GAPS IN THE DIMENSIONS OF TRANSFORMATION GROUPS

BY

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1. Introduction

If G is a compact Lie group of homeomorphisms acting effectively on a connected m-manifold M, then by a theorem of Montgomery and Zippin in [7]

$$\dim G \leq r(r + 1)/2 \leq m(m + 1)/2$$

where r is the maximal dimension of the orbits of G on M. It has been observed [12, Chapter IV], [9] that certain dimensions of G less than $m(m + 1)/2$ are also excluded. In particular, these results show that the dimension of G cannot be in the following two ranges:

$$(m - 1)m/2 + 1 < \dim G < m(m + 1)/2 \qquad (m \neq 4)$$

$$(m - 2)(m - 1)/2 + 3 < \dim G < (m - 1)m/2 \qquad m \text{ large.}$$

It is the purpose of this paper to demonstrate that the above is only a special case of a more general phenomenon. In fact, Theorem 2 indicates the general pattern of gaps in the dimensions of G of which the above ranges are a part. The corollary to Theorem 2 transcribes the result into a statement concerning homogeneous spaces.

The main tool used is Theorem 1, a generalized version of the Montgomery-Zippin result cited above. Roughly speaking, Theorem 1 states that the dimension of G is further restricted by the dimension of its center and the number of simple factor groups in its semisimple part; the larger that either of these are, the smaller is the dimension of G.

2. Preliminaries

Consider a compact transformation group K on a locally compact Hausdorff space X. The *isotropy subgroup* at a point x in X, denoted by G_x, is defined as the subgroup

$$K_x = \{k \in K \mid kx = x\}.$$

The action of K on X is said to be *effective* if

$$\bigcap_{x \in X} K_x = e,$$

the identity element of K; the action is said to be *free* if

$$K_x = e, \qquad \text{all} \quad x \in X.$$

The *orbit* of K at a point x, denoted by $K(x)$, is defined as

$$K(x) = \bigcup_{k \in K} kx.$$

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$K(x)$ is homeomorphic to the left coset space K/K_x. The action of K on X is said to be *transitive* if

$$X = K(x)$$

for some x in X.

Following [1, IX], we define a *principal isotropy subgroup* as an isotropy subgroup of the lowest possible dimension with the fewest possible components. The isotropy subgroups of points on the same orbit are all conjugate and, accordingly, we define a *principal orbit* as an orbit with principal isotropy subgroups. Roughly speaking, a principal orbit is an orbit of maximal dimension which is maximal in an extended sense. All manifolds considered will be assumed to be without boundary although the main results apply equally well to manifolds with boundary by using the standard doubling trick.

LEMMA 1. *Suppose G is a compact connected Lie group acting on a compact connected m-manifold M such that the orbit space M/G is a manifold. If*

$$t = \textit{maximal dimension of the orbits of } G \textit{ on } M,$$

then

$$\dim M/G = m - t.$$

Proof. By [1, IX] the union M^* of the principal orbits is an open dense connected subset of M. It follows, moreover, from the connectedness of M^* that the principal isotropy subgroups are all conjugate. By Gleason [5], M^* is a fibre bundle over M^*/G with t-dimensional fibres. Since M^*/G is an open subset of M/G,

$$\dim M/G = \dim M^*/G = m - t.$$

Consider again a transformation group K acting on a space X. The subset $\tilde{K}$ of all elements of K which act as the identity on X form a normal subgroup of K, and $K/\tilde{K}$, in a natural fashion, acts effectively on X. The action of K on X is said to be *almost effective* if $\tilde{K}$ is finite. An almost effective action is said to be *almost free* if every isotropy subgroup K_x, $x \in X$, is contained in $\tilde{K}$.

The proof of the following lemma is based on a technique of Montgomery and Samelson in [6].

LEMMA 2. *Let $G = G_1 \oplus G_2$ be a direct sum of 2 compact connected Lie groups. If G acts almost effectively on a compact manifold M and if G_1 acts transitively on M, then G_2 acts almost freely on M.*

Proof. Let $H_2 = (G_2)_x$, $x \in M$, be an isotropy subgroup of the almost effective action of G_2 on M. Let y be any point in M. Then there exists

$$g_1 \in G_1 \quad \text{such that} \quad g_1(x) = y.$$

But

$$H_2\, y = g_1 H_2\, g_1^{-1}(y) = y.$$

Therefore H_2 acts trivially on M.

