BOUNDED HOLOMORPHIC FUNCTIONS AND PROJECTIONS

BY

FRANK FORELLI[1]

1.1. Let R be a Riemann surface and let $H^\infty(R)$ be the algebra of bounded holomorphic functions on R. I will assume that the universal covering surface of R is the open unit disc D, as it must be if $H^\infty(R)$ contains nonconstant functions, and then, because of this assumption, there are analytic maps t from D onto R such that the pair (D, t) is a regular covering surface of R [1]. Let t be one of these maps and use t to represent $H^\infty(R)$ as a subalgebra of $H^\infty(D)$ by composing the functions in $H^\infty(R)$ with t. My aim is to show, when R is conformally equivalent to the interior of a compact bordered Riemann surface, that there is a projection P of $H^\infty(D)$ onto $H^\infty(R)$ with the property

$$P(fg) = fPg$$

for all f in $H^\infty(R)$ and g in $H^\infty(D)$. By projection I mean linear and idempotent.

1.2. Let G be the group of cover transformations of (D, t). G is the group of fractional linear transformations T that take D onto D with

$$t \circ T = t,$$

and G is isomorphic to the fundamental group of the surface R. The group G acts on $H^\infty(D)$ in the standard way by composing the functions in $H^\infty(D)$ with the transformations in G, and $H^\infty(R)$ is the algebra of functions in $H^\infty(D)$ that are invariant under G. For let f be in $H^\infty(R)$ and let $g = f \circ t$ be the function in $H^\infty(D)$ that is obtained by lifting f to D. Then g is invariant under the group G,

$$Tg = g \circ T = g$$

for all T in G, and every function in $H^\infty(D)$ that is invariant under G is obtained in this way.

Each function in $H^\infty(D)$ has a radial limit at almost every point of the unit circle Γ. Let H^∞ be the algebra of functions defined almost everywhere on Γ that are radial limits of functions in $H^\infty(D)$, and let H^∞/G be the subalgebra of functions in H^∞ that are invariant under G. The radial limit map is an algebra isomorphism between $H^\infty(D)$ and H^∞ and between $H^\infty(R)$ and H^∞/G, and it is within the framework of H^∞ and H^∞/G that I will get the projection P. The arguments I will give are intrinsic in the sense that everything will take place on Γ with an occasional trip into D, and we will not need

Received November 20, 1964; received in revised form March 4, 1966.

[1] Supported by a National Science Foundation grant.

to know anything about Riemann surfaces. The assumption that R is conformally equivalent to the interior of a compact bordered Riemann surface will appear as an assumption about the group G. The tools I will use to get the projection P are measure and Hilbert space theory, and the harmonic analysis that goes with the Hilbert space H^2.

1.3. Part 2 of this paper contains things about H^∞/G and related spaces. Here no assumptions are made about the group G. In part 3, with a condition on G, I get the projection P. Part 4 contains an application.

2.1. I will begin by recalling the interplay between Lebesgue measure on the unit circle Γ and fractional linear transformations that take the open unit disc D onto itself. I will denote by σ the Lebesgue measure on Γ given by

$$\int f \, d\sigma = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) \, dx$$

and I will write L^p for $L^p(\sigma)$. Let T be a fractional linear transformation that takes D onto D. Then T takes Γ onto Γ and

$$\int_{T^{-1}X} d\sigma = \int_X P_T \, d\sigma \tag{1}$$

for all Borel sets X contained in Γ where P_T is the Poisson kernel

$$P_T(e^{ix}) = \operatorname{Re}\left(\frac{e^{ix} + T(0)}{e^{ix} - T(0)}\right). \tag{2}$$

Because of (1), when X is a Borel set contained in Γ,

$$\sigma(T^{-1}X) = 0 \quad \text{if and only if} \quad \sigma(X) = 0. \tag{3}$$

Let Σ be the sigma-field of σ-measurable sets and let L be the algebra of Σ-measurable functions. When f is a function on Γ, Tf is the function on Γ given by

$$Tf(e^{ix}) = f(T(e^{ix})).$$

Because of (3), $T^{-1}X$ is in Σ when X is, and therefore Tf is in L when f is for

$$(Tf)^{-1}Y = T^{-1}(f^{-1}Y).$$

Moreover T preserves the equivalence classes in L obtained by identifying functions that differ only on sets of measure zero,

