EQUIOPEN TRANSFORMATION GROUPS

BY

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1.01. *Introduction.* In this paper, the property of "equiopeness" of a transformation group is defined and its relationship with equicontinuity and certain recursion properties is discussed. It is then shown that in the particular case in which the transformation group is a continuous flow defined by the solutions of an autonomous system of differential equations, the results yield an alternate proof and a weakening of the hypotheses of a theorem proved by Hartman and Wintner [5] concerning the equivalence of Minding-Dirichlet stability and Bohr almost periodicity for a solution of the system.

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1.02. STANDING HYPOTHESIS. Throughout this paper, (X, T) will denote a transformation group where X is a separated uniform space and T is abelian.

1.03. DEFINITION. The transformation group (X, T) is said to be "equicontinuous" at a point $x \in X$ if for each index α of X, there exists an index β of X such that $x\beta t \subset xt\alpha$ for each $t \in T$.

The transformation group (X, T) is said to be "equiopen" at x if for each index α of X, there exists an index β of X such that $x\alpha t \supset xt\beta$ for each $t \in T$.

The transformation group (X, T) is said to be "uniformly equicontinuous" (on X) if for each index α of X, there exists an index β of X such that for each $x \in X$, $x\beta t \subset xt\alpha$ for each $t \in T$.

The transformation group (X, T) is said to be "uniformly equiopen" (on X) if for each index α of X, there exists an index β of X such that for each $x \in X$, $x\alpha t \supset xt\beta$ for each $t \in T$.

1.04. *Remark.* The transformation group (X, T) is uniformly equicontinuous if and only if it is uniformly equiopen.

1.05. DEFINITION. Let $\mathfrak{D}$ be any class of subsets of T. The transformation group (X, T) is said to be "$\mathfrak{D}$-equicontinuous" at a point $x \in X$ if for each index α of X, there exist an index β of X and a member D of $\mathfrak{D}$ such that $x\beta d \subset xd\alpha$ for each $d \in D$.

The transformation group (X, T) is said to be "$\mathfrak{D}$-equiopen" at x if for each index α of X, there exist an index β of X and a member D of $\mathfrak{D}$ such that $x\alpha d \supset xd\beta$ for each $d \in D$.

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tinuous" (on X) if for each index α of X, there exist an index β of X and a member D of $\mathfrak{D}$ such that for each $x \in X$, $x\beta d \subset xd\alpha$ for each $d \in D$.

The transformation group (X, T) is said to be "uniformly $\mathfrak{D}$-equiopen" (on X) if for each index α of X, there exist an index β of X and a member D of $\mathfrak{D}$ such that for each $x \in X$, $x\alpha d \supset xd\beta$ for each $d \in D$.

1.06. *Remark.* Let $\mathfrak{D}$ be a class of subsets of T. The transformation group (X, T) is uniformly $\mathfrak{D}$-equicontinuous if and only if it is uniformly $\mathfrak{D}$-equiopen, provided that $\mathfrak{D}$ is "closed under inversion", that is for every $D \in \mathfrak{D}$, the inverse set $D^{-1} = \{d^{-1} / d \in D\}$ is also in $\mathfrak{D}$.

1.07. *Remark.* Let $x \in X$ and let $\mathfrak{D}$ be a class of subsets of T. If (X, T) is equicontinuous at x, then (X, T) is $\mathfrak{D}$-equicontinuous at x.

If (X, T) is equiopen at x, then (X, T) is $\mathfrak{D}$-equiopen at x.

If (X, T) is uniformly equicontinuous (or, equivalently, uniformly equiopen), then (X, T) is uniformly $\mathfrak{D}$-equicontinuous and uniformly $\mathfrak{D}$-equiopen.

1.08. DEFINITION. The transformation group (X, T) is said to be "distal" at a point $x \in X$ if for each $y \neq x$, there exists an index β of X such that for each $t \in T$, $(yt, xt) \notin \beta$. This property is discussed in [4, Chapter 10] under the name "separated".

1.09. *Remark.* If (X, T) is equiopen at x, then (X, T) is distal at x.

1.10. DEFINITION. Let α be an index of X. A subset C of X is said to be "α-large" if for some $x \in C$, $x\alpha \subset C$ (i.e. if C contains an α-neighborhood).