LEMMA 3. *Let G be a compact connected Lie group acting almost effectively and transitively on a compact connected m-manifold M. If*

$$G = G_1 \oplus G_2$$

where G_2 is isomorphic to the q-torus T^q, then G_1 is almost effective on the $(m - q)$-manifold M/G_2.

Proof. Since G_1 acts transitively on M/G_2, M/G_2 is a compact connected manifold. Now G_2 acts on M with only finitely many distinct isotropy subgroups [1, VI]. Using this fact it is easy to verify that the principal isotropy subgroup of the action of G_2 on M is trivial. Hence the dimension of a principal orbit is q and applying, Lemma 1 we obtain

$$\dim M/G_2 = m - q.$$

We must now verify that G_1 is almost effective on M/G_2. Suppose N is a closed subgroup of G_1 which acts trivially on M/G_2. If $\dim N > 0$, there exists a circle subgroup H of N. We consider the almost effective action of the $(q + 1)$-torus

$$H \oplus G_2$$

on M. Since H acts trivially on M/G_2,

$$M/(H \oplus G_2) = M/G_2.$$

It follows, however, from our earlier discussion that

$$\dim M/(H \oplus G_2) = m - (q + 1).$$

This is of course impossible.

Let G be a compact connected Lie group. Then G can be expressed in the following form

$$\text{(A)} \qquad G = (T^q \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_a)/N = \bar{G}/N$$

where T^q is a q-torus, $q \geq 0$ (T^0 is assumed to be trivial), each S_j is a compact connected simply-connected simple Lie group and N is a finite normal subgroup of $\bar{G}$.

Now each S_j of dimension 3 is isomorphic to Spin(3), the universal covering group of $SO(3)$. We employ the isomorphism

$$\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \oplus \mathrm{Spin}(3)$$

to combine pairs of 3-dimensional S_j's. With this convention, we may rewrite G in the form

$$\text{(B)} \qquad G = (T^q \oplus S_1' \oplus S_2' \oplus \cdots \oplus S_b')/N = \bar{G}/N$$

where each S_j' is either simple or isomorphic to Spin(4) and where there is at most one S_j' of dimension 3.

THEOREM 1. *Let G be a compact connected Lie group acting almost effectively on a connected manifold M and let t denote the maximal dimension of the orbits. Then if G has a structure of the form* (B), *there exist integers $t_1, t_2, \cdots, t_b$ such that*

$$\dim S_j' \leq t_j(t_j + 1)/2, \qquad j = 1, 2, \cdots, b$$

and

$$\sum t_j \leq t - q.$$

Proof. Since

$$\dim G = q + \sum \dim S_j' \leq q + \sum t_j(t_j + 1)/2$$
$$\leq q + (\sum t_j)[(\sum t_j) + 1]/2 \leq q + (t - q)(t - q + 1)/2 \leq t(t+1)/2,$$

Theorem 1 generalizes the Montgomery-Zippin result. Proceeding with the proof we consider the almost effective action of $\bar{G}$ on M. Now $\bar{G}$ acts transitively on a principal orbit $\tilde{M}$ of dimension t. Since $\bar{G}$ and M are connected, the action of $\bar{G}$ on $\tilde{M}$ may be shown to be almost effective. By Lemma 3, $G_0 = \bar{G}/T^q$ is transitive and almost effective on the compact connected $(t - q)$-manifold $M_0 = \tilde{M}/T^q$. We consider from now on this action of G_0 on M_0.

The semisimple group G_0 corresponds, of course, to the semisimple part of $\bar{G}$. We shall suppose, however, in the ensuing argument the following decomposition of G_0 into strictly simple groups S_j

$$G_0 = S_1 \oplus S_2 \oplus \cdots \oplus S_a. \tag{C}$$

Let V be a direct sum of S_j's such that V acts almost freely on M_0 and V is of maximal dimension. If $V = G_0$,

$$\dim G_0 \leq \dim M_0 = t - q.$$

Let $W = G_0/V$. Now W is the direct sum of all S_j's which do not belong to V.

W acts invariantly (but, possibly, not almost effectively) on $M_1 = M_0/V$. Since G_0 is transitive on M_0, W is transitive on M_1. Now V acts almost freely on M_0 and it follows from Gleason's theorem [5] that M_1 is a compact connected manifold with

$$\dim M_1 = \dim M_0 - \dim V.$$

If M_1 is a point, V is transitive on M_0 and by Lemma 2, W acts almost freely on M_0. Hence

$$\dim G_0 = \dim V + \dim W \leq 2 \dim M_0. \tag{δ}$$

If M_1 is not a point, since W acts transitively on M_1, some factor group S_{j_1} of W must act non-trivially on M_1. Since S_{j_1} is simple, S_{j_1} must be almost effective on M_1. Let

$$t_{j_1} = \text{maximum dimension of the orbits of } S_{j_1} \text{ on } M_1.$$

Then

$$\dim S_{j_1} \leq t_{j_1}(t_{j_1} + 1)/2$$

and by Lemma 1,

$$M_2 = M_1/S_{j_1}$$

is a compact manifold with

$$\dim M_2 = \dim M_1 - t_{j_1} = \dim M_0 - \dim V - t_{j_1} .$$

If M_2 is not a point, we continue the process. Suppose then that

$$S_{j_1}, S_{j_2}, \cdots, S_{j_k}$$

have been chosen such that

$$M_{k+1} = M_1/(S_{j_1} \oplus S_{j_2} \oplus \cdots \oplus S_{j_k})$$

is a point. Then

$$0 = \dim M_{k+1} = \dim M_0 - \dim V - \textstyle\sum_{l=1}^{k} t_{j_l}$$

or

$$\textstyle\sum_{l=1}^{k} t_{j_l} = \dim M_0 - \dim V \tag{1}$$

and

$$\dim S_{j_l} \leq t_{j_l}(t_{j_l} + 1)/2, \qquad l = 1, 2, \cdots, k. \tag{2}$$

Let $Q = S_{j_1} \oplus S_{j_2} \oplus \cdots \oplus S_{j_k}$. (The situation of equation (δ) corresponds to Q being trivial.) Since $V \oplus Q$ is transitive on M_0,

$$R = G_0/(V \oplus Q)$$

acts almost freely on M_0 by Lemma 2. By our initial assumption

$$\dim R \leq \dim V.$$

Now

$$G_0 = R \oplus V \oplus Q \tag{3}$$

and for each S_j in Q we have relation (2).

For each S_j in $R \oplus V$, not of dimension 3, observe that

$$\dim S_j < [\tfrac{1}{2} \dim S_j][\tfrac{1}{2}\dim S_j + 1]/2.$$

This follows since a simple group, not of dimension 3, is of dimension at least 8. Now pair the S_j's of dimension 3 in $R \oplus V$ and observe that

$$\dim (S_\alpha \oplus S_\beta) = \dim S_\alpha + \dim S_\beta = \frac{\left[\dfrac{\dim S_\alpha + \dim S_\beta}{2}\right]\left(\left[\dfrac{\dim S_\alpha + \dim S_\beta}{2}\right] + 1\right)}{2}$$

for

$$\dim S_\alpha = 3 = \dim S_\beta .$$

If there are an even number of S_j's of dimension 3 in $R \oplus V$, we obtain

$$\dim (R \oplus V) \leq \textstyle\sum_{l=1}^{u} t_{h_l}(t_{h_l} + 1)/2 \tag{4}$$

where

$$\sum_{l=1}^{u} t_{h_l} \leq \tfrac{1}{2} \dim (R \oplus V) \leq \dim V \tag{5}$$

and

$u =$ (number of simple factor groups in $R \oplus V$ of dimension greater than 3) $+ \frac{1}{2}$ (number of simple factor groups in $R \oplus V$ of dimension 3).

Combining (1) and (5) we have

$$\sum_{l=1}^{k} t_{j_l} + \sum_{l=1}^{u} t_{h_l} \leq \dim M_0 = t - q \tag{6}$$

and in light of (3) and (4) and the connection between the decomposition of G_0 in (C) and that of $\bar{G}$ in (B), the desired conclusion follows.

Consider finally the case that there are an odd number of S_j's of dimension 3 in $R \oplus V$. If $\dim R < \dim V$, observe that

$$\tfrac{1}{2} \dim (R \oplus V) + \tfrac{1}{2} \leq \dim V$$

and, hence, we may let the t_{h_l} corresponding to the extra S_j of dimension 3 be

$$t_{h_l} = (\tfrac{3}{2} + \tfrac{1}{2}).$$

If $\dim R = \dim V$, then some S_j, say S_α, in $R \oplus V$ must be of odd dimension larger than 3 (actually at least 15). But now if S_β is the extra S_j of dimension 3 we may let

$$t_\alpha = (\dim S_\alpha - 1)/2, \qquad t_\beta = (\dim S_\beta + 1)/2.$$

Remark. The statement and proof of Theorem 1 is of course somewhat awkward due to the special consideration paid to the simple groups of dimension 3. This is, to some extent, unavoidable as demonstrated by the following example: There is an almost effective action of

$$\mathrm{Spin}(4) = S_1 \oplus S_2$$

on S^3 where

$$S_i \cong \mathrm{Spin}(3), \qquad i = 1, 2$$

and, of course, if we let

$$t_1 = 2 = t_2$$

we would have

$$t_1 + t_2 > \dim S^3.$$

It is worthwhile noting that in the above example each S_i acts both transitively and almost freely on S^3.

As mentioned in the introduction, Theorem 1 purports to demonstrate that a transformation group of high dimension does not consist of many simple or circle factor groups. This fact, perhaps, becomes more evident if we reformulate the conclusion of Theorem 1 in the following manner.

THEOREM 1′. *Let G and M be as in Theorem* 1. *Then,*

$$\dim G \leq q + (t - q)(t - q + 1)/2 - \tfrac{1}{4} \sum_{i<j} (\sqrt{1 + 8 \dim S_i'} - 1)(\sqrt{1 + 8 \dim S_j'} - 1).$$

Proof.

$$\dim G = q + \textstyle\sum_{j=1}^{b} \dim S'_j .$$

Now

$$\dim S'_j \leq t_j(t_j + 1)/2, \qquad j = 1, 2, \cdots, b.$$

Therefore,

$$t_j \geq (\sqrt{1 + 8 \dim S'_j} - 1)/2$$

and

$$(1) \qquad \textstyle\sum_{j=1}^{b} (\sqrt{1 + 8 \dim S'_j} - 1) \leq 2 \sum_{j=1}^{b} t_j \leq 2(t - q).$$

Squaring both sides of (1) and using (1) a second time to simplify, we obtain our result.

In some sense, Theorem 1 is "best possible". For example, consider the effective action of

$$G = SO(m_1 + 1) \oplus SO(m_2 + 1) \oplus \cdots \oplus SO(m_s + 1)$$

on

$$M = S^{m_1} \times S^{m_2} \times \cdots \times S^{m_s}$$

where $m_j \geq 3, j = 1, 2, \cdots, s$.

3. Main results

THEOREM 2. *Let G be a compact connected Lie group acting almost effectively on a connected m-manifold M. Then if the dimension of G falls into one of the following ranges:*

$$(m - k)(m - k + 1)/2 + k(k + 1)/2 < \dim G < (m - k + 1)(m - k + 2)/2, \quad k = 1, 2, 3, \cdots$$

we have only three possibilities:

(i) $m = 4$, *G is locally isomorphic to the special unitary group $SU(3)$, M is homeomorphic to the complex projective plane $P^2(C)$ and G acts transitively on M.*

(ii) $m = 6$, *G is isomorphic to the exceptional group G_2, M is homeomorphic* to either the sphere *S^6 or real projective space $P^6(R)$ and G acts transitively on M.*

(iii) $m = 10$, *G is locally isomorphic to $SU(6)$, M is homeomorphic to $P^5(C)$ and G acts transitively on M.*

Before proceeding directly with the proof of Theorem 2 we shall establish a lemma which in turn depends upon knowing the maximal dimensions of proper closed subgroups of the compact simple real Lie groups. We use the standard notation: A_r $(r \geq 2, r \neq 3)$, B_r $(r \geq 1)$, C_r $(r \geq 3)$, D_r $(r \geq 3)$, G_2, F_4, E_6, E_7 and E_8 for the classification of the compact connected simple Lie groups. In the following table H_M denotes a closed connected proper subgroup of maximal dimension of a given compact connected simple Lie group G; S^1 denotes the circle group.

type G	dim G	dim H_M	type H_M
A_r	$r(r+2)$	r^2	$A_{r-1} \oplus S^1$
B_r	$r(2r+1)$	$r(2r-1)$	D_r
C_r	$r(2r+1)$	$(r-1)(2r-1)+3$	$C_{r-1} \oplus B_1$
D_r	$r(2r-1)$	$(r-1)(2r-1)$	B_{r-1}
E_6	78	52	F_4
E_7	133	79	$E_6 \oplus S^1$
E_8	248	136	$E_7 \oplus B_1$
F_4	52	36	B_4
G_2	14	8	A_2

As indicated in the table, H_M is unique up to type. As the information in this table does not appear to be written down explicitly in the literature (although, presumably well known), the author employed the following procedure in generating the table. Consider all *maximal* closed connected proper subgroups H of a compact simple Lie group G. By a *maximal* connected subgroup we mean a subgroup which is not properly contained in any closed connected proper subgroup of G. These maximal subgroups have been widely studied in the literature. The subgroups H_M are simply those maximal connected subgroups of maximal dimension.