$$Tf = Tg \text{ a.e.} \quad \text{when} \quad f = g \text{ a.e.},$$

for

$$[Tf \neq Tg] = T^{-1}[f \neq g].$$

When f is in L^1, we get from (1),

$$\int Tf \, d\sigma = \int f P_T \, d\sigma. \tag{4}$$

2.2. Let Σ/G be the sigma-field of sets in Σ that are invariant under the group G and let L/G be the algebra of functions in L that are invariant under G. X is in Σ/G if and only if X is in Σ and

$$\sigma(X \triangle T^{-1}X) = 0$$

for each T in G, and f is in L/G if and only if f is in L and

$$Tf = f \quad \text{a.e.}$$

for each T in G. There is an obvious but for this paper very important relation between the algebra of functions L/G and the sigma-field Σ/G:

L/G is the algebra of Σ/G-measurable functions.

For

$$f^{-1}Y \triangle T^{-1}(f^{-1}Y) \subset [Tf \neq f] \tag{5}$$

when Y is any Borel subset of the plane, and

$$[Tf \neq f] \subset \cup(f^{-1}Y \triangle T^{-1}(f^{-1}Y)) \tag{6}$$

where the union is taken over any countable collection of Borel subsets of the plane with the property that given any two distinct points in the plane there is a set in the collection that contains one of the points and not the other. (5) says that functions in L/G are Σ/G-measurable and (6) that Σ/G-measurable functions belong to L/G.

Let L^p/G be the linear space of functions in L^p that are invariant under G. Then, because of what has just been said, L^p/G is also the linear space of functions in L^p that are Σ/G-measurable. I will denote by E the conditional expectation given Σ/G. E is the projection of L^1 onto L^1/G given by

$$\int_X Ef\, d\sigma = \int_X f\, d\sigma \tag{7}$$

where f is in L^1 and X is in Σ/G. For a given f in L^1 the right side of (7) defines a bounded complex measure on Σ/G that is absolutely continuous with respect to σ, and thus by the Radon-Nikodym theorem there is a unique function Ef in L^1/G that satisfies (7).

The conditional expectation E is a projection of L^p onto L^p/G for $1 \leq p \leq \infty$, and the norm of E as a linear transformation on L^p is one. In particular, E takes bounded functions into bounded functions. As a linear transformation on the Hilbert space L^2, E is just the orthogonal projection of L^2 onto L^2/G. The conditional expectation E, in addition to being linear and idempotent, has the property

$$E(fg) = fEg \tag{8}$$

when f is in L^∞/G and g is in L^∞. A thorough discussion of conditioning can be found in [7, Ch. 7].

H^∞ was defined in the introduction as the algebra of functions on Γ that are radial limits of functions bounded and holomorphic on D. H^∞ can also be defined, without referring to functions on the open unit disc, as the algebra of functions in L^∞ whose Fourier coefficients vanish for negative indices. H^∞/G is the algebra of functions in H^∞ invariant under G. Now E carries H^∞ into L^∞/G, and if it were true that E carried H^∞ into H^∞/G, then because of (8) I could take E restricted to H^∞ for the projection P. However, when R is conformally equivalent to the interior of a compact bordered Riemann surface that is not simply connected, E just does not take H^∞ into H^∞/G (for a proof see Lemma 4 in part 3). Nevertheless P will be got from E. I will show in part 3 with an assumption about the group G that there is a function ϕ in H^∞ with the property

$$E(\phi) = 1 \tag{9}$$

and

$$E(\phi H^\infty) = H^\infty/G. \tag{10}$$

Then the linear transformation P given by

$$Pf = E(\phi f)$$

is a projection of H^∞ onto H^∞/G with the property

$$P(fg) = fPg$$

when f is in H^∞/G and g is in H^∞.

The remainder of this part of the paper contains things that will be needed in part 3, but that are true without any assumptions about the group G.

2.3. H^2 is the space of functions in L^2 whose Fourier coefficients vanish for negative indices. Let f be in L^2. The conjugate of f is the L^2 function f^* with Fourier series

$$\textstyle\sum_{n<0} ic_n\, e^{inx} + \sum_{n>0} - ic_n\, e^{inx}$$

where

$$\textstyle\sum c_n\, e^{inx}$$

is the Fourier series of f. $f + if^*$ is in H^2, and when f is a real function in L^2, f^* is the unique real function in L^2 with mean value zero such that $f + if^*$ belongs to H^2.

