SOME REPRESENTATION THEOREMS FOR INVARIANT PROBABILITY MEASURES

BY

S. CATER

Throughout this paper X will be a set, $\mathcal{R}$ will be a σ-algebra of subsets of X (for a definition of σ-algebra and σ-ring of subsets of X see [3]), T will be a mapping of X into X, and m will be a measure on $\mathcal{R}$. We say that $\mathcal{R}$ is *T-invariant* if $A \in \mathcal{R}$ implies $T^{-1}A \in \mathcal{R}$, and a set $A \in \mathcal{R}$ is *T-invariant* if $A = T^{-1}A$. If $\mathcal{R}$ is T-invariant, and if $m(A) = m(T^{-1}A)$ for all $A \in \mathcal{R}$, we say that m is *T-invariant.* We say that m is a probability measure on $\mathcal{R}$ if $m(X) = 1$. If m is a T-invariant probability measure, and if $m(A) = 0$ or 1 for every T-invariant set $A \in \mathcal{R}$, we say that m is *ergodic.* If m is a measure on $\mathcal{R}$ and if $E \in \mathcal{R}$, the measure m_1 defined by $m_1(A) = m(A \cap E)$, all $A \in \mathcal{R}$, is called the *contraction* of m to the set E.

In [1] Blum and Hanson studied the problem of expressing a T-invariant probability measure as a "combination" of some sort of ergodic measures. The following proposition can be inferred from their work.

PROPOSITION 1. *Let T be a 1-1 mapping of X onto X, let $\mathcal{R}$ be a T-invariant σ-algebra of subsets of X, let m be a T-invariant probability measure on $\mathcal{R}$, and let $\mathcal{E}$ be the set of all ergodic measures on $\mathcal{R}$. Suppose that for any T-invariant set $A \in \mathcal{R}$ for which there is a T-invariant probability measure m_0 with $m_0(A) > 0$ there is a $p \in \mathcal{E}$ for which $p(A) > 0$. Then m has an integral representation on $\mathcal{E}$; i.e., there is a probability measure μ on a σ-algebra of subsets of $\mathcal{E}$ such that for any set $A \in \mathcal{R}$, we have that $p(A)$, regarded as a function of p, is measurable on $\mathcal{E}$ and $m(A) = \int_{p\in\mathcal{E}} p(A)\, d\mu$.*

Employing methods similar to those in [1], Farrell [2] studied situations in which X is a topological space and $\mathcal{R}$ consists of the Baire subsets of X. The following proposition can be inferred from the work of Farrell.

PROPOSITION 2. *Let X be a compact Hausdorff space, let $\mathcal{R}$ consist of the Baire subsets of X, and let T be a continuous mapping of X into X. Then any T-invariant probability measure m on $\mathcal{R}$ has an integral representation as in Proposition 1.*

The purpose of the present paper is to construct analogues of Proposition 2 in which X is not required to be compact (or locally compact or σ-compact or metrizable) and to apply these analogues to several concrete examples to which the results stated in [2] are not applicable.

Now let $\mathfrak{F}$ be a real vector lattice of bounded real-valued functions on X. We say that $\mathfrak{F}$ is *T-invariant* if $f(x) \in \mathfrak{F}$ implies $f(Tx) \in \mathfrak{F}$. If $\mathfrak{F}$ is T-invariant,

Received February 21, 1963.

and if $\mathcal{R}$ is the smallest σ-algebra containing all sets of the form $X(f \geqq 1), f \in \mathfrak{F}$, it follows that $\mathcal{R}$ is also T-invariant.

As in Loomis [5, p. 34] we say that the vector lattice $\mathfrak{F}$ is *Stonian* if $f \in \mathfrak{F}$ implies min $(1, f) \in \mathfrak{F}$. We will establish

THEOREM I. *Let X be a set, let $\mathfrak{F}$ be a Stonian vector lattice of bounded functions on X, let $\mathcal{R}$ be the smallest σ-algebra containing all sets of the form $X(f \geqq 1), f \in \mathfrak{F}$, and let T be a mapping of X into X for which $\mathfrak{F}$ is T-invariant. Suppose $\mathfrak{F}$ satisfies*

(1) *if $\{f_n\}$ is a nondecreasing sequence of functions in $\mathfrak{F}$ converging pointwise to* 0, *then $\{f_n\}$ converges uniformly.*

Then any T-invariant probability measure m on $\mathcal{R}$ has an integral representation as in Proposition 1.

The next result generalizes Proposition 2.

THEOREM II. *Let X be a topological space, let $\mathcal{R}$ be the smallest σ-algebra containing all sets of the form $X(f \geqq 1)$ where f is real-valued, continuous, and the closure of $X(f \neq 0)$ is countably compact, and let T be a continuous mapping of X into X for which $T^{-1}A$ is countably compact for any closed countably compact set A. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation as in Proposition* 1.

Note that Theorem II does not exclude the degenerate case in which there is no nonzero continuous function f on X for which the closure of $X(f \neq 0)$ is countably compact; in this event $\mathcal{R}$ is composed of only the sets X, $\emptyset$, and the only probability measure m on $\mathcal{R}$ is given by $m(X) = 1$, $m(\emptyset) = 0$. No restrictions on the topology of X are needed in Theorem II.