The table on p. 219 of [2] immediately provides the type of all maximal closed subgroups H of maximal rank. If

$$\operatorname{rank} H < \operatorname{rank} G,$$

then by a result of J. de Siebenthal [8, p. 233] it follows that H has finite center and must, therefore, be semisimple. However, the maximal complex semisimple subalgebras of the complex simple Lie algebras have been completely determined by Dynkin [3], [4]. Briefly, we may apply Dynkin's result to our problem as follows. Suppose H is semisimple and a maximal connected subgroup of G. Corresponding to G is a simple complex Lie algebra $\mathfrak{G}$. Corresponding to H is a real Lie algebra $\mathfrak{H}$ whose complexification $\mathfrak{H}^C$ is a semisimple subalgebra of $\mathfrak{G}$. Moreover

$$\dim_C \mathfrak{H}^C = \dim_R \mathfrak{H} = \dim H.$$

A *maximal semisimple* complex subalgebra of $\mathfrak{G}$ is a proper semisimple subalgebra of $\mathfrak{G}$ which is not properly contained in any proper semisimple complex subalgebra of $\mathfrak{G}$. Now we may as well suppose that $\mathfrak{H}^C$ is a *maximal semisimple* complex subalgebra of $\mathfrak{G}$. For if $\mathfrak{A}$ is a complex semisimple subalgebra with

$$\mathfrak{H}^C \subset \mathfrak{A} \subset \mathfrak{G}$$

then $\mathfrak{A}$ would have a compact real form of real dimension larger than that of $\mathfrak{H}$.

Using the terminology of [3], $\mathfrak{H}^C$ is either an *R-subalgebra* or an *S-subalgebra.* If $\mathfrak{H}^C$ is an *R-subalgebra*, it must be a *regular* subalgebra by [3, Theorem 7.7]. We may now refer to Tables 12 and 12a of [3] for a complete listing of all *maximal semisimple regular* subalgebras. If $\mathfrak{H}^C$ is an *S-subalgebra* and G is an exceptional simple group, we may refer to Table 39 of [3]. Suppose finally that $\mathfrak{H}^C$ is an *S-subalgebra* and G is a classical group. By [3, Theorem 7.3], $\mathfrak{H}^C$ is maximal in the class of all subalgebras (not just the semisimple ones) of $\mathfrak{G}$. If $\mathfrak{H}^C$ is not simple, we refer to the table on p. 238 of [3]. If $\mathfrak{H}^C$ is simple, we are led to the case where $\mathfrak{H}^C$ is a maximal *irreducible* subalgebra of $\mathfrak{G}$. (See the discussion on p. 238 of [3].) We may now refer to the results of [4]. If $\mathfrak{G}$ is an A_r then by [4, Theorem 4.2] it is sufficient to consider the case where $\mathfrak{H}^C$ is irreducible with respect to the standard linear representation of A_r; that is, $SL(r+1)$. Therefore the minimum dimension of a faithful linear representation of $\mathfrak{H}^C$ cannot exceed $r+1$. One may now completely investigate the possibilities for $\mathfrak{H}^C$ by referring, say, to Table 30 of [4]. If $\mathfrak{G}$ is a C_r then by [4, Theorem 5.2] we need consider only the case where $\mathfrak{H}^C$ is irreducible with respect to the $\mathrm{Sp}(2r)$ symplectic representation of C_r. However, Table 12 of [4] provides the minimum dimensions of the faithful linear symplectic representations of the simple Lie algebras. Similar techniques may be employed if $\mathfrak{G}$ is of type B or D.

It is possible, by the way, to determine the type of H_M when G is of type B or D directly by a transformation group argument (e.g. Lemmas 3 and 4 of [6]).

Let $k_0 = k_0(m)$ denote the maximum integer k for which the inequality of Theorem 2 is still meaningful. One easily computes that

$$k_0 = \{\sqrt{9+8m} - 5)/2\}$$

where $\{x\}$ denote the smallest integer $\geq x$.

