{\{|t-z|\leq 1, |r\pm s|>1\}}\, ds\, dz. \end{aligned} \tag{3.15}$$

The first expression on the right-hand side of (3.15) is bounded by

$$C(1+|t|)^{-(\frac{1}{p}+\delta)} \left(\int_{\mathbb{R}} \int_0^{\infty} |g(s,z)|^p \chi_{\{|r\pm s|\leq 1\}}\, ds\, dz \right)^{1/p}.$$

Moreover, by the similar computations used for the integral $\mathcal{P}(r,t)$, the second expression in (3.15) is bounded by

$$C r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_0^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\leq 1\}}\, ds\, dz \right)^{1/p}.$$

Thus,

$$\begin{aligned} \mathcal{R}(r,t) \leq\ & C(1+|t|)^{-(\frac{1}{p}+\delta)} \left(\int_{\mathbb{R}} \int_0^{\infty} |g(s,z)|^p \chi_{\{|r\pm s|\leq 1\}}\, ds\, dz \right)^{1/p} \\ & + C r^{-(n+1+2\delta)/2} \left(\int_{\mathbb{R}} \int_0^{\infty} |g(s,z)|^p s^{n-1} \chi_{\{|t-z|\leq 1\}}\, ds\, dz \right)^{1/p}. \end{aligned} \tag{3.16}$$

Next, we consider the integral IV_b. If $|r\pm s| > 1$, we have the same results with IV_a and IV_c. If $|r-s| \leq 1$, we have the following expressions for the integrals IV_b:

$$\iint_{G_4} + \iint_{G_5} |g(s,z)||H(r,t,s,z)|\, ds\, dz. \tag{3.17}$$

Since $||t-z|-|r-s|| \approx |t-z|$ and $1/p+1/q=1$, the first expression above in (3.17) is bounded by

$$
\begin{aligned}
(3.18)\quad & C\int_{\mathbb{R}}\int_1^\infty \left(\frac{s}{r}\right)^{(n-1)/2} |t-z|^{-N}\, |g(s,z)|\chi_{\{|r-s|\le 1, |t-z|>1\}}\, ds\, dz \\
&\le C\left(\int_{\mathbb{R}}\int_1^\infty |g(s,z)|^p\chi_{\{|r-s|\le 1\}}\, ds\, dz\right)^{1/p} \\
&\quad \times\left(\int_{|z|\le\frac{1}{2}|t|} + \int_{\frac{1}{2}|t|<|z|<2|t|} + \int_{|z|\ge 2|t|}\int_1^\infty \chi_{\{|r-s|\le 1\}}|t-z|^{-Nq}\, ds\, dz\right)^{1/q} \\
&\le C\,(1+|t|)^{-(\frac{1}{p}+N-1)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r-s|\le 1\}}\, ds\, dz\right)^{1/p}.
\end{aligned}
$$

Finally, with $1/p+1/q=1$ the second expression in (3.17) is bounded by

$$
\begin{aligned}
(3.19)\quad & C\int_{\mathbb{R}}\int_1^\infty |g(s,z)|\chi_{\{|r-s|\le 1,\, |t-z|\le 1, r/2<s<2r\}}\, ds\, dz \\
&\le C\left(\int_{\mathbb{R}}\int_1^\infty |g(s,z)|^p\chi_{\{|r-s|\le 1, |t-z|\le 1, r/2<s<2r\}}\, ds\, dz\right)^{1/p} \\
&\quad \times\left(\iint_{\{|r-s|\le 1, |t-z|\le 1\}} ds\, dz\right)^{1/q} \\
&\le C\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r-s|\le 1, |t-z|\le 1, r/2<s<2r\}}\, ds\, dz\right)^{1/p}.
\end{aligned}
$$

Thus, from (3.13)–(3.16) and (3.18)–(3.19),

$$
\begin{aligned}
(3.20)\quad IV \le\ & C\,r^{-(n+1+2\delta)/2}\,(1+|t|)^{-(\frac{1}{p}+N-1)}\,\|f\|_p \\
&+ C\,r^{-(n+1+2\delta)/2}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p s^{n-1}\,\chi_{\{|t-z|\le 1\}}\, ds\, dz\right)^{1/p} \\
&+ C\,(1+|t|)^{-(\frac{1}{p}+\delta)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r\pm s|\le 1\}}\, ds\, dz\right)^{1/p} \\
&+ C\,(1+|t|)^{-(\frac{1}{p}+N-1)}\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r-s|\le 1\}}\, ds\, dz\right)^{1/p} \\
&+ C\left(\int_{\mathbb{R}}\int_0^\infty |g(s,z)|^p\chi_{\{|r-s|\le 1, |t-z|\le 1, r/2<s<2r\}}\, ds\, dz\right)^{1/p}.
\end{aligned}
$$

Combining (3.12) and (3.20), we obtain (3.10) for $p>1$.