THEOREM III. *Let X be a normal topological space, let $\mathcal{R}$ be the smallest σ-algebra containing every open F_σ set which has countably compact closure. Let T be a continuous mapping of X into X for which $T^{-1}A$ is countably compact for any closed countably compact set A. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation as in Proposition* 1.

Until Theorem I is proved we will assume its hypotheses are satisfied. Then by [2, pp. 451–452] we have the following two lemmas.

LEMMA 1. *If m_1 and m_2 are T-invariant probability measures such that $m_1(A) = m_2(A)$ for all T-invariant $A \in \mathcal{R}$, then $m_1 = m_2$ on $\mathcal{R}$.*

LEMMA 2. *If $A \in \mathcal{R}$ and $0 \leqq c \leqq 1$, there is a T-invariant $B \in \mathcal{R}$ such that $p(A) \leqq c$ if and only if $p(B) = 1$ for every ergodic measure p on $\mathcal{R}$.*

Also the following lemma can be established as in [1, Theorem 2].

LEMMA 3. *Let m be a T-invariant probability measure on $\mathcal{R}$, and let A be a*

T-invariant set in $\mathcal{R}$ for which $0 < m(A) < 1$. Then there exist T-invariant probability measures m_1 and m_2 on $\mathcal{R}$, absolutely continuous with respect to m, for which $m_1(A) = 1$, $m_2(A) = 0$, and $m = m(A)m_1 + [1 - m(A)]m_2$.

Now we employ our hypothesis (1) to establish the decisive link in the development of Theorem I.

LEMMA 4. *For any T-invariant set $A \in \mathcal{R}$ for which there is a T-invariant probability measure m on $\mathcal{R}$ with $m(A) > 0$, there exists an ergodic measure p on $\mathcal{R}$ with $p(A) > 0$.*

Proof. Let m be a T-invariant probability measure, and let A be a T-invariant set in $\mathcal{R}$ for which $m(A) > 0$.

Let $\mathcal{R}_0$ be the smallest σ-ring containing all sets of the form $X(f \geqq 1)$, $f \in \mathfrak{F}$ (we will find it necessary to consider $\mathcal{R}_0$ as well as $\mathcal{R}$). Note that

$$T^{-1}X(f(x) \geqq 1) = X(f(Tx) \geqq 1),$$

and it follows that $E \in \mathcal{R}_0$ implies $T^{-1}E \in \mathcal{R}_0$. Elementary arguments show that any set E in $\mathcal{R}$ is either in $\mathcal{R}_0$ or is the complement of some set in $\mathcal{R}_0$; if $E \in \mathcal{R}$, $E_0 \in \mathcal{R}_0$, then $E \cap E_0 \in \mathcal{R}_0$.

Select $B_1 \in \mathcal{R}_0$ so that $m(B_1) = \sup \{m(E); E \in \mathcal{R}_0\} \leqq 1$, and put

$$B_0 = B_1 \cup T^{-1}B_1 \cup T^{-1}(T^{-1}B_1) \cup \cdots .$$

Then $B_0 \in \mathcal{R}_0$, $T^{-1}B_0 \subset B_0$, and $m(E) = m(E \cap B_0)$ for all $E \in \mathcal{R}_0$. We claim that there is no set $B \in \mathcal{R}$ for which $m(B \cap B_0) = 0$ and $0 < m(B) < m(X - B_0)$. Assume such a set B exists. Set $E = X - (B_0 \cup B)$; then $m(E) > 0$. Since $E \cap B_0 = \emptyset$, it is plain that there is a set $G \in \mathcal{R}_0$ for which $E = X - G$. It follows that $E = E - B_0 = X - (G \cup B_0) = X - B_0$ and $B \subset B_0$ modulo m-null sets; hence $m(B - B_0) = 0$. Then $m(B) = m(B - B_0) + m(B \cap B_0) = 0$, which is impossible.

Let Y be the union $\bigcup_{n=0}^{\infty} T^{-n}(X - B_0)$ where T^{-n} denotes $(T^n)^{-1}$ and T^0 denotes the identity mapping of X onto X, and let m_1 be the contraction of m to Y. We claim that m_1 is T-invariant. Suppose $E \subset Y$; then

$$T^{-1}E \subset T^{-1}Y \subset Y,$$

and $m_1(T^{-1}E) = m(T^{-1}E) = m(E) = m_1(E)$. Suppose $E \subset X - Y$; then $(T^{-1}E) \cap Y = (T^{-1}E) \cap (X - B_0)$, and $E \in \mathcal{R}_0$ because $E \subset B_0$. Consequently $T^{-1}E \in \mathcal{R}_0$, $0 = m[(T^{-1}E) \cap Y] = m_1(T^{-1}E) = m(E \cap Y) = m_1(E)$, and consequently m_1 is T-invariant.