When T is in G let

$$v_T = E(P_T^*)$$

where P_T is the Poisson kernel (2). P_T^* is a real function with an absolutely convergent Fourier series since P_T is. In particular P_T^* is a real function in L^∞, and hence v_T is a real function in L^∞/G. I will denote by N the complex linear span of the functions v_T (T in G).

LEMMA 1.

$$T(f^*) - f^* = -\int f v_T \, d\sigma \tag{11}$$

for all f in L^2/G and T in G.

Proof. Let T be a fractional linear transformation that takes D onto D, and let f be a real function in L^2. $Tf + iT(f^*)$ belongs to H^2 for T takes H^2 into H^2, and thus $T(f^*)$ and $(Tf)^*$ differ by the mean value of $T(f^*)$. This is also true when f is complex since T and * are linear transformations, and we have for all f in L^2

$$T(f^*) - (Tf)^* = \int T(f^*)\, d\sigma.$$

From (4) and the fact that conjugation is self-adjoint except for a sign change we get

$$\int T(f^*)\, d\sigma = \int f^* P_T\, d\sigma = -\int f P_T^*\, d\sigma,$$

and therefore

$$T(f^*) - (Tf)^* = -\int f P_T^*\, d\sigma. \tag{12}$$

Now let f be in L^2/G and let T be in G. Since $Ef = f$ and E is self-adjoint we have

$$\int f P_T^*\, d\sigma = \int f E(P_T^*)\, d\sigma = \int f v_T\, d\sigma. \tag{13}$$

We get (11) from (12) and (13) since $Tf = f$.

LEMMA 2. *The map that takes T into v_T is a homomorphism of the group G into the additive group N:*

$$v_I = 0 \tag{14}$$

and

$$v_{ST} = v_S + v_T \tag{15}$$

for all S and T in G.

Proof. (14) is clear. To get (15) let f be in L^2/G. Then

$$ST(f^*) - f^* = S(f^*) - f^* + T(f^*) - f^* \tag{16}$$

for

$$ST(f^*) - f^* = T(S(f^*) - f^*) + T(f^*) - f^*$$

and $S(f^*) - f^*$ is a constant. From (11) and (16) we get

$$\int f v_{ST}\, d\sigma = \int f v_S\, d\sigma + \int f v_T\, d\sigma,$$

and since this is true for all f in L^2/G we get (15).

The Hilbert space L^2 has the orthogonal decomposition

$$L^2 = \overline{H_0^2} \oplus C \oplus H_0^2$$

where C is the one-dimensional space of constant functions and H_0^2 is the space of functions in H^2 with mean value zero. The next lemma gives an analogous decomposition for L^2/G. H^2/G is the space of functions in H^2 invariant under G, i.e. H^2/G is the intersection of H^2 with L^2/G, and H_0^2/G is the space of functions in H^2/G with mean value zero.

LEMMA 3. *The Hilbert space L^2/G has the orthogonal decomposition*

$$L^2/G = N^2 \oplus \overline{H_0^2/G} \oplus C \oplus H_0^2/G \tag{17}$$

where N^2 is the L^2 closure of N.

Proof. (11) shows that the orthogonal complement of N in L^2/G consists of those functions in L^2/G whose conjugates also belong to L^2/G. There is another way to describe the space of functions in L^2/G whose conjugates are invariant under G. This space is the same as

$$\overline{H_0^2/G} \oplus C \oplus H_0^2/G.$$

For when f is in L^2,

$$2f = \left(f - \int f\, d\sigma - if^*\right) + 2\int f\, d\sigma + \left(f - \int f\, d\sigma + if^*\right),$$

and the first component in this decomposition belongs to $\overline{H_0^2}$, the third to H_0^2. Putting these things together gives (17).

2.4. Let G' be the commutator subgroup of G and let $r(G/G')$ be the smallest number of elements that will generate the homology group G/G'. A corollary of Lemma 2 is that

$$d(N) \leq r(G/G')$$

where $d(N)$ is the dimension of N, for G' is contained in the kernel of any homomorphism of G into an Abelian group. The assumption about the group G that I will make in part 3 in that $r(G/G')$ is finite and is equal to $d(N)$. I want to explain here why this is true when R is the interior of a compact bordered Riemann surface that is not simply connected.

