THE WANG SEQUENCE FOR HALF-EXACT FUNCTORS[1]

BY
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Introduction

Suppose we are given a Hurewicz fiber space E over the n-sphere S^n, $n > 1$, with Y the fiber over the south pole of S^n. This fiber space determines, as will shortly be explained, two maps;

$$\gamma : S^{n-1} \times Y \to Y \quad \text{and} \quad \varphi : E \to S(S^{n-1} \times Y).$$

Using these, the projection $p : S^{n-1} \times Y \to Y$, and the inclusion j of Y into E, one can now state the following theorem of K-theory.

THEOREM. *Let (E, S^n, Y) be a Hurewicz fiber space, E and Y CW-complexes, $n > 1$. Then the following sequence is exact:*

$$\cdots \tilde{K}^{-k-1}(Y) \xrightarrow{S^{k+1}\gamma^* - S^{k+1}p^*} K^{-k-n}(Y) \xrightarrow{S^k\varphi^*} \tilde{K}^{-k}(E) \xrightarrow{S^kj^*} \tilde{K}^{-k}(Y)$$

$$\cdots \to K^{-n}(Y) \xrightarrow{\varphi^*} \tilde{K}(E) \xrightarrow{j^*} \tilde{K}(Y) \xrightarrow{\gamma^* - p^*} K^{-n+1}(Y).$$

Moreover, the set of maps $S^k\gamma^* - S^kp^*$, $k = 0, 1, \cdots$, defines a derivation

$$\theta : \tilde{K}^*(Y) \to K^*(Y);$$

if $x \in \tilde{K}^{-m}(Y)$ then $\theta(xy) = \theta(x)y + (-1)^{(n-1)m}x\theta(y)$.

This theorem is a direct corollary of the theorem we shall prove, and is seen to be the analog of Wang's exact sequence for ordinary cohomology. The proof given is entirely elementary, following from the Puppe sequence of the inclusion of Y in E. Our theorem is a slight sharpening, found necessary for the application given in [3], of the theorem proved in [5, p. 455]. The proof of parts (a) and (b) will generalize immediately to a fiber space over a suspension of a connected CW-complex. With proper care a similar theorem for bundles over circles can be proved using these techniques or a Mayer-Vietoris argument. The author wishes to thank D. W. Anderson and A. Dold for useful conversations.

Notation

We shall be working in the category of CW-complexes with basepoint and basepoint-preserving maps. If X is a space with basepoint x' and I the unit interval we can define the cone on X as

$$CX = X \times I/X \times 0 \cup x' \times I$$

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and the suspension of X as

$$SX = X \times I/X \times 0 \cup X \times 1 \cup x' \times I.$$

The suspension can also be defined as $X \wedge S^1$, where S^1 is the unit circle and $X \wedge S^1$ is the quotient of $X \times S^1$ by the one-point union $X \vee S^1$. (See [6] for details of the $\wedge$ operation.) If $f : X \to V$ we denote by $V \cup CX$ the space formed by identifying $(x, 1)$ in CX to $f(x)$ in V for all x in X.

If $f : X \to V$ and t is any half-exact functor in the sense of Dold (see [1], in particular Ch. 5) to the category of sets with basepoint, then we have the following exact sequence: (see also [4])

$$\cdots \to tSV \xrightarrow{Sf^*} tSX \xrightarrow{Qf^*} tV \cup CX \xrightarrow{Pf^*} tV \xrightarrow{f^*} tX.$$

Here Pf includes V in $V \cup CX$, Qf shrinks V in $V \cup CX$ to a point, and Sf is the suspension of f. Except possibly for the last three sets and maps, all are groups and homomorphisms. As the main example of a half-exact functor one may think of $[\ \ , B]$; the homotopy classes of basepoint-preserving maps into a fixed space B.

The group tSX operates on $tV \cup CX$. This operation satisfies, among others, the following rules. Let $\alpha, \beta \in tSX$, $a, b \in tV \cup CX$.

(i) $\alpha \top Qf^*\beta = Qf^*(\alpha + \beta)$
(ii) $Qf^*\alpha = Qf^*\beta$ if and only if $-\alpha + \beta \in$ kernel Qf^*
(iii) $Pf^*a = Pf^*b$ if and only if there exists α such that $\alpha \top a = b$.