1.11. DEFINITION. Let α be an index of X. A subset B of X is said to be "α-dense in X" if B interesects each α-large subset of X.

1.12. *Examples.* A subset B of X is dense in X if and only if B is α-dense in X for each index α of X.

If X is the set of real numbers with the usual (metric) uniformity, then a subset B of X is relatively dense in X if and only if B is α-dense in X for some index α of X.

1.13. LEMMA. *The following statements are equivalent:*

(A) *X is totally bounded (i.e. for each index α of X, there is a finite subset F of X such that $X \subset F\alpha = \bigcup_{f \in F} f\alpha$).*

(B) *For each index α of X, there is a finite subset F of X such that F is α-dense in X.*

(C) *For each index α of X, there is a positive integer k such that no disjoint class of α-large sets in X has more than k members.*

(D) *For each index α of X, each disjoint class of α-large sets in X is finite.*

Proof. It will be shown that (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (A). Note that in each of the conditions, (A), (B), (C), (D), it may be assumed without loss of generality that α is symmetric.

(1) To show (A) $\Rightarrow$ (B). Suppose X is totally bounded. Let α be a symmetric index of X and let F be a finite subset of X such that $X \subset F\alpha$. It is enough to show that F is α-dense in X. Let C be any α-large subset of X and let $x \in C$ be such that $x\alpha \subset C$. For some $d \in F$, $x \in d\alpha$ and therefore $d \in x\alpha \subset C$, since α is symmetric. Hence F intersects each α-large subset C of X which was to be shown.

(2) To show (B) $\Rightarrow$ (C). Assume (B). Let α be an index of X, let F be a finite subset of X such that F is α-dense in X, and let k be the cardinal number of F. Then for any disjoint class $\mathcal{B}$ of α-large subsets of X, each $B \in \mathcal{B}$ contains at least one member of F and no two members of $\mathcal{B}$ can contain the same member of F. Therefore $\mathcal{B}$ has at most k members which establishes (C).

(3) It is clear that (C) $\Rightarrow$ (D).

(4) To show (D) $\Rightarrow$ (A). We will show the contrapositive. Suppose X is not totally bounded. Then there exists an index β of X, such that for each finite subset F of X, $X \not\subset F\beta$. It is therefore possible to form a sequence, $x_1, x_2, x_3, \cdots$ in X such that for each $n > m$, $x_n \notin x_m\beta$. Let α be a symmetric index of X for which $\alpha^2 \subset \beta$. Then for each pair $n \neq m$, $x_n\alpha \cap x_m\alpha = \emptyset$ (since otherwise $x_n \in x_m\alpha^2 \subset x_m\beta$). Hence $\{x_n\alpha / n = 1, 2, 3, \cdots\}$ is an infinite, disjoint class of α-large sets and (D) is therefore false. The proof is completed.

1.14. *Remark.* The implication (A) $\Rightarrow$ (B) of the preceding lemma can be used to show that every totally bounded metric space is separable.

1.15. Definition. A subset D of T is said to be "replete" if for each compact subset K of T, there exists a $t \in T$ for which $tK \subset D$. In particular, D is "discretely replete" if it is discrete with respect to the discrete topology of T (that is, if D contains some translate of each finite subset F of T).

1.16. *Examples.* If T is euclidean n-space E^n with the usual (coordinatewise) addition as group operation and the usual (metric) uniformity, then a subset D of T is replete if and only if D is α-large for each index α of T.

In the special case in which T is the additive group of real numbers with the usual topology, a subset D of T is replete if and only if D "contains arbitrarily long intervals" (that is, for each positive number k, D contains an interval of length k).

1.17. Definition. A subset A of T is said to be "syndetic" if A intersects each replete subset D of T (or, equivalently, if there exists a compact subset K of T such that $T = AK = \{ak / a \in A \ \& \ k \in K\}$). In particular, A is "discretely syndetic" if it is syndetic with respect to the discrete topology of T (that is, if there exist a finite number of translates of A which cover T).

1.18. *Examples.* If T is the additive group E^n with the usual uniformity, then a subset A of T is syndetic if and only if A is α-dense for some index α of T.

In the special case in which T is the additive group of real numbers with the usual uniformity, a subset A of T is syndetic if and only if A is relatively dense.

1.19. DEFINITION. Let $x \in X$. Then (X, T) is said to be "regionally almost periodic at x" if for each neighborhood U of x, there exists a syndetic subset A of T such that $Ua \cap U \neq \emptyset$ for each $a \in A$.

The transformation group (X, T) is said to be "almost periodic" (on X) if for each index α of X, there exists a syndetic subset A of T such that for each $x \in X$, $xa \in x\alpha$ for each $a \in A$.

The terms "discretely regionally almost periodic" and "discretely almost periodic" then refer to these recursion properties when T is provided with its discrete topology (that is, when A, in each case, can be chosen to be a discretely syndetic subset of T). Thus these discrete recursion properties are the strongest almost periodicity properties for any topology on T.

1.20. THEOREM. *Let X be totally bounded, let $\mathfrak{D}$ be the class of all replete subsets of T, and suppose that (X, T) is $\mathfrak{D}$-equiopen at a point $x \in X$. Then (X, T) is discretely regionally almost periodic at x.*

Proof. Let α be an index of X and let $A = \{t \in T \,/\, x\alpha t \cap x\alpha \neq \emptyset\}$. It is enough to show that the assumption that A is not discretely syndetic leads to a contradiction. Assume that A is not discretely syndetic and hence that $A' = T - A$ is discretely replete. Then a sequence $t_1, t_2, t_3, \cdots$ can be defined inductively in T such that for each $n > 1$,

$$t_n \cdot \{t_i^{-1} \,/\, i = 1, 2, \cdots, n-1\} \subset A'.$$

It follows that for $i > j$, $t_i t_j^{-1} \in A'$ and hence $x\alpha t_i t_j^{-1} \cap x\alpha = \emptyset$ and consequently $x\alpha t_i \cap x\alpha t_j = \emptyset$. Also, by assumption, there exist a symmetric index β of X and a replete subset D of T such that $x\alpha d \supset xd\beta$ (and therefore $x\alpha d$ is β-large) for each $d \in D$. Now let m be any positive integer. There exists an $s \in T$ such that $\{t_i / i = 1, \cdots, m \cdot s \subset D$. Therefore, for each $i = 1, 2, \cdots, m$, $t_i s \in D$ and hence $x\alpha t_i s$ is a β-large set and for $i \neq j$,

$$x\alpha t_i s \cap x\alpha t_j s = (x\alpha t_i \cap x\alpha t_j)s = \emptyset.$$

This means that the collection $\{x\alpha t_i s \,/\, i = 1, \cdots, m\}$ is a disjoint class of m β-large sets. But since m was arbitrary, this contradicts the assumption that X is totally bounded by statement (C) of Lemma 1.13, and the theorem is proved.

1.21. *Remark.* The strongest form of the preceding theorem is achieved when T is assumed to have its discrete topology and when X is taken to be the orbit xT, since this is the case in which the assumption that (X, T) is $\mathfrak{D}$-equiopen at x and that X is totally bounded are weakest and the conclusion that (X, T) is discretely regionally almost periodic at x is strongest. Note that, in general, the equicontinuity and equiopeness properties are weakest for the discrete topology on T and for the space xT, whereas the recursion properties, regionally almost periodic and almost periodic, are strongest for the discrete topology on T and for the space xT.

1.22. DEFINITION. Let Y be an invariant subset of X. Then (X, T) is said to be "equicontinuous on Y", "equiopen on Y", "$\mathfrak{D}$-equicontinuous on Y", "$\mathfrak{D}$-equiopen on Y", or "regionally almost periodic on Y" according as the transformation group (Y, T) is equicontinuous, equiopen, $\mathfrak{D}$-equicontinuous, $\mathfrak{D}$-equiopen, or regionally almost periodic at each $y \in Y$.

Likewise, (X, T) is said to be "uniformly equicontinuous on Y", "uniformly equiopen on Y", "uniformly $\mathfrak{D}$-equicontinuous on Y", "uniformly $\mathfrak{D}$-equiopen on Y", or "almost periodic on Y" according as the transformation group (Y, T) is uniformly equicontinuous, uniformly equiopen, uniformly $\mathfrak{D}$-equicontinuous, uniformly $\mathfrak{D}$-equiopen, or almost periodic.