Let $d(R)$ be the greatest number of bounded harmonic functions on R with cohomologously independent conjugate differentials. Then

$$d(N) = d(R).$$

There is a one-to-one linear correspondence between functions g in L^∞/G and bounded harmonic functions f on R that is given by $\hat{g} = f \circ t$ where $\hat{g}$ is the harmonic extension of g to D and t is the analytic map that defines G. Let df be the differential of f and $*df$ the conjugate differential of f. $*df$ is cohomologous to zero if and only if g^* belongs to L^2/G. Thus if g belongs to N and $*df$ is cohomologous to zero, then g is orthogonal to N(Lemma 1) and $g = 0$. This shows that

$$d(N) \leq d(R).$$

To get the reverse inequality we may assume that $d(N)$ is finite. Then N is closed in L^2/G and $g = g_1 + g_2$ where g_1 is in N and g_2 is orthogonal to N. $g_1 = 0$ implies that $*df$ is cohomologous to zero for $*df = *df_1 + *df_2$ and $*df_2$ is cohomologous to zero. This shows that

$$d(R) \leq d(N).$$

Now let R be the interior of a compact bordered Riemann surface and let the n cycles C_j be a homology basis for R.

$$n = 2g + b - 1$$

where g is the genus of R and b is the number of boundary components. Given any n complex numbers z_j there is an analytic differential φ on the Schottky double with

$$\int_{C_j} \varphi = z_j$$

[9, p. 172]. This implies that $d(R) = n$. For let φ_k be such that

$$\int_{C_j} \varphi_k = i\delta_{jk}\,.$$

Then $\operatorname{Re} \varphi_k = df_k$ where f_k is harmonic on $\bar{R}$ since

$$\int_{C_j} \operatorname{Re} \varphi_k = 0,$$

and the n differentials $*df_k$ are a cohomology basis for R since

$$\int_{C_j} *df_k = \delta_{jk}\,.$$

When R is not simply connected $(n \geq 1)$, the homology group G/G' is a free Abelian group with n independent generators, and therefore $r(G/G') = n$. Thus when R is not simply connected

$$r(G/G') = d(R) = d(N).$$

When R is simply connected $(n = 0)$, G/G' is trivial and hence $r(G/G') = 1$. On the other hand $d(N) = d(R) = 0$.

The converse is also true. When $r(G/G')$ is finite and is equal to $d(R)$, then R is conformally equivalent to the interior of a compact bordered Riemann surface.

3.1. From now on I will assume that *$r(G/G')$ is finite and is equal to $d(N)$:*

$$r(G/G') = d(N) = n.$$

G' is the commutator subgroup of G, $r(G/G')$ is the smallest number of elements that will generate the homology group G/G', and $d(N)$ is the dimension of the vector space N defined at the beginning of **2.3.** I will show, with this assumption, that there is a function ϕ in H^∞ such that (9) and (10) hold.

Because N is finite-dimensional, the orthogonal decomposition of L^2/G given by Lemma 3 becomes

$$L^2/G = N \oplus \overline{H_0^2/G} \oplus C \oplus H_0^2/G.$$

Moreover $d(N) \geq 1$ for $r(G/G')$ is a positive integer.

LEMMA 4.

$$E(H^\infty) = N + H^\infty/G.$$

Proof. Let f be in H^∞. Ef is orthogonal to $\overline{H_0^2/G}$ because f is, and thus Ef belongs to $N \oplus H^2/G$. As Ef is bounded and N contains only bounded functions, Ef must in fact belong to $N + H^\infty/G$, and therefore $E(H^\infty)$ is contained in $N + H^\infty/G$.

To get the reverse inclusion, let T be in G. $EP_T = 1$, for when f is in L^2/G we get from (4)

$$\int fEP_T \, d\sigma = \int fP_T \, d\sigma = \int f \, d\sigma.$$

Hence

$$iv_T = E(P_T + iP_T^* - 1)$$

and N is contained in $E(H^\infty)$. Thus $N + H^\infty/G$ is contained in $E(H^\infty)$ and the proof is complete.

Let f be a nonzero vector in H^∞ and let χ be a homomorphism of G into the circle group (the multiplicative group of unimodular complex numbers). f is an eigenvector of G with eigenvalue χ if

$$Tf = \chi(T)f$$

for all T in G. (I prefer this terminology to the one that calls an eigenvector a multiplicative function.) The lemma that follows is a paraphrase of the theorem that given a homology basis for a compact bordered Riemann surface, there is an analytic differential on the Schottky double with prescribed periods along the cycles in the basis [9, p. 172]. Because of this lemma, when f is an eigenvector of G, there is a unit g in H^∞ such that fg is invariant under G.