Suppose C is a T-invariant set in $\mathcal{R}$. Then

$$T^{-n}[C \cap (X - B_0)] = (T^{-n}C) \cap T^{-n}(X - B_0) = C \cap T^{-n}(X - B_0)$$

for all $n \geqq 0$, and

$$\begin{aligned} m_1(C) = m_1(C \cap Y) &= m_1[\textstyle\bigcup_{n=0}^{\infty} C \cap T^{-n}(X - B_0)] \\ &= m_1[\textstyle\bigcup_{n=0}^{\infty} T^{-n}[C \cap (X - B_0)]]. \end{aligned}$$

Now either $m[C \cap (X - B_0)] = 0$ or $m[C \cap (X - B_0)] = m(X - B_0)$; in the former case $m_1(C) = 0$, and in the latter case $m_1(C) = m_1(Y)$. We can assume without loss of generality in the proof of Lemma 4 that $m(A \cap Y) = 0$; for if $m(A \cap Y) > 0$, then $m_1/m_1(Y)$ would be an ergodic measure p on $\mathcal{R}$ for which $p(A) = p(A \cap Y) > 0$ as is required.

In all that follows suppose $m(A \cap Y) = 0$. Set

$$U = A \cap B_0, \qquad V = A \cap (X - B_0).$$

Now $T^{-1}U = (T^{-1}A) \cap (T^{-1}B_0) \subset A \cap B_0 = U$, and $m(T^{-n}V) = m(V) = 0$ for all $n \geqq 0$. Hence

$$m[U - \textstyle\bigcup_{n=0}^{\infty} T^{-n}V] > 0.$$

Because $T^{-1}U \subset U$ and $U \cup V = (T^{-1}U) \cup (T^{-1}V)$, it follows that $\bigcup_{n=0}^{\infty} T^{-n}V$ and $A - \bigcup_{n=0}^{\infty} T^{-n}V = U - \bigcup_{n=0}^{\infty} T^{-n}V$ are T-invariant. Without loss of generality we can assume that $A \subset B_0$. Then $A \in \mathcal{R}_0$.

For each $f \in \mathfrak{F}$ let I_f denote a copy of the real line under the usual topology, and let $\times_{f \in \mathfrak{F}} I_f$ denote the Cartesian product of all the I_f. Indeed $\times_{f \in \mathfrak{F}} I_f$ is a real topological vector space under coordinatewise addition and scalar multiplication. Note that no nonzero vector in $\times_{f \in \mathfrak{F}} I_f$ is annihilated by every continuous linear functional on $\times_{f \in \mathfrak{F}} I_f$.

Let V be the set of all measures v on $\mathcal{R}_0$ for which $v(E) \leqq 1$ and $v(E) = v(T^{-1}E)$ for all $E \in \mathcal{R}_0$. Then V is a convex set where $[\alpha v_1 + (1 - \alpha)v_2](E)$ is defined to be $\alpha v_1(E) + (1 - \alpha)v_2(E)$ for $0 \leqq \alpha \leqq 1$. Note that every $f \in \mathfrak{F}$ is $\mathcal{R}_0$-measurable. We construct a mapping ϕ of V into $\times_{f \in \mathfrak{F}} I_f$ as follows: for $v \in V$, $\phi(v)_f = \int f\, dv$. Clearly ϕ is affine; i.e.,

$$\phi(\alpha v_1 + (1 - \alpha)v_2) = \alpha\phi(v_1) + (1 - \alpha)\phi(v_2) \qquad \text{for } 0 \leqq \alpha \leqq 1.$$

We claim that $\phi(V)$ is closed in $\times_{f \in \mathfrak{F}} I_f$. To see this let $(a_f, f \in \mathfrak{F})$ be a point in the closure of $\phi(V)$; we must find a $u \in V$ for which $\phi(u)_f = a_f$, all $f \in \mathfrak{F}$. Clearly the mapping $\bar{u}(f) = a_f$ is a nonnegative linear functional on the vector space $\mathfrak{F}$, and $|\bar{u}(f)| \leqq \sup |f|$ for each $f \in \mathfrak{F}$. Now suppose $\{f_n\}$ is a nonincreasing sequence of functions in $\mathfrak{F}$ converging pointwise to 0 on X; by hypothesis (1), $\{f_n\}$ converges uniformly, and consequently $\lim_{n \to \infty} \bar{u}(f_n) = 0$.

We extend $\bar{u}$ to the class of $\bar{u}$-summable functions employing Daniell's Theory [5, Chapter III]. Then $0 \leqq \bar{u}(g) \leqq 1$ for any $\bar{u}$-summable function g for which $0 \leqq g \leqq 1$. Because $\mathfrak{F}$ is Stonian, it follows that for any $f \in \mathfrak{F}$ and any real number $c > 0$ the characteristic function of $X(f > c)$ is $\bar{u}$-summable. By the Monotone Convergence Theorem and the fact that the class of $\bar{u}$-summable functions is closed under the lattice operations (see [5]) we have that the characteristic function of any set in $\mathcal{R}_0$ is $\bar{u}$-summable. We now define a set function u on $\mathcal{R}_0$ as follows: $u(E) = \bar{u}(\chi_E)$ for each $E \in \mathcal{R}_0$. Then u is a measure on $\mathcal{R}_0$, and $\bar{u}(f) = \int f\, du$ for all $f \in \mathfrak{F}$ by [5, Corollary 3, p. 35]. Hence $\int f\, du = a_f$ for all $f \in \mathfrak{F}$; to show that $u \in V$ it suffices to prove that $u(E) \leqq 1$ and $u(T^{-1}E) = u(E)$ for any $E \in \mathcal{R}_0$. But $u(E) \leqq 1$ because