We are now left to consider the case $p = 1$, that is, $\delta = (n-1)/2$. The only expressions we treat differently for this case are the integrals $\mathcal{P}(r,t)$ and $\mathcal{Q}(r,t)$. The other parts are the same as for the case $p > 1$. Thus, consider the integral $\mathcal{P}(r,t)$. If $s < r/2$ and $s > 1$, then $|r \pm s| \approx r$ and $||t-z| - |r \pm s|| \approx r$ and we see that

$$C \int_{\mathbb{R}} \int_1^{r/2} \left(\frac{s}{r}\right)^{(n-1)/2} r^{-(\delta+1+N)} \chi_{\{|t-z| \le 1\}} |g(s,z)|\, ds\, dz \\ \le C r^{-n} \int_{\mathbb{R}} \int_1^{r/2} |g(s,z)| s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz.$$

If $s > 2r$ and $r > 1$, then $|r \pm s| \approx s$ and $||t-z| - |r \pm s|| \approx s$ and we obtain

$$C \int_{\mathbb{R}} \int_{2r}^{\infty} \left(\frac{s}{r}\right)^{(n-1)/2} s^{-(\delta+1+N)} \chi_{\{|t-z|\le 1\}} |g(s,z)|\, ds\, dz \\ \le C r^{-(n-1)/2} \int_{\mathbb{R}} \int_{2r}^{\infty} |g(s,z)| s^{-(N+1)} \chi_{\{|t-z|\le 1\}}\, ds\, dz \\ \le C r^{-(n-1)/2} \int_{\mathbb{R}} \int_{2r}^{\infty} |g(s,z)| s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz\, r^{-n}.$$

Recalling that $r > 1$, it follows that

$$\mathcal{P}(r,t) \le C r^{-n} \int_{\mathbb{R}} \int_0^{\infty} |g(s,z)| s^{n-1} \chi_{\{|t-z|\le 1\}}\, ds\, dz.$$

Now consider the integral $\mathcal{Q}(r,t)$. If $s < r/2$ and $s > 1$, then we have

$$C \int_{\mathbb{R}} \int_1^{r/2} \left(\frac{s}{r}\right)^{(n-1)/2} r^{-(\delta+1)} |t-z|^{-N} |g(s,z)| \chi_{\{|t-z|>1\}}\, ds\, dz \\ \le C r^{-n} \int_{\mathbb{R}} \int_1^{r/2} |t-z|^{-N} |g(s,z)| s^{n-1}\, ds\, dz \\ \le C r^{-n} (1+|t|)^{-N} \|f\|_1.$$

If $s > 2r$ and $r > 1$, then the integral $\mathcal{Q}(r,t)$ is bounded by

$$C \int_{\mathbb{R}} \int_{2r}^{\infty} \left(\frac{s}{r}\right)^{(n-1)/2} s^{-(\delta+1)} |t-z|^{-N} |g(s,z)| \chi_{\{|t-z|>1\}}\, ds\, dz \\ \le C r^{-(n-1)/2} \left(\int_{\mathbb{R}} \int_{2r}^{\infty} |t-z|^{-N} |g(s,z)| s^{-1}\, ds\, dz \right) \\ \le C r^{-(n-1)/2} (1+|t|)^{-N} \int_{\mathbb{R}} \int_{2r}^{\infty} |g(s,z)| s^{n-1}\, ds\, dz\, r^{-n}.$$

Thus,

$$\mathcal{Q}(r,t) \le C r^{-n} (1+|t|)^{-N} \|f\|_1.$$

We may now proceed as in the case of $p > 1$. This proves Proposition 2. □

Now we turn to the proof of Theorem 1.

Proof. Since $|U^\delta g(r,t)| = |U^\delta g(r,t)|\chi_{\{0<r\leq 1\}} + |U^\delta g(r,t)|\chi_{\{r>1\}}$, by using Propositions 1 and 2 we have

$$\begin{aligned}\iint_{\{(r,t):\ |U^\delta g(r,t)|>\alpha\}} r^{n-1}\,dr\,dt &\leq \iint_{\{(r,t):\ |U^\delta g(r,t)|\chi_{\{0<r\leq 1\}}>\alpha/2\}} \\ &\quad + \iint_{\{(r,t):\ |U^\delta g(r,t)|\chi_{\{r>1\}}>\alpha/2\}} r^{n-1}\,dr\,dt \\ &\leq C\alpha^{-p}\|f\|_p^p. \qquad \square\end{aligned}$$