LEMMA 5. *Let χ be a homomorphism of G into the circle group. Then there is a unit in the algebra H^∞ that is an eigenvector of G with eigenvalue χ.*

Proof. Let the n transformations T_j in G be such that the cosets T_jG' generate G/G'. Let T be in G. Then

$$T = (\prod T_j^{k_j})S$$

where S belongs to G', and

$$\chi(T) = \prod \chi(T_j)^{k_j} \tag{18}$$

and by Lemma 2

$$v_T = \sum k_j v_j \tag{19}$$

where I have written v_j for v_{T_j}. The n real functions v_j not only span N, but because of my assumption about the group G are a basis for N, and hence there is a real function v in N with

$$\chi(T_j) = \exp\left(-i\int vv_j\,d\sigma\right). \tag{20}$$

$\exp(v + iv^*)$ is a unit in H^∞ and is moreover an eigenvector of G with eigenvalue χ, for we get from (11)

$$T(\exp(v + iv^*)) = \exp\left(v + iv^* - i\int vv_T\,d\sigma\right)$$

and from (18), (19), and (20)

$$\chi(T) = \exp\left(-i\int vv_T\,d\sigma\right).$$

The next lemma is both a consequence and a generalization of the well known fact that the index of a nilpotent linear transformation cannot exceed the dimension of the vector space on which it acts [4, p. 162].

LEMMA 6. *Let A_1 through A_m be m commuting linear transformations of a vector space V, and suppose there are m positive integers e_1 through e_m such that*

$$\prod_1^m A_j^{k_j} = 0$$

if and only if $e_j \le k_j$ for $1 \le j \le m$. Then $\sum_1^m e_j$ does not exceed the dimension of V.

Proof. The proof is by induction on m. When $m = 1$ the assertion of the lemma is the just mentioned fact about nilpotent linear transformations. Assume that the lemma is true when the number of transformations is $m - 1$. Let X be the range of $A_1^{e_1}$ and let Y be the range of $\prod_2^m A_j^{e_j}$. X is invariant under A_2 through A_m, and the conditions of the lemma are satisfied with these $m - 1$ transformations acting on X. By the induction hypothesis

$$\sum_2^m e_j \le d(X).$$

Moreover Y is invariant under A_1, and A_1 acting on Y is nilpotent of index e_1. Therefore

$$e_1 \le d(Y).$$

Finally Y is contained in the null space of $A_1^{e_1}$ and so by the rank-nullity theorem

$$d(X) + d(Y) \le d(V).$$

Combining these inequalities gives the assertion of the lemma.

LEMMA 7. *There is a function k in H^∞/G with the property*

$$k(N + H^\infty/G) = H^\infty/G. \tag{21}$$

Proof. I will use Beurling's invariant subspace theorem to get the function k, and then use Lemma 6 to show that k does what is claimed. Let M be the collection of all f in H^2 such that fN is contained in $H^?$. M is a closed subspace of H^2 invariant under multiplication by H^∞, and we will see later that M is not trivial. By Beurling's theorem describing such subspaces there is an inner function w, determined by M up to a constant factor, such that

$$M = wH^2$$

[5, Lec. 2], [6, Ch. 7]. w moreover is an eigenvector of G. For N and H^2 are invariant under G, and therefore M is also. Thus when T is in G

$$(Tw)H^2 = wH^2$$

and the inner function Tw is a constant multiple of w,

$$Tw = \chi(T)w.$$

This in turn implies that χ is a homomorphism of G into the circle group, and w is an eigenvector of G with eigenvalue χ. Now by Lemma 5 there is a unit u in H^∞ such that

$$k = wu$$

belongs to H^∞/G, and we have

$$kH^\infty/G \subset k(N + H^\infty/G) \subset H^\infty/G.$$

The codimension of the first space in the second is n, and I will show that the codimension of the first in the third does not exceed n. This will give (21).