$0 \leqq \chi_E \leqq 1$. For each $v \in V$ we have $\int f(x)\, dv = \int f(Tx)\, dv$ for all $f \in \mathfrak{F}$, and consequently $\bar{u}[f(x)] = \bar{u}[f(Tx)]$ for all $f \in \mathfrak{F}$. It follows from the Daniell Theory that for any $\bar{u}$-summable function g we have that $g(Tx)$ is also $\bar{u}$-summable and $\bar{u}[g(x)] = \bar{u}[g(Tx)]$; by setting $g = \chi_E$ it follows that $u(E) = u(T^{-1}E)$. Consequently $u \in V$, $\phi(u) = (a_f, f \in \mathfrak{F})$, and $\phi(V)$ is closed in $\times_{f \in \mathfrak{F}} I_f$.

Now let $v \in V$, and put $\bar{v}(f) = \int f\, dv$ for all $f \in \mathfrak{F}$. Then $|\bar{v}(f)| \leqq \sup |f|$ for all $f \in \mathfrak{F}$, and, as in the argument above, $\bar{v}$ can be extended to the class of all $\bar{v}$-summable functions by the Daniell Theory. The characteristic function of any set in $\mathfrak{R}_0$ is $\bar{v}$-summable. Let $E \in \mathfrak{R}_0$, and select a number ε, $0 < \varepsilon < 1$. There are a $\bar{v}$-summable function g and a nondecreasing sequence $\{g_n\}$ of nonnegative functions in $\mathfrak{F}$ converging pointwise to g for which $g \geqq \chi_E$ and $\bar{v}(g) < \bar{v}(\chi_E) + \varepsilon$. Then

$$(1 - \varepsilon)v[E \cap X(g_n > 1 - \varepsilon)] \leqq \bar{v}(g_n) \qquad \text{for all } n,$$

and since $\bigcup_{n=1}^{\infty} E \cap X(g_n > 1 - \varepsilon) = E$, we have that

$$(1 - \varepsilon)v(E) = (1 - \varepsilon)\lim_{n\to\infty} v[E \cap X(g_n > 1 - \varepsilon)]$$
$$\leqq \lim_{n\to\infty} \bar{v}(g_n) = \bar{v}(g) < \bar{v}(\chi_E) + \varepsilon,$$

and $v(E) \leqq \bar{v}(\chi_E)$. Let $f \in \mathfrak{F}$, and let c be a positive number. Put

$$h = f - \min(c, f) \quad \text{and} \quad h_n = \min(1, nh)$$

for each integer n. Then $\{h_n\}$ is a nondecreasing sequence of functions in $\mathfrak{F}$ converging pointwise to $\chi_{X(f>c)}$, $vX(f > c) \geqq \bar{v}(h_n)$ for all n, and by the Monotone Convergence Theorem $vX(f > c) \geqq \bar{v}(\chi_{X(f>c)})$. Let $\mathcal{S}$ be the family of all sets $E \in \mathfrak{R}_0$ for which $v(E) = \bar{v}(\chi_E)$. If $E \in \mathcal{S}$ and $A \in \mathfrak{R}_0$, then $A \cap E \in \mathcal{S}$; for $\bar{v}(\chi_{A\cap E}) \geqq v(A \cap E)$, $\bar{v}(\chi_{E-A\cap E}) \geqq v(E - A \cap E)$, $\bar{v}(\chi_E) = v(E)$ imply that $\bar{v}(\chi_{A\cap E}) = v(A \cap E)$. But $X(f > c) \in \mathcal{S}$ for $f \in \mathfrak{F}$ and $c > 0$. Then $\mathcal{S}$ is closed under finite intersections, differences, finite unions, and (by the Monotone Convergence Theorem) countable unions. Consequently $\mathcal{S}$ is a σ-ring and $\mathcal{S} = \mathfrak{R}_0$. For any $E \in \mathfrak{R}_0$, $\bar{v}(\chi_E) = v(E)$. Furthermore if g is any bounded $\mathfrak{R}_0$-measurable function on X, g is the uniform limit of a monotone sequence of $\bar{v}$-summable functions, g is $\bar{v}$-summable, and $\bar{v}(g) = \int g\, dv$.

Consequently ϕ is 1-1 on V. If $v_1, v_2 \in V$, $\phi(v_1) = \phi(v_2)$, then $\int g\, dv_1 = \int g\, dv_2$ for any function g which is the pointwise limit on X of a monotone sequence of functions in $\mathfrak{F}$; it follows from the Daniell Theory that $v_1(E) = \int \chi_E\, dv_1 = \int \chi_E\, dv_2 = v_2(E)$ for any $E \in \mathfrak{R}_0$.