1.23. *Remark.* Let $x \in X$. Then the transformation group (xT, T) is uniformly equicontinuous if and only if (xT, T) is both equicontinuous at x and equiopen at x. Hence, if the original transformation group (X, T) is both equicontinuous at x and equiopen at x, then (X, T) is uniformly equicontinuous on xT.

1.24. *Remark.* The following three statements are valid. The second and third are proved in [2], and the first follows from theorems 4.34 and 4.61 of [4].

(1) If (X, T) is almost periodic on an orbit xT, then (X, T) is discretely almost periodic on $\overline{xT}$.
(2) If (X, T) is regionally almost periodic at x and equicontinuous at x, then (X, T) is almost periodic on $\overline{xT}$.
(3) If (X, T) is regionally almost periodic at each point of X and $\mathfrak{D}$-equicontinuous at x (where $\mathfrak{D}$ is the class of all replete subsets of T), then (X, T) is equicontinuous at x.

The following theorem is a consequence of these results and Theorem 1.20.

1.25. THEOREM. *Let $x \in X$ and consider the transformation group $(\overline{xT}, T)$. Let $\mathfrak{D}$ be the class of all replete subsets of T and suppose that $(\overline{xT}, T)$ is both $\mathfrak{D}$-equicontinuous at x and $\mathfrak{D}$-equiopen at x. Then the following statements are equivalent.*

(A) *$\overline{xT}$ is totally bounded.*
(B) *$(\overline{xT}, T)$ is discretely regionally almost periodic at x.*
(C) *$(\overline{xT}, T)$ is regionally almost periodic at x.*
(D) *$(\overline{xT}, T)$ is discretely almost periodic.*
(E) *$(\overline{xT}, T)$ is almost periodic.*

Proof. It will be shown that (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (E) $\Rightarrow$ (D) $\Rightarrow$ (A).

(1) (A) $\Rightarrow$ (B) by Theorem 1.20.

(2) It is clear that (B) $\Rightarrow$ (C).

(3) To show (C) $\Rightarrow$ (E). Assume (C). By [4, Theorem 3.27], $(\overline{xT}, T)$ is regionally almost periodic at each point of $\overline{xT}$. Statement (E) then follows from the results (3) and (2) mentioned in the preceding remark.

(4) (E) $\Rightarrow$ (D) by statement (1) of the preceding remark.

(5) To show (D) $\Rightarrow$ (A). Assume (D). Let α be any index of $\overline{xT}$. By statement (3) of the preceding remark, $(\overline{xT}, T)$ is equicontinuous at x and hence there exists an index β of $\overline{xT}$, such that $x\beta t \subset xt\alpha$ for each $t \in T$. Since $(\overline{xT}, T)$ is discretely almost periodic, there exists a discretely syndetic subset A of T such that $xA \subset x\beta$. Let F be a finite subset of T for which $T = AF$. Then $xT = xAF \subset x\beta F \subset xF\alpha$. Since xF is a finite subset of xT, this proves that xT is totally bounded. It follows that $\overline{xT}$ is totally bounded and the implication (D) $\Rightarrow$ (A) is established. This completes the proof.

1.26. COROLLARY. *If, in the preceding theorem, it is also assumed that x has a compact neighborhood (relative to X or relative to $\overline{xT}$), then the following can be added to the list of equivalent statements in the conclusion.*

(F) *$\overline{xT}$ is compact (and hence $(\overline{xT}, T)$ is uniformly equicontinuous).*

Proof. It will be shown that (F) $\Rightarrow$ (A) and (D) $\Rightarrow$ (F).

(1) It is clear that (F) $\Rightarrow$ (A).

(2) To show (D) $\Rightarrow$ (F). Assume (D). Let U be a (closed) compact neighborhood of x. There exists a discretely syndetic subset A of T such that $xA \subset U$. Let F be a finite subset of T for which $T = AF$. Then $xT = xAF \subset UF$. But UF is a finite union of translates of U, each of which is closed and compact. Hence UF is closed and compact and $\overline{xT} \subset UF$. Therefore $\overline{xT}$ is compact as was to be shown.