Let F be the orthogonal projection of L^2/G onto N, and when f is in H^∞/G let A_f be the linear transformation of N given by

$$A_f v = F(fv).$$

When v is in N and f is in H^∞/G, fv is in $N + H^\infty/G$ for fv is orthogonal to $\overline{H^2/G}$, and hence *the linear space* $N + H^\infty/G$ *is invariant under multiplication by* H^∞/G. Because of this

$$A_{fg} = A_f A_g$$

and therefore the linear transformation that takes f to A_f is an algebra homomorphism of H^∞/G onto a commutative subalgebra of the algebra of linear transformations of N. Let M/G be the kernel of this homomorphism. M/G is the collection of all f in H^∞/G such that fN is contained in H^∞/G. M/G is not trivial for H^∞/G is infinite-dimensional and M/G has finite codimension in H^∞/G. We have

$$M/G = M \cap L^\infty/G \tag{22}$$

and in particular M is not trivial. From (22) we get

$$M/G = kH^\infty/G \tag{23}$$

for

$$M = kH^2.$$

$(H^\infty/G)/(M/G)$ is isomorphic to a commutative subalgebra of the algebra of linear transformations on the n-dimensional space N, and it is tempting to believe that such an algebra has dimension not greater than n. This however is not true, and all that can be said is that the dimension of a commutative algebra of linear transformations on an n-dimensional space does not exceed $1 + [n^2/4]$. To see that the codimension of M/G in H^∞/G does not exceed n I must look at the inner function w.

Suppose w contains a singular part, and let $w = bs$ be the factorization of w into a Blaschke product b and a singular inner function s [5, p. 10], [6, p. 63]. Because w is an eigenvector of G, so are b and s. Let m be a positive integer and let r be an inner function with $r^m = s$. Then r is also an eigenvector of G, and by Lemma 5 there are units f and g in H^∞ such that bf and rg are invariant under G. But we now have an ascending chain of $m - 1$ distinct ideals

$$bf(rg)^j H^\infty/G \qquad (j = m - 1, \cdots, 1)$$

between M/G and H^∞/G, and this of course is not possible when m is large for the codimension of M/G in H^∞/G is finite. Hence *the inner function w is a Blaschke product.* For the same reason w is a *finite* Blaschke product in the sense that

$$w = \prod_1^p b_j$$

where b_j is the irreducible Blaschke product determined by the orbit under G of a point in the open unit disc. Renumbering, if necessary, let b_1 through b_m be the distinct elements among the b_j.

Let u_j be a unit in H^∞ such that

$$f_j = b_j u_j$$

belongs to H^∞/G (b_j is an eigenvector of G). Then by (23)

$$M/G = \left(\prod_1^m f_j^{e_j}\right) H^\infty/G.$$

The codimension of M/G in H^∞/G is

$$\sum_1^m e_j$$

for the codimension of $f_j H^\infty/G$ in H^∞/G is 1. On the other hand the conditions of Lemma 6 are satisfied with $V = N$ and $A_j = A_{f_j}$, and we find that the codimension of M/G in H^∞/G does not exceed n. This completes the proof.

When R is the interior of a compact bordered Riemann surface, k can be any function continuous on $\bar{R}$ and holomorphic on R whose zeros are the critical points of the Green's function with pole at $t(0)$. Moreover by working on R and taking k to be such a function, one can give a shorter proof of Lemma 7.

THEOREM 1. *There is a projection P of H^∞ onto H^∞/G with the property*

$$P(fg) = fPg \tag{24}$$

for all f in H^∞/G and g in H^∞.

Proof. Let k be the function given by Lemma 7. By Lemma 4

$$kE(H^\infty) = H^\infty/G. \tag{25}$$

Let h in H^∞ be such that $kEh = 1$ and let $\phi = kh$. Then by (8) and (25)

$$E(\phi) = 1 \quad \text{and} \quad E(\phi H^\infty) = H^\infty/G.$$

The linear transformation P given by

$$Pf = E(\phi f)$$

is a projection of H^∞ onto H^∞/G with the property (24).

3.2. I have proved more than I have claimed. Pf in fact is defined for f in L^1 and P is a bounded projection of L^1 onto L^1/G with the property

$$P(fg) = fPg$$

for all f in L^∞/G and g in L^∞. On the other hand it is easy to see that such a projection must be given by

$$Pf = E(\phi f) \tag{26}$$

where ϕ belongs to L^∞ and $E\phi = 1$ [8]. Moreover if P (given by (26)) takes H^∞ onto H^∞/G and in addition if

$$\int Pf\, d\sigma = \int f\, d\sigma$$

for all f in H^∞ (a property the projection given by the proof of Theorem 1 has), then it is easy to see that ϕ belongs to H^∞ and is divisible by the inner function w, i.e. ϕ belongs to the subspace M defined at the beginning of the proof of Lemma 7. If I drop the mean value assumption and assume only that P (given by (26)) takes H^∞ onto H^∞/G, then $b\phi$ must belong to M where b is the Blaschke product determined by the orbit under G of the center of the open unit disc. Conversely if $b\phi$ belongs to M, then again it is easy to see that the projection P given by (26) takes H^∞ onto H^∞/G. The last two sentences are relevant for there are functions ϕ in L^∞ with $E\phi = 1$ and $b\phi$ in M that are not in H^∞.