Now $\phi(V)$ is closed in $\times_{f \in \mathfrak{F}} I_f$ and is bounded in each component I_f. By the Tychonoff Product Theorem $\phi(V)$ is a compact subset of $\times_{f \in \mathfrak{F}} I_f$. The restriction of m to $\mathfrak{R}_0$ is a measure $v_1 \in V$ for which $v_1(A) > 0$ (remember that $A \in \mathfrak{R}_0$). By the Daniell Theory there are an $\mathfrak{R}_0$-measurable function g with $0 \leqq g \leqq \chi_A$ and a nonincreasing sequence $\{f_n\}$ of functions in $\mathfrak{F}$ converging pointwise to g such that $\int g\, dv_1 > 0$. Let V_n denote the subset of V

composed of all $v \in V$ for which $\int f_n \, dv \geqq \int g \, dv_1$. Let U be the set of all $v \in V$ for which $\int g \, dv \geqq \int g \, dv_1$. Then $\bigcap_{n=1}^{\infty} V_n = U$, and $\phi(U)$ is a nonvacuous convex compact subset of $\times_{f \in \mathfrak{F}} I_f$ because each $\phi(V_n)$ is convex, compact, and $v_1 \in U$.

Let k be the supremum of the set of numbers $\{\int g \, dv; v \in V\}$. For each integer $n > 0$ let S_n be the set of $v \in V$ for which $\int g \, dv \geqq k - n^{-1}$. By essentially the same argument given in the preceding paragraph each $\phi(S_n)$ is nonvacuous, convex, compact. Let $S = \bigcap_{n=1}^{\infty} S_n$; it follows that $\phi(S)$ is nonvacuous, convex, compact, and $\int g \, dv = k$ for any $v \in S$. By [4, Theorem 2.6.4, p. 28] $\phi(S)$ has an extreme point, say $\phi(p_0)$. Because ϕ is affine and 1-1, p_0 must be extremal in S. In fact p_0 is extremal in V, for if $v_2, v_3 \in V$, $\alpha v_2 + (1 - \alpha)v_3 = p_0$, $0 < \alpha < 1$, then

$$k = \int g \, dp_0 = \alpha \int g \, dv_2 + (1 - \alpha) \int g \, dv_3,$$

and plainly $v_2, v_3 \in S$, $v_2 = v_3 = p_0$.

Let B_1 be a set in $\mathfrak{R}_0$ for which $p_0(E) = p_0(E \cap B_1)$ for every $E \in \mathfrak{R}_0$. Define the measure p on $\mathfrak{R}$ as follows: $p(E) = p_0(E \cap B_1)$ for $E \in \mathfrak{R}$. Now $p_0(B_1) = 1$; for if $p_0(B_1) < 1$, then $p_0/p_0(B_1)$ is a measure v in V for which $\int g \, dv = \int g \, dp_0/p_0(B_1) > \int g \, dp_0 = k$, which is impossible. Consequently p is a probability measure on $\mathfrak{R}$.

We claim that p is T-invariant. If $E \in \mathfrak{R}_0$, then

$$p(E) = p_0(E \cap B_1) = p_0(E) = p_0(T^{-1}E) = p_0[(T^{-1}E) \cap B_1] = p(T^{-1}E),$$

and

$$p(X - E) = 1 - p(E) = 1 - p(T^{-1}E) = p(X - T^{-1}E) = p[T^{-1}(X - E)].$$

Since every set in $\mathfrak{R}$ is either in $\mathfrak{R}_0$ or is the complement of some set in $\mathfrak{R}_0$, we have that p is T-invariant.

We claim that p is ergodic. Assume that p is not ergodic, and let E be a T-invariant set in $\mathfrak{R}$ for which $0 < p(E) < 1$. Then by Lemma 3 there exist T-invariant probability measures m_1 and m_2 on $\mathfrak{R}$, absolutely continuous with respect to p, for which

$$m_1(E) = 1, \qquad m_2(E) = 0 \quad \text{and} \quad \mathrm{p} = p(E)m_1 + [1 - p(E)]m_2.$$

Because p_0 is extremal in V, it follows that m_1 must coincide with m_2 on $\mathfrak{R}_0$. In particular

$$m_1(E \cap B_1) = m_2(E \cap B_1), \quad \text{and} \quad m_1(E - E \cap B_1) \neq m_2(E - E \cap B_1).$$

Since m_1 and m_2 are absolutely continuous with respect to p, we have $p(E - E \cap B_1) > 0$. But $p(E - E \cap B_1) = p_0[(E - E \cap B_1) \cap B_1] = 0$, which is impossible.

Hence p is ergodic and $p(A) = p_0(A \cap B_1) = \int \chi_A \, dp_0 \geqq \int g \, dp_0 = k > 0$. This concludes the proof of Lemma 4.

Theorem I now can be developed by the same argument as in [1, pp. 1127–1128]. For the sake of completeness we briefly sketch the proof.

Proof of Theorem I. For each T-invariant set A in $\mathcal{R}$ let

$$\pi_A = \{p \in \mathcal{E}; p(A) = 1\}.$$

Then the collection of all such sets is a σ-algebra Π of subsets of $\mathcal{E}$.

Let m be a T-invariant probability measure on $\mathcal{R}$. Define a set function μ on Π as follows: $\mu(\pi_A) = m(A)$ for all T-invariant sets $A \in \mathcal{R}$. Routine arguments employing Lemma 4 show that μ is a probability measure on Π.