Fiber spaces

Let (E, S^n, Y) be a Hurewicz fiber space with $n > 1$, Y the fiber over the south pole, $y_0 \in Y$ the basepoint. If E happens to be a fiber bundle with group G [7] there is a characteristic map

$$\gamma_0 : S^{n-1} \to G.$$

Define $\gamma : S^{n-1} \times Y \to Y$ by $\gamma(x, y) = \gamma_0(x)y$. (In this case of a bundle, γ can even be used to reconstruct E. For let D^n be the n-disc. Then E is formed from $D^n \times Y$ by identifying $S^{n-1} \times Y$ to Y by γ. [2, pp. 204, 5].) Such a map γ can be constructed for a fiber space also. Let $\rho : D^n \to S^n$ be the wrap-around map that sends S^{n-1} to the south pole. Let ρ induce a fiber space E' over D^n with $\rho' : E' \to E$. As D^n is contractible, E' is fiber homotopy equivalent to $D^n \times Y$ by a map $\sigma : D^n \times Y \to E'$ which is well defined up to homotopy by choosing a basepoint $x_0 \in S^{n-1} \subset D^n$ and requiring that σ be the identity map when restricted to $x_0 \times Y$. Let λ be the composite $\rho' \circ \sigma$, and let γ be the restriction of λ to $S^{n-1} \times Y$.

Now consider the Puppe sequences of $j : Y \to E$ and the inclusion

$$i : S^{n-1} \times Y \to D^n \times Y.$$

They can be joined at the first two terms by γ and λ, giving a commutative square. The third terms can be joined by a map Ω which is λ on $D^n \times Y$ and ($\gamma \times$ identity) on the cone. Using suspensions we have a transformation of the sequences:

$$\begin{tikzcd} S^{n-1}\times Y \arrow[r, "i"] \arrow[d, "\gamma"] & D^n \times Y \arrow[r, "Pi"] \arrow[d, "\lambda"] & D^n \times Y \cup C(S^{n-1}\times Y) \arrow[r, "Qi"] \arrow[d, "\Omega"] & S(S^{n-1}\times Y) \arrow[r] \arrow[d, "S\gamma"] & \cdots \\ Y \arrow[r, "j"] & E \arrow[r, "Pj"] & E \cup CY \arrow[r, "Qj"] & SY \arrow[r] & \cdots. \end{tikzcd}$$

The map Ω is easily seen to be a homotopy equivalence. For σ induces a homotopy equivalence between $D^n \times Y/S^{n-1} \times Y$ and $E'/S^{n-1} \times Y$, $E'/S^{n-1} \times Y$ is homeomorphic to E/Y, and E/Y and $E \cup CY$, $D^n \times Y/S^{n-1} \times Y$ and $D^n \times Y \cup C(S^{n-1} \times Y)$ are naturally homotopy equivalent. Let Ω' be a homotopy inverse to Ω.

We define $\varphi : E \to S(S^{n-1} \times Y)$ by $\varphi = Qi \circ \Omega' \circ P_j$. In the case of a bundle φ is homotopic to the map which shrinks the fibers over the north and south poles and the line from (x_0, y_0) to $(0, y_0)$ in $D^n \times Y$ to a point.

One more observation before stating the theorem. The exact sequence of the pair $(S^{n-1} \times Y, S^{n-1} \vee Y)$ is also short and split:

$$0 \to tS^{k+n}Y \to tS^{k+1}(S^{n-1} \times Y) \to tS^{k+1}Y \oplus tS^{k+n} \to 0.$$

The maps to and from $tS^{k+1}Y$ are given by $S^{k+1}l^*$ and $S^{k+1}p^*$ respectively, where l is the inclusion of Y in $S^{n-1} \times Y$ by $l(y) = (x_0, y)$, and

$$p : S^{n-1} \times Y \to Y$$

is the projection. We shall think of $tS^{k+n}Y \oplus tS^{k+n}$ as a subgroup of $tS^{k+1}(S^{n-1} \times Y)$.

The Wang sequence

THEOREM. *Let t be a half-exact functor, E and Y CW-complexes, (E, S^n, Y) a Hurewicz fiber space with $n > 1$. Then*

(a) *the following sequence is exact:*

$$\cdots tS^{k+1}Y \xrightarrow{S^{k+1}\gamma^* - S^{k+1}p^*} tS^{k+n}Y \oplus tS^{k+n} \xrightarrow{S^k\varphi^*} tS^kE$$
$$\xrightarrow{S^kj^*} tS^kY \cdots \xrightarrow{S\gamma^* - Sp^*} tS^nY \oplus tS^n \xrightarrow{\varphi^*} tE$$
$$\xrightarrow{j^*} tY \underset{p^*}{\overset{\gamma^*}{\rightrightarrows}} tS^{n-1} \times Y,$$

(b) *$\varphi^*a = \varphi^*b$ if and only if a and b are in the same coset of the image of $S\gamma^* - Sp^*$,*

(c) *if t is multiplicative, then the set of maps $S^k\gamma^* - S^kp^*$, $k = 0, 1, \cdots$, defines a derivation*

$$\theta : \sum_{k\geq 0} tS^kY \to \sum_{k\geq 0} tS^{k+n-1}Y \oplus tS^{k+n-1};$$

if $x \in tS^mY$ then $\theta(xy) = \theta(x)y + (-1)^{(n-1)m}x\theta(y)$.