LEMMA 4. *For $m \geq 17$, if G acts almost effectively on a connected m-manifold M and G is isomorphic to an A_r or an exceptional group, then*

$$\dim G < (m - k_0)(m - k_0 + 1)/2.$$

Proof. For $m \geq 17$

$$(m - k_0)(m - k_0 + 1)/2 \geq 91$$

and, checking the dim G column of our table, we see immediately that we need concern ourselves only with E_7, E_8 or A_r for $r \geq 9$.

If G acts almost effectively on M

$$\dim G/H \leq m$$

for some proper subgroup H of G. Hence

$$\dim G \leq m + \dim H. \tag{1}$$

If G is isomorphic to an A_r we may use the table and (1) to conclude that

$$\dim G \leq m(m+4)/4.$$

It is easily checked, however, that

$$m(m+4)/4 < (m-k_0)(m-k_0+1)/2$$

for $m \geq 17$. If G is isomorphic to an E_7, we find using (1) and the table that

$$m \geq 54.$$

However,

$$\dim G = 133 < (m-k_0)(m-k_0+1)/2$$

for $m \geq 54$. If G is isomorphic to E_8, we reach a similar conclusion.

Proof of Theorem 2. Suppose now

$$(1)\quad (m-k)(m-k+1)/2 < \dim G < (m-k+1)(m-k+2)/2$$

where $k \leq k_0$. Also assume $m \geq 17$ so that the last lemma applies. We choose to investigate separately the cases $m < 17$ at the end of the proof. Let G be in the form (B) of the last section. Now $\bar{G}$ acts almost effectively on M. Let S_0' be a simple factor group of $\bar{G}$ of maximal dimension and let l_0 denote the least integer such that

$$(2)\qquad \dim S_0' \leq l_0(l_0+1)/2.$$

If S_0' is an A_r or an exceptional group, then we know from Lemma 4 that

$$l_0 \leq m - k_0 \leq m - k.$$

On the other hand if S_0' is a B_r, C_r or D_r we know from the form of $\dim S_0'$ (see the table) and (1) that

$$l_0 \leq m - k.$$

Therefore, in any case,

$$(3)\qquad l_0 \leq m - k.$$

Letting

$$q = (1\cdot 2)/2 + (1\cdot 2)/2 + \cdots + (1\cdot 2)/2$$

we obtain from Theorem 1 that

$$(4)\qquad \dim G \leq l_0(l_0+1)/2 + \textstyle\sum_{j=1}^{s} t_j(t_j+1)/2$$

where

(i) $1 \leq t_j \leq l_0$, all j
(ii) $\sum_{j=1}^{s} t_j \leq m - l_0$.

Suppose

$$(5)\qquad l_0 = m - k - u, \qquad 0 \leq u \leq m - k - 2.$$

From (4),

(6) $\dim G \leq (m - k - u)(m - k - u + 1)/2 + \sum_{j=1}^{s} t_j(t_j + 1)/2.$

If $\sum_{j=1}^{s} t_j \leq u$,

$$\dim G \leq (m - k - u)(m - k - u + 1)/2 + u(u + 1)/2$$
$$\leq (m - k)(m - k + 1)/2$$

which contradicts our assumption (1). Hence,

$$\sum_{j=1}^{s} t_j > u. \tag{7}$$

But now from (5), (6), (7) and (i) we obtain,

$$\dim G \leq (m - k)(m - k + 1)/2 + \sum_{j=1}^{s} t'_j(t'_j + 1)/2 \tag{8}$$

where

$$t'_j \geq 0, \text{ all } j \quad \text{and} \quad \sum_{j=1}^{s} t'_j = \left(\sum_{j=1}^{s} t_j\right) - u. \tag{9}$$

From (ii), (5) and (9),

$$\sum_{j=1}^{s} t'_j \leq m - l_0 - u = k \tag{10}$$

and finally from (8) and (10),

$$\dim G \leq (m - k)(m - k + 1)/2 + k(k + 1)/2. \tag{11}$$

Hence for $m \geq 17$, the dimension of G cannot be in any of the ranges indicated in the statement of Theorem 2.

We now investigate the situation for $m \leq 16$. From the nature of the proof it is clear that we need be concerned only with the cases where the maximal simple factor group S'_0 is an A_r or an exceptional group. (Note we used Lemma 4 only to obtain (3) which holds for all m if S'_0 is a B_r, C_r or D_r.) Moreover since S'_0 is almost effective on M and now $m \leq 16$, we may use the table and the techniques of Lemma 4 to conclude that S'_0 must be isomorphic to G_2, F_4 or $SU(r)$, $3 \leq r \leq 9$ ($r \neq 4$). We may now proceed in a straightforward fashion to investigate the cases from $m = 1$ to $m = 16$. The results turn up just three possibilities:

(i) $m = 4$, G is locally isomorphic to $SU(3)$ and G acts transitively on M.
(ii) $m = 6$, G is isomorphic to G_2 and G acts transitively on M.
(iii) $m = 10$, G is locally isomorphic to $SU(6)$ and G acts transitively on M.

As a completely representative example, we indicate the details for $m = 10$. To conclude the proof we must of course later determine the topological structure of M. If $m = 10$, the only possibilities are:

$$S'_0 \cong SU(3), \quad G_2, \quad SU(5), \quad SU(6).$$

Now $\bar{G}$ acts almost effectively and transitively on a principal orbit $\tilde{M}$. Suppose

$$\bar{G} = S_0' \oplus S. \tag{12}$$

Now S_0' acts transitively and invariantly on the compact connected manifold $\tilde{M}/S$. If S_0' acts trivially on $\tilde{M}/S$, then S is transitive on $\tilde{M}$ and by Lemma 2, S_0' acts almost freely on $\tilde{M}$. In this case,

$$\dim S_0' \leq \dim \tilde{M} \leq \dim M \leq 10$$

and S_0' must be isomorphic to $SU(3)$. If S_0' acts non-trivially on $\tilde{M}/S$, then it must act almost effectively on $\tilde{M}/S$ since S_0' is simple. Now suppose

$$t = \text{maximal dimension of the orbits of } S \text{ on } \tilde{M}.$$

By Lemma 1,

$$\dim \tilde{M}/S = \dim \tilde{M} - t \leq 10 - t. \tag{13}$$

Moreover, by applying the Montgomery-Zippin Theorem to the almost effective action of S on $\tilde{M}$ we find that

$$t \geq (\sqrt{1 + 8 \dim S} - 1)/2. \tag{14}$$

For $m = 10$, we compute

$$N(m) = (m - k_0)(m - k_0 + 1)/2 + k_0(k_0 + 1)/2 = 34. \tag{15}$$

We now investigate the four cases for S_0'.

Case (a) *S_0' is isomorphic to $SU(6)$.* If $\dim S > 0$, we find from (14) and (13) that

$$\dim \tilde{M}/S < 10.$$

But this is impossible since S_0' is almost effective on $\tilde{M}/S$. Hence $\dim S = 0$, $\bar{G} = S_0'$ and G is locally isomorphic to $SU(6)$. Since $\dim \tilde{M} = 10 = \dim M$, G is transitive on M and we have possibility (iii).

Case (b) *S_0' is isomorphic to $SU(5)$.* If $\dim S > 3$,

$$\dim \tilde{M}/S < 8$$

which is impossible for reasons similar to those above. Hence

$$\dim G = \dim \bar{G} = \dim S_0' + \dim S \leq 24 + 3 < N(10).$$

Case (c) *S_0' is isomorphic to G_2.* We find that

$$\dim S \leq 10$$

and, hence,

$$\dim G \leq 14 + 10 < N(10).$$

Case (d) S_0' *is isomorphic to* $SU(3)$. We now have 2 subcases to consider. If S_0' is almost effective on $\tilde{M}/S$, then

$$\dim S \leq 21 \quad \text{and} \quad \dim G \leq 8 + 21 < N(10).$$

On the other hand, suppose S_0' acts almost freely on $\tilde{M}$. Then S acts transitively on the 2-manifold $\tilde{M}/S_0'$. Break S up into a direct sum of simple groups and circle groups and let T be a partial direct sum of these groups (that is, T is a subgroup of S) which acts transitively and almost effectively on $\tilde{M}/S_0'$. Now

$$\dim T \leq (2\cdot 3)/2 = 3.$$

Since $S_0' \oplus T$ acts transitively on $\tilde{M}$, S/T acts almost freely on $\tilde{M}$ by Lemma 2. Hence

$$\dim G = \dim S_0' + \dim T + \dim S/T \leq 8 + 3 + 10 < N(10).$$

The procedures for handling $m \leq 16$, $m \neq 10$, are completely analogous.

We may now employ methods of H. C. Wang [11, p. 184–185] to determine the topological structure of M for each of the 3 possibilities. Let $\tilde{G}$ be the universal covering group of G. Then $\tilde{G}$ acts almost effectively and transitively on M. If $\tilde{H}$ is an isotropy subgroup of this action,

$$M \approx \tilde{G}/\tilde{H}.$$

If $m = 4$ and G is locally isomorphic to $SU(3)$, then

$$\tilde{G} = SU(3)$$

and $\tilde{H}^*$, the identity component of $\tilde{H}$, must be of type $A_1 \oplus S^1$, as indicated in the table. By Wang [10, Theorem III], $\tilde{H}^*$ is, up to an inner automorphism of $\tilde{G}$, the standard unitary subgroup $U(2)$ in $SU(3)$. By verifying that $U(2)$ is its own normalizer in $SU(3)$ we obtain

$$\tilde{H} = \tilde{H}^*.$$

Hence

$$M \approx \tilde{G}/\tilde{H} = \tilde{G}/\tilde{H}^* = SU(3)/U(2) = P^2(C).$$

Similarly for $m = 10$ we conclude that M is homeomorphic to $P^5(C)$. Suppose now $m = 6$ and G is isomorphic to G_2. Then

$$\tilde{M} \approx \tilde{G}/\tilde{H}^* = G_2/SU(3) = S^6$$

is a covering space of

$$M \approx \tilde{G}/\tilde{H}.$$

Since the Euler characteristic of $\tilde{M}$ is two, the order of the covering is one or two. If the order is one,

$$M \approx S^6.$$

If the order is two, since the deck transformation is an isometry of S^6,

$$M \approx P^6(R).$$

If G is effective on M, it follows from [10, (2.1)] that G has trivial center and, therefore, in cases (i) and (iii) G is isomorphic to $SU(3)/Z$ and $SU(6)/Z$ respectively where Z denotes the full centers of the groups.

Remarks. For each m and k there is, of course, an effective action of

$$G = SO(m - k + 1) \oplus SO(k + 1)$$

on

$$M = S^{m-k} \times S^k.$$

Let G be a compact connected Lie group and H a proper closed subgroup of G. We say G/H is an *almost effective homogeneous space* if every normal subgroup of G contained in H is finite.

COROLLARY. *Suppose G/H is an almost effective homogeneous space with*

$$\dim G/H \leq m$$

and

$$(m - k)(m - k + 1)/2 + k(k + 1)/2 < \dim G < (m - k + 1)(m - k + 2)/2, \quad k = 1, 2, \cdots, k_0(m).$$

Then there exist just three possibilities:

(i) $m = \dim G/H = 4$, *G is locally isomorphic to $SU(3)$ and G/H is homeomorphic to $P^2(C)$.*

(ii) $m = \dim G/H = 6$, *G is isomorphic to G_2 and G/H is homeomorphic to S^6 or $P^6(R)$.*

(iii) $m = \dim G/H = 10$, *G is locally isomorphic to $SU(6)$ and G/H is homeomorphic to $P^5(C)$.*

Proof. If

$$\dim G/H = m$$

the corollary is an immediate consequence of Theorem 2 by considering the almost effective action of G on G/H. If

$$\dim G/H < m$$

let

$$q = m - \dim G/H \quad \text{and} \quad M = G/H \times S^q.$$

Now G acts almost effectively on M by

$$g(x, y) = (gx, y).$$

Again apply Theorem 2.

Theorem 1 could be strengthened by taking into account the types of the simple factor groups S_j'. For example, if S_j' is of type A,

$$\dim S_j' \leq t_j(t_j + 4)/4$$

as is easily verified by referring to the proof of Lemma 4 and checking through the proof of Theorem 1.

As for Theorem 2, it seems quite likely that there exists an even more general pattern of gaps. For example, consider the dimension of G in the following range

$$(m - k)(m - k + 1)/2 + (k - 1)k/2 + 1 < \dim G < (m - k)(m - k + 1)/2 + k(k + 1)/2, \quad (k \leq k_0(m)).$$

For $m \geq 15$ letting

$$G = SO(m - 3) \oplus SU(3) \quad \text{and} \quad M = S^{m-4} \times P^2(C),$$

we obtain examples of almost effective actions in the above range. Are there any other examples?

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