I do not know if it is possible to represent the bounded holomorphic function Pf directly in terms of the group G and the bounded holomorphic function f.

4. The projection given by Theorem 1 can be used to relate ideals in H^∞/G to ideals in H^∞. Here is what I have in mind. The extension of an ideal J in H^∞/G is the ideal J^e in H^∞ generated by J, and the contraction of an ideal K in H^∞ is the ideal K^c in H^∞/G obtained by intersecting K with H^∞/G. Thus J is contained in J^{ec}, and J^{ec} is the smallest contracted ideal that contains J.

THEOREM 2. *Every ideal in H^∞/G is the contraction of its extension.*

Proof. Let f be in J^{ec}. Then

$$f = \sum f_j g_j$$

where f_j is in J and g_j is in H^∞, and

$$f = Pf = \sum P(f_j g_j) = \sum f_j Pg_j .$$

Thus f belongs to J.

Theorem 2 is equivalent to the statement that every homomorphism of the algebra H^∞/G can be extended to a homomorphism of the algebra H^∞ in the sense that the image of H^∞/G is a subalgebra of the image of H^∞.

COROLLARY. *The corona conjecture is true for R.*

Proof. Let k functions f_j in $H^\infty(R)$ be such that

$$\sum |f_j| \geq \varepsilon > 0$$

on R. Regard f_1 through f_k as functions in $H^\infty(D)$ by composing them with t (t is the analytic map that defines G). Then

$$\sum |f_j| \geq \varepsilon > 0$$

on D. Carleson [3] has shown that the corona conjecture is true for the open unit disc, and therefore the ideal in H^∞ (i.e. in $H^\infty(D)$) generated by f_1 through f_k is H^∞. By Theorem 2 or by its proof the ideal in H^∞/G (i.e. in $H^\infty(R)$) generated by the k functions f_j is H^∞/G.

The corollary is not new but I believe the proof is. Alling [2] and Stout [10] have shown that the corona conjecture is true for the interior of a compact bordered Riemann surface, and the referee tells me that Steven Sheinberg in a 1963 Princeton thesis and others have done this.

COROLLARY. *Every maximal ideal in H^∞/G is the contraction of a maximal ideal in H^∞, and every prime ideal in H^∞/G is the contraction of a prime ideal in H^∞.*

Proof. Let J be an ideal in H^∞/G. There is an ideal K in H^∞ that is maximal among all ideals in H^∞ that contract to J. If J is maximal or prime in H^∞/G, then K has the same property in H^∞ [11, p. 259].

REFERENCES

1. L. V. AHLFORS AND L. SARIO, *Riemann Surfaces*, Princeton, Princeton University Press, 1960.
2. N. L. ALLING, *Extensions of meromorphic function rings over non-compact Riemann surfaces, I*, Math. Zeitschrift, vol. 89 (1965), pp. 273–299.
3. L. CARLESON, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2), vol. 76 (1962), pp. 547–559.
4. P. R. HALMOS, *Finite dimensional vector spaces*, Princeton, Princeton University Press, 1948.
5. H. HELSON, *Lectures on invariant subspaces*, New York, Academic Press, 1964.
6. K. HOFFMAN, *Banach spaces of analytic functions*, Englewood Cliffs, Prentice-Hall, 1962.

7. M. LOÈVE, *Probability theory*, New York, Van Nostrand, 1955.
8. SHU-TEH CHEN MOY, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math., vol. 4 (1954), pp. 47–63.
9. A. PFLUGER, *Theorie der Riemannschen Flächen*, Berlin, Springer, 1957.
10. E. L. STOUT, *Bounded holomorphic functions on finite Riemann surfaces*, Trans. Amer. Math. Soc., vol. 120 (1965), pp. 255–285.
11. O. ZARISKI AND P. SAMUEL, *Commutative algebra, vol. I*, Princeton, Van Nostrand, 1958.

UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN