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AN OPENNESS THEOREM FOR HARMONIC 2-FORMS ON 4-MANIFOLDS

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ABSTRACT. Let M be a closed, oriented 4-manifold with $b_2^\pm > 0$. In this paper we show that the space of transverse intrinsically harmonic 2-forms in a fixed cohomology class is open in the space of closed 2-forms, subject to a condition which arises from cohomological considerations of a singular differential ideal.

1. Introduction

In this paper we address the following question: When is a closed i-form ω on a closed manifold M of dimension n *intrinsically harmonic*, that is, when does there exist a Riemannian metric g with respect to which ω is harmonic? In the case of 1-forms, an answer was given by Calabi in [2]:

THEOREM 1 (Calabi). *Let ω be a closed 1-form on M. Assume that it is transverse to the zero section of T^*M. Then ω is intrinsically harmonic if and only if* (i) *ω does not have any zeros of index 0 or n, and* (ii) *given any two points $p, q \in M$ which are not zeros of ω, there exists a path $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$, such that $\omega(\gamma(t))(\dot{\gamma}(t)) > 0$ for all $t \in [0,1]$.*

Let $\Omega^i(M)$ be the space of i-forms, and $\Omega^i_\alpha(M)$ the subspace consisting of i-forms in the cohomology class $\alpha \in H^i(M;\mathbf{R})$. Denote by $\mathcal{H}^i_\alpha \subset \Omega^i(M)$ the space of intrinsically harmonic i-forms in the class α, and let $\widetilde{\mathcal{H}}^i_\alpha \subset \mathcal{H}^i_\alpha$ be the harmonic i-forms in α which are transverse to all the strata of $\bigwedge^i(\mathbf{R}^n)^*$ under the action of $SO(n)$. Call elements in $\widetilde{\mathcal{H}}^i_\alpha$ *transverse*. For transversality results for harmonic forms we refer the reader to [8]. Calabi's theorem implies the following:

PROPOSITION 1. *$\widetilde{\mathcal{H}}^1_\alpha \subset \Omega^1_\alpha(M)$ is an open subset.*

In the case of 2-forms, the situation is quite subtle. There is no known analog of Calabi's theorem for 2-forms, and an intrinsic characterization of harmonic 2-forms is rather elusive. In this paper we prove an openness theorem for transverse harmonic 2-forms on a 4-manifold, which will illustrate some of the obstructions which may arise in the general case.

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Let M be a closed, oriented 4-manifold with $b_2^{\pm} > 0$. Then the generic harmonic 2-form ω in the class $\alpha \in H^2(M; \mathbf{R})$ (generic in the space of metrics) is neither self-dual (SD) nor anti-self-dual (ASD) (cf. Section 4.3 of [3]), and is transverse. In particular, recalling the stratification of $\bigwedge^2(\mathbf{R}^4)^*$ under the action of $SO(4)$, we have the following:

(i) ω has no zeros.
(ii) The locus where ω is SD/ASD consists of a union of circles $C = \bigcup S^1$.
(iii) The locus where ω has rank 2 is a 3-manifold N (possibly disconnected).

Note that C and N are disjoint. For a proof, we refer to [8].

In order to state the theorem, it would be convenient to introduce the following:

Connectivity Condition. Let $\{N_j\}_{j=1}^r$ be the set of connected components of N. We call ω *semi-contact* on N_j if the pullback to N_j of $*\omega$ is zero. Let N' be the union of all the semi-contact N_j. Then ω satisfies the *connectivity condition* if $M - N'$ is connected.

We then have the following result.

THEOREM 2. *$\widetilde{\mathcal{H}}_\alpha^2 \subset \Omega_\alpha^2$ is open on the set of transverse intrinsically harmonic 2-forms ω satisfying the connectivity condition.*

On the way to proving this theorem we encounter the cohomology of the singular differential ideal $\mathcal{I} = (*\omega)$, which gives rise naturally to our connectivity condition. We will compute the *infinitesimal harmonic perturbations* of a harmonic form ω (see Section 2), via the singular differential ideal, and pass from infinitesimal to local using the Nash-Moser iteration technique.

We remark that the SD harmonic 2-forms are quite interesting in their own right—for a discussion see [9].

2. Infinitesimal harmonic perturbations

Let $\mathrm{Met}(M)$ be the space of C^∞-metrics on an n-dimensional manifold M, $T_g\mathrm{Met}(M) = \Gamma(\mathrm{Sym}^2(TM))$ be its tangent space at $g \in \mathrm{Met}(M)$, and $\Omega^i(M)$ consist of C^∞ i-forms. Then define

$$\Phi_\alpha\colon \mathrm{Met}(M) \to \Omega_\alpha^i(M),$$

which sends the metric g to the i-form ω with $\Delta_g\omega = 0$ and $[\omega] = \alpha$.

The derivative of Φ_α is the *infinitesimal harmonic perturbation map*

$$d\Phi_\alpha(g)\colon \Gamma(\mathrm{Sym}^2(TM)) \to d\Omega^{i-1}(M),$$

which we shall now compute.

Consider a 1-parameter family (ω_t, g_t) of harmonic i-forms on M, with $g_0 = g$, $[\omega_t] \in \alpha$, $h = \frac{d}{dt} g_t|_{t=0}$, and $\eta = \frac{d}{dt}\omega_t|_{t=0}$ exact. We differentiate

$$d\omega_t = 0, d^*\omega_t = 0$$

to obtain

$$\begin{aligned} &\text{(i)} \quad d\eta = 0, \\ &\text{(ii)} \quad d^*\eta = \pm d^*(*\dot{*}_{g_t}\omega). \end{aligned}$$

The Hodge decomposition gives

$$\Omega^i = d\Omega^{i-1} \oplus d^*\Omega^{i+1} \oplus \mathcal{H}^i,$$

so we find that $\eta = \pm\pi_1(*\dot{*}_{g_t}\omega)$, where π_1 is the projection onto the $d\Omega^{i-1}$ factor. Hence, $d\Phi_\alpha(g)$ is the composite map

$$\Gamma(\mathrm{Sym}^2(TM)) \xrightarrow{*\dot{*}\omega} \Omega^i(M) \xrightarrow{\pi_1} d\Omega^{i-1}(M)$$
$$h \mapsto *\dot{*}_{g+th}\omega \mapsto \pi_1(*\dot{*}_{g+th}\omega).$$

Hence, in order to compute the image of $d\Phi_\alpha(g)$, we solve the equation

$$\eta + *\eta' + \mu = *\dot{*}_{g+th}\omega, \tag{1}$$

where the exact form η is the given candidate for an infinitesimal harmonic perturbation, and we determine η' exact, μ harmonic, and h, the metric perturbation.

