CYCLIC BRANCHED COVERINGS OF DOUBLED CURVES IN 3-MANIFOLDS

BY

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Let C be a tame, simple closed curve in the interior of the 3-manifold M and suppose $h : S^1 \times D^2 \to M$ is a tame imbedding into the interior of M with $h(S^1 \times 0) = C$. Let K be the simple closed curve in $V = S^1 \times D^2$ indicated by the figure. Then $h(K)$ is a *double* of C in M.

The fundamental group $Q = \pi_1(V - K)$ is presented by

$$Q = | a, m, l : a = la\bar{l}\bar{a}la\bar{l}\bar{a}la\bar{l}, m = l\bar{a}lal\bar{a} |,$$

where of course m and the second relation may be deleted ($\bar{x} = x^{-1}$). We keep them in the presentation since m, l generate $\pi_1(T)$ where $T = S^1 \times S^1$, and therefore $ml = lm$. The mapping $a \mapsto t, m \mapsto 1, l \mapsto 1$ of generators to elements of the infinite cyclic (multiplicative) group Z generated by t extends to a unique epimorphism $Q \to Z$ and the composition

$$Q \to Z \to Z/Z^r = Z_r$$

is an epimorphism $\varepsilon : Q \to Z_r$ to the cyclic group of order r.

Applying Fox's version [1] of the Reidemeister-Schreier algorithm, we obtain a presentation for $\tilde{Q} = \pi_1(\tilde{V})$, where $\tilde{V}$ is the r-fold cyclic branched covering space of V branched along K, corresponding to the kernel of ε:

$$\tilde{Q} = | m_i, l_i : [l_i, \bar{l}_{i+1}][\bar{l}_{i+2}, l_{i+1}] = 1, m_i = l_i \bar{l}_{i-1} l_i \bar{l}_{i+1} |$$

where the subscripts $i \in \mathbf{Z}_r$ (the integers mod r), and $[x, y] = xy\bar{x}\bar{y}$. Again the generators m_i and second class of relations may be deleted, but they are left in the presentation, since for each i the m_i, l_i generate the fundamental group of a boundary component $\tilde{T}_i$ of the boundary $\tilde{T}$ of $\tilde{V}$.

Now we see that each of the l is non-trivial since the image $[l_i]$ of l_i under the epimorphism

$$\tilde{Q} \to \tilde{Q}/[\tilde{Q}, \tilde{Q}] = H = \text{free abelian group on} \quad r \quad \text{generators}$$

is non-trivial (the $[l_i]$ form a basis for H). Actually we see much more; namely,

$$m_i^p l_i^q = (l_i \bar{l}_{i-1} l_i \bar{l}_{i+1})^p l_i^q$$

has image $[l_i]^{2p+q}[l_{i-1}]^{-p}[l_{i+1}]^{-p}$ in H. If $r > 1$ this is trivial iff $p = q = 0$. Hence if $r > 1$, $m_i^p l_i^q$ is trivial iff $p = q = 0$. This means that for $r > 1$ the inclusion induced homomorphisms $\pi_1(\tilde{T}_i) \to \pi_1(\tilde{V})$ are monomorphisms for each i.

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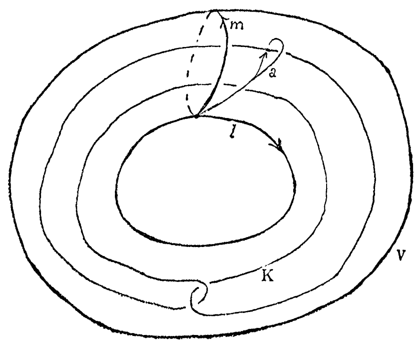
Note now that $\pi_1(M - h(K))$ is the direct limit of the following diagram of groups and homomorphisms:



$$\begin{array}{ccccc} \pi_1(M - h(V^\circ)) & \leftarrow & \pi_1(h(T)) & \rightarrow & \pi_1(h(V - K)) \\ \| & & \| & & \| \\ \pi_1(M - C) & & \pi_1(T) & & \pi_1(V - K) = Q. \end{array}$$

The homomorphisms

$$\pi_1(M - C) \to 1, \quad \pi_1(T) \to 1, \quad \pi_1(V - K) = Q \to Z_r$$

yield a unique compatible epimorphism $\varepsilon' : \pi_1(M - h(K)) \to Z_r$. This leads in turn to a branched r-fold cyclic covering $\tilde{M}$ of M branched over $h(K)$.

Because of the choice of ε', $\tilde{M}$ may be viewed as $\tilde{V}$ with r copies $\tilde{N}_i$ of $M - h(V^\circ)$ attached to $\tilde{V}$ along its corresponding boundary components $\tilde{T}_i$.

We call a simple closed curve C in the interior of M *non-trivial* if the inclusion-induced homomorphism

$$\pi_1(h(T)) \to \pi_1(M - h(V^\circ)) = \pi_1(M - C)$$

is a monomorphism. For C in a simply connected M this is equivalent to saying C is knotted.

THEOREM. *If C is non-trivial in M and K is any double of C as above with $\tilde{M}$ the r-fold cyclic branched covering of M over K as above, $r > 1$, then $\pi_1(\tilde{M})$ contains r pairwise non-conjugate copies of $\pi_1(M - C)$. These may be chosen to be the images of the inclusion-induced homomorphisms $\pi_1(\tilde{N}_i) \to \pi_1(\tilde{M})$ which are monomorphisms.*

Proof. First note that $\pi_1(\tilde{M})$ is the direct limit of the following diagram of groups and homomorphisms:

$$\begin{array}{ccccc} \pi_1(\tilde{N}_1) & \leftarrow & \pi_1(\tilde{T}_1) & \searrow & \\ & \cdots & & & \pi_1(\tilde{V}). \\ \pi_1(\tilde{N}_r) & \leftarrow & \pi_1(\tilde{T}_r) & \nearrow & \end{array}$$

But since C is non-trivial, the homomorphisms $\pi_1(\tilde{T}_i) \to \pi_1(\tilde{N}_i)$ are monomorphisms; furthermore, we showed above that the other homomorphisms $\pi_1(\tilde{T}_i) \to \pi_1(\tilde{V})$ are monomorphisms since $r > 1$. Therefore the canonical homomorphisms $\pi_1(\tilde{N}_i) \to \pi_1(\tilde{M})$ are monomorphisms, where each $\pi_1(\tilde{N}_i) = \pi_1(M - C)$. The statement on non-conjugacy will follow if we show that l_i is not conjugate to any $m_j^p\, l_j^q$ in $\tilde{Q} = \pi_1(\tilde{V})$ where $i \neq j$. If these elements are conjugate in $\tilde{Q}$, then they must have equal images in H. Hence we must have

$$[l_i] = [l_j]^{2p+q}[l_{j-1}]^{-p}[l_{j+1}]^{-p}$$

and therefore $p = 0$, $q = 1$, $i = j$. The theorem follows.

COROLLARY. *No double of a knotted simple closed curve in a simply connected 3-manifold has a simply connected r-fold cyclic branched covering for $r > 1$.*

Proof. Observe that a knotted simple closed curve in a simply-connected 3-manifold is non-trivial and apply the theorem.

Hence it is impossible to find counterexamples to the Smith conjecture by looking at the cyclic branched coverings of doubled knots.

REFERENCE

1. R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds and related topics, M. K. Fort, editor, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

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