Appendix

Let $1 \leq p < 2$. Denote the quasi norm $(\sup_{\alpha>0}\alpha^p|\{x \in \mathbb{R}^k : |f(x)| > \alpha\}|)^{1/p}$ of f in $L^{p,\infty}$ by $\|f\|_{L^{p,\infty}}$. Let $T_m f = m^\vee * f$. We define the class of Fourier multipliers $\mathcal{M}(L^p_{rad}, L^{p,\infty})$ to be the set of all bounded measurable functions m so that for all $f \in C_0^\infty(\mathbb{R}^k) \cap L^p_{rad}(\mathbb{R}^k)$,

$$\|T_m f\|_{L^{p,\infty}} \leq C\|f\|_{L^p_{rad}}.$$

The best constant C is the norm of the operator T_m, and we write $\|m\|_{\mathcal{M}}$ for this quantity.

We now split $\mathbb{R}^{k+l}$ into $\mathbb{R}^k \oplus \mathbb{R}^l$. Let $p \geq 1$. Denote by $L^p_{\text{rad}}(L^p)$ the space of all measurable functions f of the form $f(x', x'') = g(|x'|, x'')$ where g is defined on $(0,\infty) \times \mathbb{R}^l$, for which

$$\|f\|_{L^p_{\text{rad}}(L^p)} = \left(\int_{\mathbb{R}^l}\int_0^\infty |g(s,z)|^p s^{k-1}\,ds\,dz\right)^{1/p}$$

is finite.

We define the class of Fourier multipliers $\mathcal{N}(L^p_{\text{rad}}(L^p), L^{p,\infty})$ as the set of all bounded measurable functions m so that for all $f(x', x'') = g(|x'|, x'') \in C_0^\infty(\mathbb{R}^{k+l})$,

$$\|T_m f\|_{L^{p,\infty}} \leq D\,\|f\|_{L^p_{\text{rad}}(L^p)}.$$

The best constant D is the quasi norm of the operator T_m, and we write $\|m\|_{\mathcal{N}}$ for this quantity.

LEMMA 5. *Suppose f_ϵ and f are measurable functions on $\mathbb{R}^{k+l}$ and $f_\epsilon \to f$ almost everywhere. Assume that $\|f_\epsilon\|_{L^{p,\infty}} \leq M^{1/p}$ for some $M > 0$ and for all $\epsilon > 0$. Let $\alpha > 0$ be fixed. Then $\alpha^p|\{x \in \mathbb{R}^{k+l} : |f(x)| > \alpha\}| \leq M$.*

Proof. Suppose $\alpha^p|\{x\colon |f(x)| > \alpha\}| > M$. Then from the right continuity of $|\{x\colon |f(x)| > \alpha\}|$, there exists $\geq\alpha$ such that $\alpha^p|\{x\colon |f(x)| > \}| > M$. Define $E =$

$\{x\colon |f(x)| > \alpha\}$. We know $|E| < \infty$. Then by Egoroff's theorem for every $\eta > 0$, there exists $F \subset E$ such that $|E \setminus F| < \eta$ and $f_\epsilon \to f$ uniformly on F. We choose ϵ small so that $|f_\epsilon(x) - f(x)| < (\beta - \alpha)/2$ for $x \in F$. Moreover on F the inequality $|f(x)| > \beta$ implies that $|f_\epsilon(x)| \geq |f(x)| - |f_\epsilon(x) - f(x)| > \beta - (\beta - \alpha)/2 > \alpha$.

From this, $\{x\colon |f(x)| > \beta\} \cap F \subset \{x\colon |f_\epsilon(x)| > \alpha\}$ and $\alpha^p|\{x\colon |f(x)| > \beta\} \cap F| \leq \alpha^p|\{x\colon |f_\epsilon(x)| > \alpha\}|$. Since $\{x\colon |f(x)| > \beta\} = (\{x\colon |f(x)| > \beta\} \cap F) \cup (\{x\colon |f(x)| > \beta\} \cap (E \setminus F))$,

$$\begin{aligned} \alpha^p|\{x\colon |f(x)| > \beta\}| - \alpha^p|E \setminus F| &\leq \alpha^p|\{x\colon |f(x)| > \beta\} \cap F| \\ &\leq \alpha^p|\{x\colon |f_\epsilon(x)| > \alpha\}| \leq M. \end{aligned}$$

When η is sufficiently small, this is a contradiction. $\square$

The following is based on de Leeuw's restriction theorem [3].

THEOREM 2. *Let $m(\xi', \xi'')$ be contained in the class $\mathcal{N}(L^p_{rad}(L^p), L^{p,\infty})$ and be continuous. Then $m_{\xi''}(\xi') \equiv m(\xi', \xi'')$ is contained in the class $\mathcal{M}(L^p_{rad}, L^{p,\infty})$ and the multiplier norm of $m_{\xi''}$ does not exceed that of m.*

Proof. Let $f_1 \in C_0^\infty(\mathbb{R}^k) \cap L^p_{rad}(\mathbb{R}^k)$ and $f_{2,\epsilon} \in C_0^\infty(\mathbb{R}^l)$ with $\widehat{f_{2,\epsilon}}(\xi'') = \epsilon^{l(1/p-1)} \frac{\phi(\frac{\xi''-a}{\epsilon})}{\|\check{\phi}\|_p}$ and $\operatorname{supp} \phi \subset B(0, 1)$ (the unit ball about the origin). So $f_{2,\epsilon}(x'') = \epsilon^{l/p} \frac{\check{\phi}(\epsilon x'')}{\|\check{\phi}\|_p} e^{i<x'',a>}$ and $\|f_{2,\epsilon}\|_p = 1$. Define $f_\epsilon(x', x'') = (f_1 \otimes f_{2,\epsilon})(x', x'')$. So $\|f_\epsilon\|_{L^p_{rad}(L^p)} = \|f_1\|_{L^p_{rad}}$. Since we have

$$\begin{aligned} (3.21)\quad & T_m(f_1 \otimes f_{2,\epsilon})(x', x'') \\ &= \frac{1}{(2\pi)^{k+l}} \int_{\xi''} \int_{\xi'} m(\xi', \xi'') \widehat{f_1}(\xi') \epsilon^{l(1/p-1)} \frac{\phi(\frac{\xi''-a}{\epsilon})}{\|\check{\phi}\|_p} e^{i<x',\xi'>+i<x'',\xi''>} d\xi' \, d\xi'' \\ &= T_{m^\epsilon}\left(f_1 \otimes \epsilon^{l/p} \frac{\check{\phi}}{\|\check{\phi}\|_p}\right)(x', \epsilon x'') e^{i<x'',a>} \end{aligned}$$

where $m^\epsilon(\xi', \xi'') = m(\xi', \epsilon\xi'' + a)$,

$$\left\| T_{m^\epsilon}\left(f_1 \otimes \frac{\check{\phi}}{\|\check{\phi}\|_p}\right)\right\|_{L^{p,\infty}} \leq \|m\|_{\mathcal{N}} \|f_1\|_{L^p_{rad}}. \tag{3.22}$$

Then for all $\alpha > 0$, we have $\alpha^p|\{(x', x'')\colon |T_{m^\epsilon}(f_1 \otimes \frac{\check{\phi}}{\|\check{\phi}\|_p})(x', x'')| > \alpha\}| \leq \|m\|^p_{\mathcal{N}} \|f_1\|^p_{L^p_{rad}}$ and by the Lebesgue Dominated Convergence Theorem, $T_{m^\epsilon}(f_1 \otimes \frac{\check{\phi}}{\|\check{\phi}\|_p})$ converges to $(T_{m_a} f_1) \otimes \frac{\check{\phi}}{\|\check{\phi}\|_p}$ as $\epsilon \to 0$ where $m_a(\xi') = m(\xi', a)$. Then from

Lemma 5, we have

$$\left\| (T_{m_a} f_1) \otimes \frac{\check{\phi}}{\|\check{\phi}\|_p} \right\|_{L^{p,\infty}} \leq \|m\|_{\mathcal{N}} \|f_1\|_{L^p_{rad}}. \tag{3.23}$$

Hence m_a is contained in the class $\mathcal{M}(L^p_{rad}, L^{p,\infty})$ and thus

$$\|m_a\|_{\mathcal{M}} \leq \|m\|_{\mathcal{N}}. \qquad \square$$

Even if we replace the continuity assumption of m by almost everywhere conditions, Theorem 2 still holds.

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REFERENCES

[1] J. Bourgain, *Estimates for the cone multipliers*, Operator Theory: Adv. Appl. **77** (1995), 41–60.

[2] S. Chanillo and G. Muckenhoupt, *Weak type estimates for Bochner-Riesz spherical summation multipliers*, Trans. Amer. Math. Soc. **294** (1986), 693–703.

[3] M. Jodeit Jr., *A note on Fourier multipliers*, Proc. Amer. Math. Soc. **27** (1971), 423–424.

[4] G. Mockenhaupt, *A note on the cone multipliers*, Proc. Amer. Math. Soc. **117** (1993), 145–152.

[5] G. Mockenhaupt, A. Seeger and C. D. Sogge, *Wave front sets, local smoothing and Bourgain's circular maximal theorem*, Ann. of Math. **136** (1992), 207–218.

[6] D. Müller and A. Seeger, *Inequalities for spherically symmetric solutions of the wave equation*, Math. Z. **3** (1995), 417–426.

[7] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

[8] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, London, 1966.

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