3. Infinitesimal computation for non-self-dual (or anti-self-dual) harmonic 2-forms on a 4-manifold

Let us now specialize to the 4-manifold M with $b_2^\pm > 0$. We prove the following microlocal result:

THEOREM 3. *Let (ω, g) be a transverse harmonic 2-form on M^4 in the class $\alpha \in H^2(M; \mathbf{R})$. If ω satisfies the connectivity condition, then $d\Phi_\alpha(g)$ is surjective, i.e., all the exact 2-forms on M are infinitesimal harmonic perturbations.*

Observe that a transverse harmonic 2-form must necessarily be non-SD/ASD, when both $b_2^+ > 0$ and $b_2^- > 0$.

In order to make use of Equation 1, we must first compute the image of the map

$$i_{\omega(x)}: \mathcal{S} \to \bigwedge\nolimits^2(\mathbf{R}^4)^*,$$
$$h \mapsto *\dot{*}_{g+th}\omega(x),$$

where $\mathcal{S}$ is the set of symmetric $n \times n$ matrices, and we assume that the bundle T^*M has been trivialized near x. In [8] we computed

1. $\operatorname{Im} i_{\omega(x)} = 0$, if $\omega(x) = 0$.
2. $\operatorname{Im} i_{\omega(x)} = \{\text{ASD (SD) 2-forms}\}$, if $\omega(x)$ is SD (ASD).
3. $\operatorname{Im} i_{\omega(x)} = (*\omega(x))^{\perp}$, otherwise.

For a transverse 2-form there are no points x where $\omega(x) = 0$. The primary difficulty with the transverse non-SD/ASD harmonic 2-form on a 4-manifold is that $\operatorname{Im} i_{\omega(x)}$ is never surjective.

It is most convenient to rewrite Equation (1) as follows: Noting that $\operatorname{Im} i_{\omega(x)} \subset (*\omega(x))^{\perp}$ whenever $\omega(x) \neq 0$, and, in particular, $\operatorname{Im} i_{\omega(x)} = (*\omega(x))^{\perp}$ when $\omega(x)$ is not SD/ASD, after taking $*$ we obtain

$$\eta' + *\eta + \mu \perp \omega, \tag{2}$$

where $\perp$ is the pointwise inner product with respect to g, and μ is some harmonic 2-form which may not be the same μ as in Equation (1). This can be rephrased as

$$(\eta' + *\eta + \mu) \wedge *\omega = 0. \tag{3}$$

We will thus compute the image of $d\Phi_\alpha(g)$ in the following fashion: Fix $\eta \in d\Omega^1(M)$, and solve for $\eta' = d\xi$ and μ harmonic in Equation (3), where on each component S^1 of C we additionally require that $(\eta' + *\eta + \mu)|_{S^1}$ be ASD whenever $\omega|_{S^1}$ is SD (and vice versa). If there exist such η' and μ, then, by linear algebra, we can find an h solving Equation (1). Neighborhoods of C require a little care when solving for h.

3.1. *Singular differential ideal.*
We want to compute the image of the composite map

$$\Omega^1(M) \xrightarrow{\mathcal{A}} \Omega^3(M) \xrightarrow{d} \Omega^4(M)$$
$$\xi \mapsto \xi \wedge *\omega \mapsto d(\xi \wedge *\omega) = d\xi \wedge *\omega.$$

We shall relate the image of this map to the cohomology $H^4(M, \mathcal{I})$ of a singular differential ideal, and compute it in this section. Let $\mathcal{I} = (*\omega)$ be the differential ideal generated by $*\omega$. The ideal has the chain complex

$$0 \to \mathcal{I}^0 = 0 \to \mathcal{I}^1 = 0 \to \mathcal{I}^2 \to \mathcal{I}^3 \to \mathcal{I}^4 \to 0.$$

Observe that $\mathcal{I}^4 = \Omega^4$: As long as $*\omega$ has no zeros, there exists a 2-form ξ such that $\xi \wedge *\omega = F\omega \wedge *\omega$ for given F. Also noting that $\mathcal{I}^3 = \{\xi \wedge *\omega | \xi \in \Omega^1\}$, we have:

LEMMA 1. $H^4(M, \mathcal{I}) = \Omega^4(M)/\operatorname{Im} d \circ \mathcal{A}$.

Hence, our problem is equivalent to computing $H^4(M, \mathcal{I})$ of a *singular differential ideal.*

PROPOSITION 2. $H^4(U, \mathcal{I}) = H^4(U, \mathbf{R})$, *if* $U \subset \{x \in M | \omega^2(x) \neq 0\}$.

Proof. This follows from observing that if ω is symplectic at x, then $\xi \mapsto \xi \wedge *\omega$ gives an isomorphism $\bigwedge^1(\mathbf{R}^4)^* \simeq \bigwedge^3(\mathbf{R}^4)^*$. $\square$

COROLLARY 1. *If ω is symplectic, then* $H^4(M, \mathcal{I}) = \mathbf{R}$.

COROLLARY 2. *If $\omega(x)$ is of generic type (i.e., nondegenerate and non-SD/ASD) for all $x \in M$, then $d\Phi_\alpha(g)$ is surjective.*

Proof. Note that $*\eta \wedge *\omega = \eta \wedge \omega$, with η exact. Hence $[*\eta \wedge *\omega] = 0 \in H^4(M; \mathbf{R})$. That is, we can let $\mu = 0$ and solve for $d\xi \wedge *\omega = *\eta \wedge *\omega$, which has a solution $d\xi$ by the proposition. $\square$

Let us now examine $\mathcal{I} = (*\omega)$ near the rank 2 submanifold N. Let $I \times N$ be a neighborhood of N, with coordinates (t, x). We can write

$$\omega = (\mu_1 + dt \wedge \tilde{\mu}_2) + t(\omega_1 + dt \wedge \widetilde{\omega}_2),$$

with μ_1, ω_1 2-forms, and $\tilde{\mu}_2$, $\widetilde{\omega}_2$ 1-forms, all without a dt-term. Here μ_1, $\tilde{\mu}_2$ do not depend on t.