Remark. The double arrow in (a) means that the image of j^* is the set of all x in tY such that $\gamma^*x = p^*x$.

Proof. We begin with the exact sequence of j:

$$\cdots \to tS^{k+1}Y \xrightarrow{S^kQj^*} tS^k(E \cup CY) \xrightarrow{S^kPj^*} tS^kE \xrightarrow{S^kj^*} tS^kY$$
$$\to \cdots \to tSY \xrightarrow{Qj^*} tE \cup CY \xrightarrow{Pj^*} tE \xrightarrow{j^*} tY.$$

We shall show in the case $k > 0$ that $tS^k(E \cup CY)$ is isomorphic to $tS^{k+n}Y \oplus tS^{k+n}$. In the case $k = 0$ we shall show there is a bijection between $tE \cup CY$ and the abelian group $tS^nY \oplus tS^n$.

Let $q : D^n \times Y \to S^{n-1} \times Y$ be the projection $q(x, y) = (x_0, y)$. Then $i \circ q$ is homotopic to the identity, so that the exact sequence for i breaks up into short split sequences

$$0 \to tS^{k+1}(D^n \times Y) \underset{S^{k+1}q^*}{\overset{S^{k+1}i^*}{\rightleftarrows}} tS^{k+1}(S^{n-1} \times Y) \xrightarrow{S^kQi^*}$$
$$tS^k(D^n \times Y \cup C(S^{n-1} \times Y)) \to 0.$$

We define as usual a splitting

$$J^k : tS^k(D^n \times Y \cup C(S^{n-1} \times Y)) \to tS^{k+1}(S^{n-1} \times Y)$$

by $J^k(a) = \alpha - S^{k+1}i^* \circ S^{k+1}q^*\alpha$ for any α such that $S^kQi^*\alpha = a$. This is well defined even for $k = 0$ by rule (ii). Since $q \circ i = l \circ p$, $J^k(a)$ can be expressed alternatively as

$$J^k(a) = \alpha - S^{k+1}p^* \circ S^{k+1}l^*\alpha$$

and we see that the image of

$$tS^k(D^n \times Y \cup C(S^{n-1} \times Y))$$

in $tS^{k+1}(S^{n-1} \times Y)$ is precisely the subgroup

$$tS^{k+n}Y \oplus tS^{k+n}.$$

We now substitute $tS^{k+n}Y \oplus tS^{k+n}$ in the sequence for j, using the isomorphism (or bijection if $k = 0$)

$$J^k \circ S^k\Omega^* : tS^k(E \cup CY) \to tS^{k+n}Y \oplus tS^{k+n}.$$

By construction the map from $tS^{k+n}Y \oplus tS^{k+n}$ to tS^kE is

$$S^k(Qi \circ \Omega' \circ Pj)^* \quad \text{or} \quad S^k\varphi^*$$

by definition.

The map from $tS^{k+1}Y$ to $tS^{k+n}Y \oplus tS^{k+n}$ is defined by

$$J^k \circ S^k\Omega^* \circ S^kQj^*.$$

But $Qj \circ \Omega = S\gamma \circ Qi$ and $\gamma \circ l$ equals the identity on Y, so we see that this map is $S^{k+1}\gamma^* - S^{k+1}p^*$.

To check exactness at the last term in sequence (a), we note first that $\gamma^* \circ j^* = \gamma^* \circ p^*$ because λ is a homotopy between $j \circ \gamma$ and $j \circ p$. For the second step, let $a \in tY$ be such that $\gamma^*a = p^*a$. Let $p_2 : D^n \times Y \to Y$ be the projection. Then $p_2 \circ i = p$, so $\gamma^*a = i^*(p_2^*\, a)$. Note that E' restricted to S^{n-1} is $S^{n-1} \times Y$. Let ψ be the inclusion $\psi : S^{n-1} \times Y \to E'$. Let σ' be the homotopy inverse to σ. Then the following diagram is commutative:

$$\begin{tikzcd}
tS^{n-1}\times Y \arrow[dr, "\sigma'^*"] & & tY \arrow[ll, "\gamma^*"'] \arrow[dl, "\rho'^*"'] \\
& tS^{n-1}\times Y & \\
tD^n \times Y \arrow[uu, "i^*"] \arrow[dr, "\sigma'^*"'] & & tE \arrow[ll, "\lambda^*"'] \arrow[uu, "j^*"'] \arrow[dl, "\rho'^*"] \\
& tE' \arrow[uu, "\psi^*"] &
\end{tikzcd}.$$

Hence $\psi^*(\sigma'^* \circ p_2^*\, a) = \rho'^*a$. Now because E is formed from E' and Y identified along the images of $S^{n-1} \times Y$ in each by ψ and ρ', the "patching axiom" of half-exact functors says there exists an element $b \in tE$ such that $j^*b = a$.

For part (b), suppose $a, b \in tS^nY \oplus tS^n \subset tS(S^{n-1} \times Y)$ are such that $\varphi^*a = \varphi^*b$. Then by definition,

$$Pj^* \circ \Omega'^* \circ Qi^*a = Pj^* \circ \Omega'^* \circ Qi^*b.$$

Hence by (iii) there exists $\alpha \in tSY$ such that

$$\alpha \top \Omega'^* \circ Qi^*a = \Omega'^* \circ Qi^*b$$

and so

$$\Omega^*(\alpha \top \Omega'^* \circ Qi^*a) = Qi^*b.$$

By the naturality of the operation with respect to the transormation of the sequences for i and j this gives

$$S\gamma^*\alpha \top Qi^*a = Qi^*b.$$

By (i),

$$Qi^*(S\gamma^*\alpha + a) = Qi^*b.$$

By (ii),

$$Qi^*(S\gamma^*\alpha - Sp^*\alpha + a) = Qi^*(b).$$

Since Qi^* is monomorphic on the subgroup $tS^nY \oplus tS^n$ we see that $S\gamma^*\alpha - Sp^*\alpha = b - a$, hence a and b are in the same coset.

Finally to prove part (c), let $x \in tS^m Y$ and $y \in tS^r Y$. Then

$$\theta(xy) \in tS^{m+r}(S^{n-1} \times Y)$$

and is equal to $S^m\gamma^*x \cdot S^r\gamma^*y - S^m p^*x \cdot S^r p^*y$. We claim

$$\theta(x)y = (S^m\gamma^*x - S^m p^*x)S^r p^*y.$$

For the diagram is commutative:

$$\begin{array}{ccccc}
(tS^{m+n-1}Y \oplus tS^{m+n-1}) & \times & tS^rY & \to & tY \wedge S^{n-1} \wedge S^m \wedge S^r \\
\downarrow & & \downarrow{\scriptstyle p^*} & & \downarrow \\
tS^m(S^{n-1} \times Y) & \times & tS^r(S^{n-1} \times Y) & \to & tS^{m+r}(S^{n-1} \times Y).
\end{array}$$

On the other hand,

$$x\theta(y) = (-1)^{(n-1)m}S^m p^*x(S^r\gamma^*y - S^r p^*y).$$

For the diagram in this case is

$$\begin{array}{ccccc}
tS^mY & \times & (tS^{r+n-1}Y \oplus tS^{r+n-1}) & \to & tY \wedge S^m \wedge S^{n-1} \wedge S^r \\
\downarrow{\scriptstyle p^*} & & \downarrow & & \downarrow \\
tS^m(S^{n-1} \times Y) & \times & tS^r(S^{n-1} \times Y) & \to & tS^{m+r}(S^{n-1} \times Y).
\end{array}$$

The last vertical arrow involves the map of $S^{m+r+n-1}$ to itself induced by the permutation $S^m \wedge S^{n-1} \to S^{n-1} \wedge S^m$, as well as the inclusion as a subgroup. The permutation map gives the sign.

Hence

$$\theta(xy) - \theta(x)y - (-1)^{(n-1)m}x\theta(y) = (S^m\gamma^*x - S^m p^*x)(S^r\gamma^*y - S^r p^*y),$$

which we will show is zero. The two factors lie in

$$tS^{m+n-1}Y \oplus tS^{m+n-1} \quad \text{and} \quad tS^{r+n-1}Y \oplus tS^{r+n-1},$$

subgroups of $tS^m(S^{n-1} \times Y)$ and $tS^r(S^{n-1} \times Y)$, so we can write this product as $(a_m + b_m)(a_r + b_r)$, displaying the direct sum decompositions. The four terms of this product are all zero since we see that the definition of multiplication involves composition with the inessential diagonal map

$$S^{n-1} \to S^{n-1} \wedge S^{n-1}$$

in order to bring them into $tS^{m+r+n-1}Y \oplus tS^{m+r+n-1}$ in $tS^{m+r}(S^{n-1} \times Y)$.

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