1.27. THEOREM. *Let X be totally bounded and suppose that (X, T) is both $\mathfrak{D}$-equicontinuous and $\mathfrak{D}$-equiopen at each point of X where $\mathfrak{D}$ is the class of all replete subsets of T. Then (X, T) is equicontinuous at each point of X.*

Proof. It follows from Theorem 1.20 that (X, T) is discretely regionally almost periodic at each $x \in X$ and hence from Remark 1.24(3) that (X, T) is equicontinuous at each $x \in X$.

1.28. THEOREM. *Let X be totally bounded and suppose that (X, T) is uniformly $\mathfrak{D}$-equicontinuous (or, equivalently, uniformly $\mathfrak{D}$-equiopen) where $\mathfrak{D}$ is the class of all replete subsets of T. Then (X, T) is uniformly equicontinuous.*

Proof. It follows from Theorem 1.20 that (X, T) is discretely regionally almost periodic at each $x \in X$. From this point on, the proof parallels that of Theorem 2 in [2]. Let α be an index of X. Choose a symmetric index β of X such that $\beta^8 \subset \alpha$. Choose an index γ of X and a replete subset D of T such that $x\gamma^2 d \subset xd\beta$ for each $d \in D$ and each $x \in X$. Let $x \in X$, let $y \in x\gamma$, and let $t \in T$. It is enough to show that $yt \in xt\alpha$. Choose an index δ of X such that $\delta \subset \gamma$ and such that $x\delta t \subset xt\beta$ and $y\delta t \subset yt\beta$. Since (X, T) is discretely regionally almost periodic at each point of X, discretely syndetic sets A_1 and A_2 can be found such that $x\delta a_1 \cap x\delta \neq \emptyset$ for each $a_1 \in A_1$ and $y\delta a_2 \cap y\delta \neq \emptyset$ for each $a_2 \in A_2$. It is easy to see that the set Dt^{-1} is discretely replete and hence that there exists an element s of T such that $s \in Dt^{-1} \cap A_1$. By similar reasoning, the set $Dt^{-1} \cap D(st)^{-1}$ is discretely replete and therefore there exists an element $r \in Dt^{-1} \cap D(st)^{-1} \cap A_2$. It follows that $s \in A_1$, $r \in A_2$, $st \in D$, $rt \in D$, and $srt = rst \in D$. By the choice of A_1 and A_2, $x\delta s \cap x\delta \neq \emptyset$ and $y\delta r \cap y\delta \neq \emptyset$ and it is therefore possible to find points w and z in X such that $w, ws \in x\delta \subset x\gamma^2$ and $z, zr \in y\delta \subset x\gamma^2$. Therefore

$$wsrt \in x\gamma^2 srt \subset xsrt\beta, \qquad wsrt \in x\gamma^2 rt \subset xrt\beta,$$

$$wst \in x\gamma^2 st \subset xst\beta, \qquad zrst \in x\gamma^2 rst \subset xrst\beta,$$

$$zrst \in x\gamma^2 st \subset xst\beta \quad \text{and} \quad zrt \in x\gamma^2 rt \subset xrt\beta.$$

Also $wst \in x\delta t \subset xt\beta$ and $zrt \in y\delta t \subset yt\beta$, and it follows from the symmetry of β that

$$xt \in wst\beta \subset xst\beta^2 \subset zrst\beta^3 \subset xrst\beta^4,$$

$$yt \in zrt\beta \subset xrt\beta^2 \subset wsrt\beta^3 \subset xsrt\beta^4,$$

and hence $yt \in xt\beta^8 \subset xt\alpha$ which proves the theorem.

1.29. *Remark.* In [1], J. D. Baum proves that a transformation group (X, T) with compact phase space X and abelian phase group T is uniformly equicontinuous if and only if it is uniformly $\mathfrak{D}$-equicontinuous where $\mathfrak{D}$ is the class of all replete semigroups of T. It was later shown by Jesse Clay [3] that it is enough to take $\mathfrak{D}$ to be the class of replete subsets of T. The preceding result shows further that the assumption of compactness for X can be weakened to total boundedness, or if X is assumed compact, it is enough that (X, T) be $\mathfrak{D}$-equicontinuous and $\mathfrak{D}$-equiopen at each $x \in X$.