On $I \times N$, we can solve for α in $d\alpha = *\eta \wedge *\omega$. Since α must satisfy $\alpha = \xi \wedge *\omega$ for some 1-form ξ, we require $\alpha|_N = 0$. Let us then modify $\alpha \mapsto \alpha - \delta\alpha$ so that $(\alpha - \delta\alpha)|_N = 0$. We write

$$\begin{aligned} \alpha &= \alpha_1(t, x) + dt \wedge \widetilde{\alpha}_2(t, x) &(4)\\ &= \alpha_1(0, x) + dt \wedge \widetilde{\alpha}_2(0, x) + \text{ h.o. in } t. &(5) \end{aligned}$$

Here, α_1 is a 3-form and $\widetilde{\alpha}_2$ is a 2-form, both without dt-terms.

If we let $\delta\alpha(t, x) = \alpha_1(0, x) + d(t\widetilde{\alpha}_2(0, x))$, then $(\alpha - \delta\alpha)|_N = 0$; since $\delta\alpha$ is closed, we still have $d(\alpha - \delta\alpha) = *\eta \wedge *\omega$. It is not difficult to see that $(\alpha - \delta\alpha)|_N = 0$ is sufficient to ensure the existence of a ξ such that $\xi \wedge *\omega = \alpha - \delta\alpha$. This follows from the transversality of ω near N. Summarizing, we have:

PROPOSITION 3. $H^4(I \times N, \mathcal{I}) = 0$.

Having taken care of the local aspects, we can pass from local to global. As before, let $\{N_j\}_{j=1}^r$ be the set of connected components of N. We say that ω is *semi-contact* on N_j if $\omega = \mu_1 + t(\omega_1 + dt \wedge \widetilde{\omega}_2)$, with μ_1 nonsingular and closed on N_j, i.e., $i_{N_j}^*(*\omega) = 0$, where $i_{N_j}\colon N_j \to M$ is the inclusion. Let N' be the union of all the semi-contact N_j. Then we have the following theorem:

THEOREM 4. *$\dim H^4(M,\mathcal{I}) = \#$ of connected components of $M - N'$.*

Proof. If $[\beta] = 0 \in H^4(M;\mathbf{R})$, then there exists a global α such that $d\alpha = \beta$.

CLAIM 1. *If $i^*_{N_j}[\alpha] = 0 \in H^3(N_j;\mathbf{R})$, then we can modify α so that $\alpha|_{N_j} = 0$.*

Proof of Claim 1. Recall Equation (4) in the proof of Proposition 3. If $i^*_{N_j}[\alpha] = 0$, then we can write $\alpha_1(0,x) = d_3\gamma_j$ on N_j. Extend γ_j to $I \times N_j$ so that $\gamma_j(t,x) = \gamma_j(0,x)$, and damp $\gamma_j + t\tilde{\alpha}_2(0,x)$ out outside of $I \times N_j$. Finally, modify $\alpha \mapsto \alpha - \delta\alpha$, where $\delta\alpha = d(\gamma_j + t\tilde{\alpha}_2(0,x))$. □

CLAIM 2. *If $M - N'$ is connected, then we can modify $\alpha \mapsto \alpha + \delta\alpha$ with $\delta\alpha \in H^3(M;\mathbf{R})$ so that $i^*_{N_j}[\alpha + \delta\alpha] = 0 \in H^3(N_j;\mathbf{R})$ for all N_j semi-contact.*

Proof of Claim 2. Consider the exact sequence

$$H^3(M) \xrightarrow{i} H^3(N') \longrightarrow H^4(M,N') \longrightarrow H^4(M) \longrightarrow 0. \tag{6}$$

Since $M - N'$ is connected, $H^4(M,N') \simeq H_0(M - N') \simeq \mathbf{R}$. This implies that i is surjective, and that there exists a $\delta\alpha \in H^3(M;\mathbf{R})$ such that $i^*_{N_j}[\alpha + \delta\alpha] = 0 \in H^3(N_j;\mathbf{R})$ for all N_j semi-contact. □

CLAIM 3. *If $i^*_{N_i}[\alpha] = 0$ for all N_i semi-contact, then there exists an $\alpha = \xi \wedge *\omega$ such that $d\alpha = \beta$.*

Proof of Claim 3. Let α satisfy $d\alpha = \beta$, with the additional condition that $i^*_{N_i}[\alpha] = 0$ for all N_i which are semi-contact. By Claim 1, we may also assume that $\alpha|_{N_i} = 0$ for all N_i semi-contact. Now assume N_j is not semi-contact. Then we can write

$$\omega = (\mu_1 + dt \wedge \tilde{\mu}_2) + t(\omega_1 + dt \wedge \tilde{\omega}_2),$$

with $\tilde{\mu}_2$ not identically zero. Then,

$$*\omega(0,x) = (*_3\tilde{\mu}_2 + dt \wedge *_3\mu_1)(0,x),$$

and there exist $\xi_j(t,x) = c_j f_j \tilde{\mu}_2(0,x)$ on N_j such that

$$\int_{N_j} c_j f_j \tilde{\mu}_2 \wedge *_3\tilde{\mu}_2 = \int_{N_j} \alpha.$$

We then damp ξ_j outside of $I \times N_j$, and solve for $\xi \wedge *\omega = \alpha - \sum \xi_j \wedge *\omega$, where the sum runs over all non-semi-contact N_j. Here, we may need to modify α using Claim 1, so that $(\alpha - \sum \xi_j \wedge *\omega)|_{N_i} = 0$ for every component N_i of N. Finally, we can write $\alpha = (\xi + \sum \xi_j) \wedge *\omega$. □

We will now complete the proof of Theorem 4. Refer back to Equation (6). Observe that $i^*_{N'}[\xi \wedge *\omega] = 0 \in H^3(N')$ if N' is the union of the semi-contact components. Hence, given β with $[\beta] = 0 \in H^4(M; \mathbf{R})$, for $\beta = d\alpha$ with $\alpha = \xi \wedge *\omega$ to be satisfied, we need $i^*_{N'}[\alpha] = 0 \in H^3(N')/i(H^3(M))$. This condition is also sufficient, since $i^*_{N'}[\alpha] = 0 \in H^3(N')/i(H^3(M))$ implies that there exists a representative α with $i^*_{N_j}[\alpha] = 0 \in H^3(N_j)$ for all N_j semi-contact, and we can apply Claim 3. Finally, $\dim H^3(N')/i(H^3(M)) = \dim H^4(M, N') - \dim H^4(M) =$ (# of components of $M - N'$) $- 1$. Thus, $\dim H^4(M, \mathcal{I}) =$ # of connected components of $M - N'$. □

Remark 1. We have two differential ideals, $\mathcal{I} = (*\omega)$ and $\mathcal{J} = (\omega)$, whose fates seem interconnected. It would be interesting to find out how they are related.