Now fix a set $B \in \mathcal{R}$, and regard $p(B)$, $p \in \mathcal{E}$, as a function of p. For any real number c, $0 \leqq c \leqq 1$, it follows from Lemma 2 that there is a T-invariant set B_c such that $p(B_c) = 1$ if and only if $p(B) \leqq c$ for all ergodic measures p; hence $\{p \in \mathcal{E}; p(B) \leqq c\} = \pi_{B_c}$, and $p(B)$ is a Π-measurable function on $\mathcal{E}$. Since $|p(B)| \leqq 1$, $p(B)$ is μ-summable on $\mathcal{E}$.

Define $m'(A) = \int_{p \in \mathcal{E}} p(A) \, d\mu$ for each $A \in \mathcal{R}$. Then m' is a T-invariant probability measure on $\mathcal{R}$. But if A is a T-invariant set in $\mathcal{R}$, then $m'(A) = \mu(\pi_A) = m(A)$. By Lemma 1, $m = m'$ and $m(A) = \int_{p \in \mathcal{E}} p(A) \, d\mu$, all $A \in \mathcal{R}$. This concludes the proof of Theorem I.

Theorems II and III follow immediately from this result.

Proof of Theorem II. Let $\mathfrak{F}$ be the family of all continuous real-valued functions on X for which the closure of $X(f \neq 0)$ is countably compact. Then $\mathfrak{F}$ is obviously a Stonian vector lattice of bounded functions. Indeed $\mathfrak{F}$ is T-invariant, for if $f(x) \in \mathfrak{F}$, then $T^{-1}X(f(x) \neq 0) = X(f(Tx) \neq 0)$, and the closure of $X(f(Tx) \neq 0)$ must be countably compact. We claim that $\mathfrak{F}$ satisfies hypothesis (1) in Theorem I. To see this, let $\{f_n\}$ be a nonincreasing sequence of functions in $\mathfrak{F}$ converging pointwise to 0 on X. Select $\varepsilon > 0$. Let A be the closure of the set $X(f_1 > 0)$. Then A is countably compact, and $A \subset \bigcup_{n=1}^{\infty} X(f_n < \varepsilon)$; hence there is an index N for which $A \subset X(f_N < \varepsilon)$ and $0 \leqq f_N < \varepsilon$. Thus $\{f_n\}$ converges uniformly, and $\mathfrak{F}$ satisfies (1). Theorem II now follows from Theorem I.

Proof of Theorem III. Let $\mathfrak{F}$ be the T-invariant Stonian vector lattice of functions composed of all real-valued functions f for which the closure of $X(f \neq 0)$ is countably compact. Let $\mathcal{R}'$ be the smallest σ-algebra of subsets of X containing all sets of the form $X(f \geqq 1)$, $f \in \mathfrak{F}$.

For every real number $c > 0$ and $f \in \mathfrak{F}$, $X(f > c)$ is an open F_σ set;

$$X(f > c) = \bigcup_{n=1}^{\infty} X(f \geqq c + n^{-1}).$$

Hence $X(f > c)$ is in $\mathcal{R}$, and $X(f \geqq 1) = \bigcap_{n=1}^{\infty} X(f > 1 - n^{-1})$ is in $\mathcal{R}$. Hence $\mathcal{R}' \subset \mathcal{R}$.

But on the other hand, suppose U is an open F_σ set with countably compact closure; say $U = \bigcup_{n=1}^{\infty} E_n$ where each E_n is closed. By Urysohn's Lemma there is a continuous real-valued function g_n for which $0 \leqq g_n \leqq 1$, $g_n(E_n) = 1$, and $g_n(X - U) = 0$. Put $f = \sum_{n=1}^{\infty} 2^{-n} g_n$. Then f is continuous on X, and $U = X(f > 0)$. For each integer $n > 0$, $X(f \geqq n^{-1})$ is in $\mathcal{R}'$, and consequently $U = \bigcup_{n=1}^{\infty} X(f \geqq n^{-1})$ is also in $\mathcal{R}'$. Hence $\mathcal{R} \subset \mathcal{R}'$ and $\mathcal{R} = \mathcal{R}'$. Theorem III now follows from Theorem II.

Having established Theorems I, II, III we turn now to some concrete applications.

Example 1. Let X be a set, let $\mathcal{R}$ be the smallest σ-algebra containing all the countable subsets of X, and let T be a mapping of X into X such that $T^{-1}x$ is at most a finite set for any $x \in X$. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation. To see this, give X the discrete topology and observe that Theorem II applies. (Note also that X is not σ-compact if X is uncountable.)

Example 2. Let X be the set of all countable ordinal numbers endowed with the order topology, let T be a continuous mapping of X into X, and let $\mathcal{R}$ be the smallest σ-algebra of subsets of X containing all the countable subsets. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation.

To see this, let $\mathcal{R}'$ be the smallest σ-algebra containing all sets of the form $X(f \geqq 1)$ where f is real-valued and continuous on X. Any continuous function f on X is constant on a final interval, and $X(f \geqq 1)$ is either a countable set or else the union of a countable set with a final interval. It follows that $\mathcal{R}' \subset \mathcal{R}$. On the other hand any set composed of one point is in $\mathcal{R}'$ and $\mathcal{R}' = \mathcal{R}$. But X is countable compact. Theorem II then gives us the conclusion immediately. Note that X is not compact or σ-compact or metrizable.

Example 3. Let $\aleph$ be a transfinite cardinal number, let Y consist of the smallest ordinal number whose power exceeds $\aleph$ and all smaller ordinal numbers, and endow Y with the order topology. Let X be the Cartesian product $Y \times Y$ with the diagonal removed, and let T be a homeomorphism of X onto X (for example, $T(a, b) = (b, a)$). Let $\mathcal{R}$ be the smallest σ-algebra containing all the compact G_δ subsets of X. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation.

To see this, let $\mathfrak{F}$ be the Stonian vector lattice composed of all continuous functions on X with compact support, and show that Theorem I applies. (The reader can also prove that X is not countably compact or σ-compact or metrizable.)

Example 4. Let Y be defined as in Example 3, and let Z be the set of all ordinal numbers in Y but the greatest one. Let X be the Cartesian product of countably infinitely many copies of Z, and let T be any continuous mapping of X into X. Let $\mathcal{R}$ be the smallest σ-algebra containing all sets of the form $X(f \geqq 1)$, f continuous on X. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation.

This conclusion follows immediately from Theorem II provided we are able to show that X is countably compact. Observe that any monotonic sequence of points in Z must converge to some limit in Z. And any sequence $\{x_n\}$ in Z has a monotonic subsequence (to see this show that if $\{x_n\}$ has no nonincreasing subsequence, then an argument by induction proves that $\{x_n\}$ has a nondecreasing subsequence). Thus every subsequence $\{x_n\}$ in Z has a convergent subsequence. With the Cantor diagonal method one can show that any sequence of points in X has a convergent subsequence. It follows that every infinite set in X has at least one accumulation point, and X is countably compact. Note that X is not locally compact or σ-compact or metrizable. Indeed every compact subset of X has void interior and no nonzero continuous function on X has compact support.

We conclude with three corollaries.

COROLLARY 1. *Let X be a locally compact Hausdorff space, and let T be a continuous mapping of X into X such that $T^{-1}A$ is compact for any compact set A. Let $\mathcal{R}$ be the smallest σ-algebra containing all the compact G_δ sets. Then $\mathcal{R}$ is T-invariant, and any T-invariant probability measure m on $\mathcal{R}$ has an integral representation.*

Proof. Let $\mathfrak{F}$ be the Stonian vector lattice composed of all continuous functions on X with compact support, and it follows at once that Theorem I applies. (Compare Corollary 1 with [2, Theorem 4] in which X is required to be σ-compact.)

COROLLARY 2. *Let X be a compact Hausdorff space, and let $\mathcal{R}$ consist of the Baire sets in X. Let T be a 1-1 mapping of X onto X for which T and T^{-1} map Baire sets into Baire sets, and suppose the graph of T is a Baire subset of $X \times X$. Then any T-invariant probability measure on $\mathcal{R}$ has an integral representation.*

Proof. For each integer n, positive, negative or zero, let X_n be a copy of X. Let Y be the Cartesian product $\times_{n=-\infty}^{\infty} X_n$; then Y is also compact Hausdorff. We define a mapping ϕ of X into Y as follows: $\phi(x)_n = T^n x$ for all $x \in X$. Obviously ϕ is 1-1. Put $X^* = \phi(X)$. Let T^* be the mapping of Y onto Y given by $(T^*y)_n = y_{n+1}$ for all $y \in Y$. Then T^* is a homeomorphism of Y onto Y, $\phi^{-1}T^*\phi = T$ on X, and $\phi T \phi^{-1} = T^*$ on X^*. Furthermore $T^{*n}[\phi(x)] = \phi(T^n x)$ and $T^{*n}X^* = X^*$ for all n.

For each index n let V_n be the set of all points $y \in Y$ for which $y_{n+1} = Ty_n$. Then V_n is a Baire set in Y because the graph of T is a Baire set in $X \times X$. Consequently $X^* = \bigcap_{n=-\infty}^{\infty} V_n$ is a Baire set in Y.

Let f be a continuous real-valued function on X, and (for some fixed index

n) put $f^*(y) = f(y_n)$ for all $y \in Y$. Then f^* is continuous on Y, and $\phi^{-1}[Y(f^* \geqq 1)] = T^{-n}X(f \geqq 1)$ is a Baire set in X because T^{-n} maps Baire sets into Baire sets. Likewise $\phi^{-1}[Y(f^* \geqq c)]$ is a Baire set in X for any real number c, and $f^*\phi$ is a Baire function on X.

By the Stone-Weierstrass Theorem the algebra of all continuous real-valued functions on Y is the smallest uniformly closed algebra containing all the functions on Y constructed from continuous functions on X as was f^* in the preceding paragraph; consequently for any continuous function g^* on Y we have that $g^*\phi$ is a Baire function on X, and $\phi^{-1}Y(g^* \geqq 1)$ is a Baire set in X. Thus if E^* is any Baire set in Y, $\phi^{-1}E^*$ is a Baire set in X. And if E^* is a Baire subset of X^*, $\phi^{-1}E^*$ is a Baire set in X.

On the other hand if f is a continuous function on X, then f^* is continuous on Y where $f^*(y) = f(y_0)$ for all $y \in Y$. Hence $\phi[X(f \geqq 1)] = X^* \cap Y(f^* \geqq 1)$ is a Baire subset of X^*, because X^* is a Baire set in Y. For any Baire set E in X, $\phi(E)$ is a Baire subset of X^*.

By Proposition 1 it suffices to show that given a T-invariant probability measure m on the Baire sets in X and a T-invariant Baire set A for which $m(A) > 0$, there exists an ergodic measure p on the Baire sets of X for which $p(A) > 0$. Clearly it suffices then to show that given a T^*-invariant probability measure m^* on the Baire subsets of X^* and a T^*-invariant Baire subset A^* of X^* for which $m^*(A^*) > 0$, there exists an ergodic (with respect to T^*) measure p^* for which $p^*(A^*) > 0$.

We extend m^* to a measure $\bar{m}$ on the Baire sets in Y as follows: for each Baire set E^* in Y put $\bar{m}(E^*) = m^*(E^* \cap X^*)$. Obviously $\bar{m}$ is a T^*-invariant probability measure. Since T^* is a homeomorphism of Y onto Y, it follows from Proposition 2 that there exists an ergodic measure $\bar{p}$ on the Baire sets in Y for which $\bar{p}(A^*) > 0$; hence $\bar{p}(A^*) = 1$. Then the contraction of $\bar{p}$ to X^* is an ergodic measure p^* on the Baire subsets of X^* for which $p^*(A^*) = 1$. This completes the proof.

COROLLARY 3. *Let X be a set, let $\mathfrak{F}$ be a Stonian vector lattice of bounded real-valued functions on X, and let $\mathfrak{R}_0$ be the smallest σ-ring containing all the sets of the form $X(f \geqq 1), f \in \mathfrak{F}$. Then the following are equivalent:*

(1) *If $\{f_n\}$ is any nonincreasing sequence of functions in $\mathfrak{F}$ converging pointwise to 0, then $\{f_n\}$ converges uniformly.*

(2) *If $\bar{u}$ is any nonnegative linear functional on $\mathfrak{F}$, bounded in the sense that $|\bar{u}(f)| \leqq M \sup |f|$ for some $M > 0$ and all $f \in \mathfrak{F}$, there is a measure u on $\mathfrak{R}_0$ for which $\bar{u}(f) = \int f\, du$ for all $f \in \mathfrak{F}$.*

(3) *If $\{f_n\}$ is a nonincreasing sequence of functions in $\mathfrak{F}$ converging pointwise to 0, and if $\bar{u}$ is any bounded nonnegative linear functional on $\mathfrak{F}$, then $\lim_{n\to\infty} \bar{u}(f_n) = 0$.*

Proof. That (1) $\Rightarrow$ (2) was established essentially in the proof of Lemma 4, so we will not repeat it here.

To show that (2) $\Rightarrow$ (3), assume (2), let $\{f_n\}$ be a nonincreasing sequence of functions in $\mathfrak{F}$ converging pointwise to 0, and let $\bar{u}$ be a bounded nonnegative linear functional on $\mathfrak{F}$. Then $0 = \lim_{n\to\infty} \int f_n\, du = \lim_{n\to\infty} \bar{u}(f_n)$ by the Monotone Convergence Theorem.

To show that (3) $\Rightarrow$ (1), let $\mathfrak{A}$ be an algebra of bounded real-valued functions on X such that $\mathfrak{F} \subset \mathfrak{A}$ and $\mathfrak{A}$ is complete in the sup norm. Then under the sup norm, $\mathfrak{A}$ is a commutative Banach algebra. There exists an isometric isomorphism p of $\mathfrak{A}$ onto $C(Y)$, the Banach algebra (under the sup norm) of all continuous functions vanishing at infinity on a certain locally compact Hausdorff space Y.

Now assume (3), and let $\{f_n\}$ be a nonincreasing sequence of functions in $\mathfrak{F}$ converging pointwise to 0, and let $y \in Y$. Then $f \to p(f)(y)$ is a bounded nonnegative linear functional on $\mathfrak{F}$, and by (3), $\lim_{n\to\infty} p(f_n)(y) = 0$. Thus $\{p(f_n)\}$ converges pointwise to 0 on Y, and it follows that $p(f_n)$ converges uniformly. Because p is isometric, $\{f_n\}$ must also converge uniformly to 0 on X. Thus (3) $\Rightarrow$ (1), and Corollary 3 is proved.

Hence in Theorem I, hypothesis (1) can be replaced by (2) or by (3).

References

1. J. R. Blum and D. L. Hanson, *On invariant probability measures I*, Pacific J. Math., vol 10 (1960), pp. 1125–1129.
2. R. H. Farrell, *Representation of invariant measures*, Illinois J. Math., vol. 6 (1962), pp. 447–467.
3. P. R. Halmos, *Measure theory*, New York, Van Nostrand, 1950.
4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, rev. ed., 1957.
5. L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, Van Nostrand, 1953.

University of Oregon
Eugene, Oregon