It should also be noted that since the topology of T is not specified, it can be assumed here and in Theorems 1.25 through 1.28 that T is discrete so that $\mathfrak{D}$ is the class of all discretely replete subsets of T. As remarked previously, this provides the weakest form of the hypotheses of $\mathfrak{D}$-equicontinuity and $\mathfrak{D}$-equiopeness.

2.01. Definition. A "continuous flow" is by definition a transformation group (X, R) where R is the additive group of real numbers with the usual topology.

2.02. *Remark.* Let $(*)$ $(dx/dt) = f(x)$ be a vector system of autonomous differential equations where $x = (x_1, \cdots, x_n)$ and $f(x) = (f_1(x), \cdots, f_n(x))$ are vectors in euclidean n-space E^n. Suppose that f satisfies conditions (e.g. Lipschitz conditions) sufficient to insure the existence of a non-vacuous open subset X of E^n with the property that for each $x \in X$, there is a unique solution π_x of the system $(*)$, defined for all real t and with values in X, such that $\pi_x(0) = x$. Then the solutions of the system $(*)$ define a continuous flow (X, R) on X where, for each $x \in X$, the solution π_x is the x-motion of the flow (i.e. $\pi_x(t) = xt$ for each $x \in X$ and each $t \in R$).

2.03. Standing Hypothesis. For the remainder of this section, let $(*)$ $(dx/dt) = f(x)$ be a vector system of autonomous differential equations such as described in the preceding remark and let (X, R) denote the continuous flow defined by the solutions of $(*)$.

2.04. Definition. Let $x \in X$ and let π_x be the corresponding solution of the system $(*)$. Then π_x is said to be "Minding-Dirichlet stable" (or simply "M-D stable") with respect to X provided that for each positive real number ε, there exists a positive real number δ, such that if π_y is any solution of $(*)$ with $\pi_y(0) = y \in X$, then $\rho(\pi_x(t_0), \pi_y(t_0)) < \delta$ for some $t_0 \in R$ implies that $\rho(\pi_x(t), \pi_y(t)) < \varepsilon$ for all $t \in R$.

2.05. *Remark.* The above definition coincides with the definition of a stable map on a general group T as given by Gottschalk and Hedlund in [4, Definition 4.68] when X is taken to be the orbit $xR = \pi_x(R)$.

2.06. *Remark.* A solution π_x of the system $(*)$ is M-D stable with respect to X in the sense of Definition 2.04 if and only if the continuous flow (X, R) is both equicontinuous at x and equiopen at x. By remark 1.23, the solution π_x is M-D stable with respect to xR if and only if (X, R) is uniformly equicontinuous on xR.

2.07. Definition. A function ϕ on R to a uniform space Y is said to be "Bohr almost periodic" provided that for each index α of Y, there exists a syndetic subset A of R such that $\phi(t + a) \in \phi(t)\alpha$ for each $t \in R$ and each $a \in A$.

2.08. *Remark.* It can be shown (see [4, Theorem 4.61]) that in the above definition, the set A can always be chosen to be a discretely syndetic subset of T.

2.09. *Remark.* Let π_x be a solution of the system $(*)$. Then the following statements are equivalent.

(A) π_x is Bohr almost periodic.

(B) (X, R) is almost periodic on xR.

(C) (X, R) is discretely almost periodic on xR.

2.10. *Remark.* It is proved by Hartman and Wintner in [5] (and also by Gottschalk and Hedlund in [4, Theorem 4.73] in a more general setting) that if X is totally bounded, then every solution π_x of $(*)$ which is M-D stable with respect to X is Bohr almost periodic. Theorem 1.25 makes it possible to replace the assumption of M-D stability for π_x in this theorem by the weaker condition that the transformation group (xR, R) is both $\mathfrak{D}$-equicontinuous at x and $\mathfrak{D}$-equiopen at x where $\mathfrak{D}$ is the class of all discretely replete subsets of R. To see this, it is enough to apply Theorem 1.25 to the transformation group (xR, R), in which case $\overline{xR} = xR$.

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