Remark 2. The computations of the singular differential ideals seem generalizable to higher dimensions, provided we have sufficient genericity.

Let us finally close this section with the following:

Conjecture. The connectivity condition is non-vacuous, i.e. there exists a transverse intrinsically harmonic form ω on a manifold M which does not satisfy the connectivity condition. Although we know of no explicit examples where the connectivity condition is necessary, the condition arises in such a natural fashion as a necessary condition for the surjectivity of the derivative map that we suspect that there indeed exist examples.

3.2. *Analysis near* $\bigcup S^1$.

In the previous section we saw that, if the connectivity condition is met, then we have a solution to $(\eta' + *\eta + \mu) \wedge *\omega = 0$. Note that we can set $\mu = 0$ since $*\eta \wedge *\omega = \eta \wedge \omega$ is exact on M. Under the conditions of Theorem 3, we find that there exists a 1-form ξ such that $d(\xi \wedge *\omega) = *\eta \wedge *\omega$ by Theorem 4, and hence we can set $\eta' = d\xi$.

Now we need to perform a more careful analysis near $C = \bigcup S^1$ in order to finish the proof of Theorem 3. Consider a connected component S^1 of C and let $N(S^1) = S^1 \times D^3$ have coordinates θ, x_1, x_2, x_3, which are orthonormal at $S^1 \times \{0\}$. Without loss of generality, let ω be SD on S^1. Fix an exact η, and we will solve for η' satisfying $(\eta' + *\eta) \wedge *\omega = 0$ on $S^1 \times D^3$, with the additional constraint that $\eta' + *\eta$ be ASD on S^1.

LEMMA 2. *There exists an exact η'_1 such that $\eta'_1 + *\eta$ is ASD on S^1.*

Proof. Let $\eta'_1 = -\eta$. Then η is exact and $-\eta + *\eta$ is ASD. □

Next, consider

$$\Omega^1(N(S^1)) \xrightarrow{A} \Omega^3(N(S^1)) \xrightarrow{d} \Omega^4(N(S^1))$$
$$\xi \mapsto \xi \wedge *\omega \mapsto d(\xi \wedge *\omega) = d\xi \wedge *\omega.$$

LEMMA 3. *Given* $F\omega \wedge *\omega \in \Omega^4(N(S^1))$ *with* $F|_{S^1} = 0$, *there exists a* $\xi \in \Omega^1(N(S^1))$ *with* $\xi|_{S^1} = 0$ *and* $d\xi|_{S^1} = 0$ *such that* $d \circ \mathcal{A}(\xi) = F\omega \wedge *\omega$.

Proof. The key is to find $\alpha = \xi \wedge *\omega$ of the form $\alpha = \widetilde{\alpha} \wedge d\theta$ with $\widetilde{\alpha} = \sum_i \alpha_i dx_{(i)}$, such that

$$\alpha_j(\theta, 0) = 0, \quad \frac{\partial}{\partial x_i}\alpha_j(\theta, 0) = 0, \text{ and } \frac{\partial}{\partial \theta}\alpha_j(\theta, 0) = 0,$$

where $1 \le i, j \le 3$, and $\theta \in S^1$. $d\alpha = d\widetilde{\alpha} \wedge d\theta = d_3\widetilde{\alpha} \wedge d\theta$, where d_3 is the differential with respect to $\{x_i\}$; on the other hand, $F\omega \wedge *\omega = f dx_1 dx_2 dx_3 d\theta$ for some f with $f|_{S^1} = 0$. Thus solving for $d\alpha = F\omega \wedge *\omega$ is equivalent to solving for $\sum_i \frac{\partial \alpha_i}{\partial x_i} = f$. It is clearly advantageous to us that this partial differential equation is very underdetermined. Let $\alpha_2 = \alpha_3 = 0$ on $N(S^1)$. Then $\frac{\partial \alpha_1}{\partial x_1} = f$ can be solved with initial condition $\alpha_1(\theta, 0, x_2, x_3) = 0$. Since $f|_{S^1} = 0$, we can choose α with $\frac{\partial \alpha_i}{\partial x_j}(\theta, 0) = \frac{\partial \alpha_i}{\partial \theta}(\theta, 0) = 0$.

Thus, $\alpha = \xi \wedge *\omega$ has $\alpha(\theta, 0)$ and all of its first partials vanish on S^1. Under the linear map $\mathcal{A}^{-1}$, α will get sent to ξ, with $\xi(\theta, 0)$ and all of the first partials of ξ equal to zero on S^1. Thus, $\xi|_{S^1} = 0$ and $d\xi|_{S^1} = 0$. □

We find an η'_1 as in Lemma 2 and an η'_2 as in Lemma 3 such that $(\eta'_2 + \eta'_1 + *\eta) \wedge *\omega = 0$ on $N(S^1)$. Let $\eta'_{N(S^1)} = \eta'_1 + \eta'_2$. This proves the following proposition:

PROPOSITION 4. *Given any exact* 2*-form* η, *there exists an exact* $\eta'_{N(S^1)}$ *on* $N(S^1)$ *such that* $\eta'_{N(S^1)} + *\eta$ *is ASD on* S^1 *and* $(\eta'_{N(S^1)} + *\eta) \wedge *\omega = 0$ *on* $N(S^1)$.

Let η be an exact 2-form on M as before. On M we have $\eta' = d\xi$ such that $(\eta' + *\eta) \wedge *\omega = 0$, and on $N(C)$ there exists an $\eta'_{N(C)}$ such that $\eta'_{N(C)} + *\eta$ is SD/ASD on the various S^1 as appropriate, and satisfies $(\eta'_{N(C)} + *\eta) \wedge *\omega = 0$ on $N(C)$.

Now write $\eta' = d\xi$ and $\eta'_{N(C)} = d\xi_{N(C)}$. Then, $d((\xi - \xi_{N(C)}) \wedge *\omega) = 0$, and $(\xi - \xi_{N(C)}) \wedge *\omega$ must be *exact* on $N(C)$. Write $(\xi - \xi_{N(C)}) \wedge *\omega = d\gamma$ on $N(C)$, with γ defined on $N(C)$. Extend γ to all of M by damping out outside of $N(C)$. Since ω is symplectic on Supp(γ), we can write $d\gamma = \xi' \wedge *\omega$, and modify $\eta' \mapsto \eta' - d\xi' = d(\xi - \xi')$. Summarizing, we have:

PROPOSITION 5. *Assume* ω *satisfies the connectivity condition. Then given an exact* 2*-form* η *on* M, *there exists an* $\eta' = d\xi$ *on* M *such that* $\eta' + *\eta$ *is SD/ASD on* C *and* $(\eta' + *\eta) \wedge *\omega = 0$ *on* M.

It remains to obtain a section h with $**_{g+th}\omega = \eta' + *\eta$. We use the following proposition with $\beta = \eta' + *\eta$ to complete our argument for Theorem 3.

PROPOSITION 6. *There exists a smooth solution h to the equation $i_\omega(h) = \beta$, provided $\beta|_{S^1}$ is ASD and $\beta \wedge *\omega = 0$ on $N(S^1) = S^1 \times D^3$.*

Proof. Decompose $\omega = \omega_+ + \omega_-$ and $\beta = \beta_+ + \beta_-$ into the SD and ASD parts. If $i_\omega(h) = \beta$, then

$$\begin{aligned} i_{\omega_+}(h) &= \beta_- \\ i_{\omega_-}(h) &= \beta_+. \end{aligned}$$

We expand $\omega_1^+ = \omega_+$ to a basis $\{\omega_1^+, \omega_2^+, \omega_3^+\}$ for the SD forms near S^1. Since $T_g\mathrm{Met}(M) \simeq \mathrm{Hom}(\bigwedge^+, \bigwedge^-)$, in order to specify h it suffices to specify

$$\begin{aligned} \omega_1^+ &\mapsto \beta_1^- = \beta_- \\ \omega_2^+ &\mapsto \beta_2^- \\ \omega_3^+ &\mapsto \beta_3^- \end{aligned}$$

in a manner consistent with $\omega_- \mapsto \beta_+$.

CLAIM. *$h\colon \bigwedge^+ \oplus \bigwedge^- \to \bigwedge^- \oplus \bigwedge^+$ satisfies $\langle h(\alpha_+), \alpha_-\rangle = -\langle \alpha_+, h(\alpha_-)\rangle$, where $\alpha_\pm \in \bigwedge^\pm$.*

The claim is an easy exercise. We then see that the consistency condition is $\langle \beta_i^-, \omega_-\rangle = -\langle \omega_i^+, \beta_+\rangle$, or, equivalently, $\beta_i^- \wedge \omega_- = \omega_i^+ \wedge \beta_+$. We check that $\beta \wedge *\omega = 0$ implies $(\beta_+ + \beta_-) \wedge (\omega_+ - \omega_-) = \beta_+ \wedge \omega_+ - \beta_- \wedge \omega_- = 0$, giving us $\beta_- \wedge \omega_- = \omega_+ \wedge \beta_+$.

Let us now show that there exist β_2^-, β_3^- satisfying the consistency conditions. Write $\omega_- = \sum_l x_l \omega_l^-$ and $\beta_i^- = \sum_j b_{ij}\omega_j^-$, $i = 2, 3$, where $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ is a basis for $\bigwedge^-$ on $N(S^1)$, $\omega_i^- \wedge \omega_j^- = a_{ij} dv_{N(S^1)}$, and $dv_{N(S^1)}$ is the volume form on $N(S^1)$. Then

$$\begin{aligned} \beta_i^- \wedge \omega_- &= \sum_{jl} b_{ij}\omega_j^- \wedge x_l \omega_l^- = \sum_{jl} b_{ij} a_{jl} x_l dv_{N(S^1)} \\ \omega_i^+ \wedge \beta_+ &= \sum_l r_{il} x_l dv_{N(S^1)} \text{ for some } r_{il}, \end{aligned}$$

and solving for β_i^- in $\beta_i^- \wedge \omega_- = \omega_i \wedge \beta_+$ would be tantamount to solving for b_{ij} in $\sum_l b_{ij} a_{jl} = r_{il}$. But here a_{ij} is invertible since $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ is a basis for $\bigwedge^-$. □

This completes the proof of Theorem 3.

3.3. *Analysis near N.*

Although it is not necessary for our theorem, it is instructive to study the neighborhood $I \times N$ of N. Assume N is connected and the metric g on $I \times N$ is the product metric for simplicity. Take coordinates (t, x) on $I \times N$. Write

$$\omega = (\mu_1 + dt \wedge *_3\mu_2) + t(\omega_1 + dt \wedge *_3\omega_2),$$

where μ_1, μ_2 do not depend on t, ω_1, ω_2 depend on t, and μ_i, ω_i are all 2-forms without a dt-term. Write d, $*$ on N as d_3, $*_3$.

It turns out that ω_1 and ω_2 are completely determined by μ_1 and μ_2 because of the harmonicity ($d\omega = 0,\ d * \omega = 0$).

PROPOSITION 7. *ω_1 and ω_2 are given by*

$$\begin{aligned}
\omega_1(t,x) &= \frac{1}{t}\left(\frac{e^{(d_3*_3)t}+e^{-(d_3*_3)t}}{2}-1\right)\mu_1+\frac{1}{t}\left(\frac{e^{(d_3*_3)t}-e^{-(d_3*_3)t}}{2}\right)\mu_2,\\
\omega_2(t,x) &= \frac{1}{t}\left(\frac{e^{(d_3*_3)t}-e^{-(d_3*_3)t}}{2}\right)\mu_1+\frac{1}{t}\left(\frac{e^{(d_3*_3)t}+e^{-(d_3*_3)t}}{2}-1\right)\mu_2,
\end{aligned}$$

provided $e^{\pm(d_3*_3)t}(\mu_1)$ and $e^{\pm(d_3*_3)t}(\mu_2)$ make sense.

Proof. (A) $d\omega = 0$ implies

(1) $d_3\mu_1 = -td_3\omega_1$,
(2) $t\dot{\omega}_1 + \omega_1 = d_3 *_3 \mu_2 + td_3 *_3 \omega_2$,

(B) $d * \omega = 0$ implies

(3) $d_3\mu_2 = -td_3\omega_2$,
(4) $t\dot{\omega}_2 + \omega_2 = d_3 *_3 \mu_1 + td_3 *_3 \omega_1$.

Observe that (1) and (3) imply that $d_3\mu_1 = d_3\mu_2 = d_3\omega_1 = d_3\omega_2 = 0$ because the μ_i are t-independent.

Let us first integrate (2) and (4) using $(tf)' = tf'(t) + f(t) = h(t)$ as the model, with $f(t) = \frac{1}{t}\left(c + \int_0^t h(s)ds\right)$ as its general solution. If we require that $f(0)$ be finite, $c = 0$, and we have $f(t) = \frac{1}{t}\int_0^t h(s)ds$. Thus,

$$\begin{aligned}
\omega_1(t,x) &= \frac{1}{t}\int_0^t [d_3 *_3 \mu_2(s,x) + sd_3 *_3 \omega_2(s,x)]\,ds\\
&= d_3 *_3 \mu_2(0,x) + \frac{1}{t}\int_0^t sd_3 *_3 \omega_2(s,x)ds,\\
\omega_2(t,x) &= d_3 *_3 \mu_1(0,x) + \frac{1}{t}\int_0^t sd_3 *_3 \omega_1(s,x)ds.
\end{aligned}$$

Plugging ω_1 into the right-hand side of ω_2 (and vice versa), and iterating, we obtain

$$\begin{aligned}
\omega_1(t,x) &= (d_3*_3)\mu_2 + \frac{t}{2}(d_3*_3)^2\mu_1 + \frac{t^2}{6}(d_3*_3)^3\mu_2 + \cdots\\
&= \frac{1}{t}\left(\frac{e^{(d_3*_3)t}+e^{-(d_3*_3)t}}{2}-1\right)\mu_1+\frac{1}{t}\left(\frac{e^{(d_3*_3)t}-e^{-(d_3*_3)t}}{2}\right)\mu_2,\\
\omega_2(t,x) &= \frac{1}{t}\left(\frac{e^{(d_3*_3)t}-e^{-(d_3*_3)t}}{2}\right)\mu_1+\frac{1}{t}\left(\frac{e^{(d_3*_3)t}+e^{-(d_3*_3)t}}{2}-1\right)\mu_2 \qquad \square
\end{aligned}$$

Example (Contact case). This is the situation where $\omega = \mu_1 + t(\omega_1 + dt \wedge *_3\omega_2)$, with $*_3\mu_1 = \xi$, a contact 1-form, and $d_3 *_3 \mu_1 = d\xi = \mu_1$. Then we obtain

$$\begin{aligned} \omega &= (e^t + e^{-t})\mu_1 + (e^t - e^{-t})dt \wedge *_3\mu_1 \\ &= d((e^t + e^{-t})\xi). \end{aligned}$$

4. Local considerations

In this section, Met(M) and $\Omega^2_\alpha(M)$ are Fréchet spaces of smooth sections, with a *grading* given by Hölder norms $|\cdot|_{C^k}$. With the help of the Nash-Moser iteration technique, we now pass from the microlocal computation to a local statement:

THEOREM 5. *Φ_α is surjective near an (ω, g) which satisfies the connectivity condition.*

It is evident that Theorem 5 implies Theorem 2. Theorem 5, in turn, follows from the following:

THEOREM 6. *Let $g_0 \in Met(M)$ be a metric for which $(\omega_0 = \Phi_\alpha(g_0), g_0)$ satisfies the connectivity condition. Then there exist constants $C_k > 0$ and $\delta > 0$ with the following property: Given $\eta \in d\Omega^1$ and $|g - g_0|_1 \leq \delta$, there exists an $h \in \Gamma(\mathrm{Sym}^2(TM))$ such that $d\Phi_\alpha(g)(h) = \eta$ and $|h|_{k-2} \leq C_k(|\eta|_k + |\eta|_0|g|_k)$.*

Theorem 6 implies Theorem 5 by the Nash-Moser iteration process, which we describe in the next two sections.

4.1. *Tame maps.*

We will use the notion of tame maps between tame Fréchet manifolds, following R. Hamilton [7]. We refer the reader to [7] for definitions and a thorough discussion. Note that a *smooth tame map L: $F \to G$* of tame Fréchet manifolds is a tame map all of whose derivatives are tame.

Let V, W be vector bundles over M, and $\Gamma(V)$, $\Gamma(W)$ be tame Fréchet spaces of C^∞-sections over M. Consider $D^r(V, W)$, whose sections are differential operators of degree r from V to W. Locally we can write a differential operator of degree r as

$$L(\phi)(f) = \sum_{|\alpha| \leq r} \phi_\alpha(D_\alpha f).$$

Here α is a multiindex $(\alpha_1, \dots, \alpha_n)$ and $D_\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_n}$. We can think of $\phi = \{\phi_\alpha\}$ as a section of $D^r(V, W)$. Then we have a map

$$\begin{gathered} L\colon\ \Gamma(D^r(V, W)) \times \Gamma(V) \to \Gamma(W), \\ (\phi, f) \mapsto L(\phi)(f). \end{gathered}$$

PROPOSITION 8. *L is a smooth tame map.*

Now consider an open set $U \subset \Gamma(D^r(V, W))$ consisting of $\phi = \{\phi_\alpha\}$ such that $L(\phi)$ is elliptic and invertible. Then we have

$$L^{-1}\colon U \times \Gamma(W) \to \Gamma(V),$$
$$(\phi, g) \mapsto [L(\phi)]^{-1}(g).$$

PROPOSITION 9. *L^{-1} is a smooth tame map of degree $-r$.*

PROPOSITION 10. *$\Phi_\alpha\colon \mathrm{Met}(M) \to \Omega^i_\alpha(M)$ is a smooth tame map of degree 0.*

Proof. By the previous proposition,

$$L^{-1}\colon U \times \Omega^i(M) \to \Omega^i(M)$$

is a smooth tame map of degree -2, where $U \subset \Gamma(D^2(\bigwedge^i, \bigwedge^i))$ consists of elliptic and invertible degree 2 operators.

Now, consider the inclusion

$$\mathrm{Met}(M) \times \mathbf{C} \to \Gamma\left(D^2\left(\bigwedge^i, \bigwedge^i\right)\right),$$
$$(g, \lambda) \mapsto \Delta_g + \lambda,$$

which is a smooth tame map of degree 2. Since the composition of tame maps is tame, it follows that

$$G\colon \left((\mathrm{Met}(M) \times \mathbf{C}) \bigcap U\right) \times \Omega^i(M) \to \Omega^i(M),$$
$$[(g, \lambda), \omega] \mapsto G_g(\lambda)\omega \overset{\text{def}}{=} (\Delta_g + \lambda)^{-1}\omega$$

is a smooth tame map of degree 0. Next, consider

$$\Pi\colon \mathrm{Met}(M) \times \Omega^i(M) \to \Omega^i(M),$$
$$(g, \omega) \mapsto \pi_g(\omega),$$

where $\pi_g\colon \Omega^i(M) \to \mathcal{H}^i_g$ is the orthogonal projection onto the harmonic space $\mathcal{H}^i_g$. Π is a smooth tame map because

$$\pi_g(\omega) = -\frac{1}{2\pi i}\int_C G_g(\lambda)\omega d\lambda,$$

and $C \subset \mathbf{C}$ can be fixed on a small neighborhood of g. Finally, composing Π with

$$i\colon \mathrm{Met}(M) \to \mathrm{Met}(M) \times \Omega^i(M),$$
$$g \to (g, \omega_0),$$

we find that Φ_α is a smooth tame map of degree 0. $\square$

4.2. *Nash-Moser iteration scheme.*
The following is the version of Nash-Moser that we will use:

THEOREM 7 (Nash-Moser). *Let F, G be tame Fréchet spaces and $U \subset F$ an open set. Suppose L: $U \to G$ is a smooth tame map, $dL(f)$ is surjective for all $f \in U$, and there exists a family of right inverses $(dL)^{-1}$: $U \times G \to F$ which is a tame map. Then L is locally surjective.*

We already know that Φ_α: $\mathrm{Met}(M) \to \Omega^2_\alpha(M)$ is a smooth tame map and that $d\Phi_\alpha$ is surjective near (ω_0, g_0). The conditions

$$|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k) \tag{7}$$

would assure us that $(dL)^{-1}$ is tame. Applying the Nash-Moser iteration process, we see that Theorem 6 would imply Theorem 5.

4.3. *Estimates.*
We will prove Estimates 7 above by carefully retracing the argument in Theorem 3. Keep in mind that $|g - g_0|_1 \leq \delta$ throughout.
The following interpolation lemma is useful in our estimates:

LEMMA 4 (Interpolation). *If f_1, f_2 are functions on a compact manifold X, then*

$$|f_1 f_2|_k \leq C(|f_1|_0|f_2|_k + |f_1|_k|f_2|_0).$$

In the proof of Theorem 4, we first solve for $d\alpha = *\eta \wedge *\omega$. Noting that $|\omega|_k \leq C(1 + |g|_k)$ since Φ_α is smooth tame of degree 0, we obtain bounds

$$|d\alpha|_k \leq C(|\eta|_k|g|_0 + |g|_k|\eta|_0) \leq C(|\eta|_k + |\eta|_0|g|_k)$$

by interpolation.

LEMMA 5. *Given an exact i-form β on a compact manifold X, there exists an $\alpha \in \Omega^{i-1}(X)$ such that $d\alpha = \beta$ and $|\alpha|_{k+1} \leq C|\beta|_k$.*

Proof. We make use of the Green's function G_{g_0} at g_0, and write $\alpha = d^{*_0} G_{g_0}\beta$. $d\alpha = \beta$, and

$$|\alpha|_{k+1} = |d^{*_0} G_{g_0}\beta|_{k+1} \leq C|G_{g_0}\beta|_{k+2} \leq C|\beta|_k. \quad \square$$

Thus, there exists an α such that $d\alpha = *\eta \wedge *\omega$ and $|\alpha|_{k+1} \leq C(|\eta|_k + |\eta|_0|g|_k)$.

Claim 2 *bounds.* Next, we bound the α modified as in Claim 2 of Theorem 4. Observe that, as long as $|g - g_0|_1 \leq \delta$, for δ small, $|\omega - \omega_0|_1$ is small, and the harmonic form remains transverse. Hence, the rank 2 subsets N remain submanifolds, and are close together, provided the $|g - g_0|_1$ are kept small.

Take a basis $\{[dv_{N_i}]\}$ for $H^3(N')$, with dv_{N_i} a volume form of unit volume on N_i. Let $[\gamma_i] \in H^3(M)$ satisfy $i^*_{N_i}[\gamma_j] = [\delta_{ij} dv_{N_i}]$. Fix representatives $\gamma_i \in [\gamma_i]$. Then $\delta\alpha = -\sum_i a_i \gamma_i$, with $|\gamma_i|_{k+1}$ fixed constants, and $a_i = \int_{N_i} \alpha$. Hence,

$$|\delta\alpha|_{k+1} \leq C \sum_i |\alpha|_0 |\gamma_i|_{k+1} \leq C|\alpha|_0 \leq C(|\eta|_k + |\eta|_0 |g|_k).$$

Claim 3 *bounds.* We now have bounds for α, where $d\alpha = *\eta \wedge *\omega$ and $i^*_{N_i}[\alpha] = 0$ for all N_i semi-contact. Take N_j not semi-contact, and we first estimate ξ_j on $I \times N_j$. $\xi_j(t, x) = c_j f_j \tilde{\mu}_2(0, x)$, with $\int_{N_j} \xi_j \wedge *_3 \tilde{\mu}_2 = \int_{N_j} \alpha$, where we are using the same $f_j \tilde{\mu}_2(0, x) = f_j \tilde{\mu}_2(g_0)(0, x)$ for all $|g - g_0| \leq \delta$, and we are simply varying the scaling factor c_j. Thus,

$$|\xi_j|_{k+1} \leq C|\alpha|_0 |\omega_0|_{k+1} \leq C|\alpha|_0,$$

on $I \times N_j$.

We now give bounds for the damping out process. Let $\phi(t)$ be a smooth function on $\mathbf{R}$ such that

$$\phi(t) = \begin{cases} 1 & \text{on } [\frac{-1}{2}, \frac{1}{2}] \\ 0 & \text{outside } [-1, 1], \end{cases}$$

and $0 \leq \phi(t) \leq 1$ on $[-1, \frac{-1}{2}] \bigcup [\frac{1}{2}, 1]$.

Then, modify $\xi_j \mapsto \xi_j \phi$. We find that

$$|\xi_j \phi|_{k+1} \leq C(|\xi_j|_{k+1} |\phi|_0 + |\phi|_{k+1} |\xi_j|_0) \leq C|\xi_j|_{k+1},$$

since ϕ is fixed throughout. With this new ξ_j,

$$|\alpha - \sum \xi_j \wedge *\omega|_{k+1} \leq C(|\eta|_k + |\eta|_0 |g|_k).$$

Claim 1 *bounds.* We may now assume that $i^*_{N_i}[\alpha] = 0$ for all N_i. We then modify $\alpha \mapsto \alpha - \delta\alpha$ so that $(\alpha - \delta\alpha)|_{N_i} = 0$. If we write $\alpha_1(0, x) = d_3 \gamma_j$ on N_j, then

$$|\gamma_j|_{k+2, N} \leq C|\alpha|_{k+1}$$

by Lemma 5. However, we can only bound $|\gamma_j + t\tilde{\alpha}_2(0, x)|_{k+1} \leq C|\alpha|_{k+1}$ because of the term $t\tilde{\alpha}_2$—we lose one derivative here unless we are careful.

Instead, use $\psi_\varepsilon(t)\tilde{\alpha}_2(0, x)$, where

$$\psi_\varepsilon(t) = \begin{cases} t & \text{on } [-\varepsilon, \varepsilon] \\ 0 & \text{outside } [-1, 1], \end{cases}$$

and ψ_ε damps out slowly to 0 on $[-1,-\varepsilon]\bigcup[\varepsilon,1]$. It is not difficult to see that for ε small, there exist ψ_ε with $|\psi_\varepsilon|_0$ arbitrarily small, and $|\psi_\varepsilon|_i \le |\psi|_i$, for $i>1$, where

$$\psi(t)=\begin{cases} t & \text{on } [-\frac{1}{2},\frac{1}{2}] \\ 0 & \text{outside } [-1,1], \end{cases}$$

and ψ damps out slowly to 0 on $[-1,-\frac{1}{2}]\bigcup[\frac{1}{2},1]$. $\psi_\varepsilon(t)\widetilde{\alpha}_2(0,x)$ will clearly do the job of $t\widetilde{\alpha}_2(0,x)$, with the advantage that we can find an ε (dependent on g) with

$$|\psi_\varepsilon\widetilde{\alpha}_2(0,x)|_{k+2} \le C|\alpha|_{k+1},$$
$$|\delta\alpha|_{k+1} = |d(\gamma_j+\psi_\varepsilon(t)\widetilde{\alpha}_2(0,x))|_{k+1} \le C|\alpha|_{k+1}.$$

As before, we do not lose any derivatives by damping out $\gamma_j+\psi_\varepsilon(t)\widetilde{\alpha}_2(0,x)$.

Thus,

$$|\alpha-\delta\alpha|_{k+1} \le C(|\eta|_k+|\eta|_0|g|_k).$$

Bounds for η'. Finally, $\alpha|_{N_i}=0$ for all N_i, and we solve for $\xi\wedge *\omega=\alpha$. We do not lose any derivatives where $\mathcal{A}$ is an isomorphism. However, near the N_i's we lose one derivative, i.e.,

$$|\xi|_k \le C(|\eta|_k+|\eta|_0|g|_k),$$

and

$$|\eta'|_{k-1} \le C|d\xi|_{k-1} \le C(|\eta|_k+|\eta|_0|g|_k).$$

Estimates near S^1. On $N(S^1)$, we have bounds

$$|\eta'|_k \le |\xi|_{k+1} \le C(|\eta|_k+|\eta|_0|g|_k).$$

Let us compute bounds on $\eta'_{N(S^1)}$ and $\xi_{N(S^1)}$. $\eta'_1=-\eta$, so $|\eta'_1|_k \le C|\eta|_k$. For bounds on η'_2 satisfying $\eta'_2\wedge *\omega = -(\eta'_1+*\eta)\wedge *\omega$ on $N(S^1)$ and $-(\eta'_1+*\eta)\wedge *\omega|_{S^1} = F\omega\wedge *\omega|_{S^1}=0$, we look to the proof of Lemma 3. Clearly, $|F|_k \le C(|\eta|_k+|\eta|_0|g|_k)$. Solving for $\alpha=\widetilde{\alpha}\wedge d\theta$ with $d\alpha=F\omega\wedge *\omega$, we have

$$|\alpha|_k \le C|F|_k \le C(|\eta|_k+|\eta|_0|g|_k),$$

and hence

$$|\xi_{N(S^1)}|_k \le C(|\eta|_k+|\eta|_0|g|_k).$$

Note that we have lost one derivative—had we worked a bit harder, that would not have been necessary, unlike the loss of derivative near N_i, which seems inherent to the problem.

Finally, we write $d\gamma=(\xi-\xi_{N(C)})\wedge *\omega$ on $N(C)$. By compactifying $S^1\times D^3$ to $S^1\times S^3$, for example, we can use Lemma 3 and obtain a γ satisfying

$$|\gamma|_{k+1} \le C|(\xi-\xi_{N(C)})\wedge *\omega|_k \le C(|\eta|_k+|\eta|_0|g|_k).$$

Damping γ out, we do not lose any derivatives, and hence

$$|\eta' - d\xi|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

In order the complete the proof of Theorem 6, we are left to prove:

LEMMA 6. *There exists an h on M such that $|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k)$.*

Proof. Consider h away from $N(S^1)$. Since $\eta + *\eta' = \{h, \omega\}$, the anticommutator of h and ω viewed as matrices, and i_ω has constant rank throughout, we are able to bound

$$|h|_{k-1} \leq C(|\eta + *\eta'|_{k-1}|\omega|_0 + |\eta + *\eta'|_0|\omega|_{k-1}) \leq C(|\eta|_k + |\eta|_0|g|_k),$$

by interpolation.

We next find h on $N(S^1)$. Writing $\beta = \eta' + *\eta$ and $\beta = \beta_+ + \beta_-$,

$$|\beta_1^-|_{k-1} = |\beta_-|_{k-1} \leq |\beta|_{k-1} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

β_2^-, β_3^- come from solving $\beta_i^- \wedge \omega_- = \omega_i^+ \wedge \beta_+$. Hence,

$$|r_{il}|_{k-2} \leq C|\omega_i^+ \wedge \beta_+|_{k-1} \leq (|\eta|_k + |\eta|_0|g|_k),$$

and we lose a derivative. Since $b_{ij}a_{jl} = r_{il}$, we have

$$|\beta_i^-|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k).$$

Hence $|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k)$ on $N(S^1)$. We finally interpolate the h that we find on $N(S^1)$ to the h on $M - N(S^1)$, while keeping $|h|_{k-2} \leq C(|\eta|_k + |\eta|_0|g|_k)$. □

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