ON THE RATIO-LIMIT THEOREM FOR MARKOV PROCESSES RECURRENT IN THE SENSE OF HARRIS

BY

RICHARD ISAAC[1]

For a recurrent, irreducible Markov chain $\{X_n\}$ with stationary transition probabilities P_{ij}^k and stationary measure $\{\pi_i\}$, the Doeblin-Chung ratio-limit theorem states [1, p. 48, Theorem 5]

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_{ij}^k}{\sum_{k=1}^{n} P_{lm}^k} = \frac{\pi_j}{\pi_m} \tag{1}$$

for i, j, l and m any arbitrary states. Let $\{X_n\}$, $n \geq 1$ be a Markov process with stationary transition probabilities $P^k(x, E)$ defined on (Ω, Σ). Using Chung's notation [2] put

$$L(x, E) = P(X_n \in E \quad \text{for some} \quad n \geq 1 \mid X_0 = x)$$

$$Q(x, E) = P(X_n \in E \quad \text{for infinitely many} \quad n \geq 1 \mid X_0 = x).$$

Throughout this paper it will be assumed that condition (C) is satisfied: there exists a measure m on Σ such that $m(E) > 0$ implies $Q(x, E) = 1$ for all $x \in \Omega$. This condition was assumed by Harris [4] and under it he proved the existence and essential uniqueness of a σ-finite stationary measure π (he assumed Σ was countably generated, but this assumption was later shown to be unnecessary [8]). Orey [9, p. 809] asked whether, under condition (C),

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} P^k(x, E)}{\sum_{k=1}^{n} P^k(y, F)} = \frac{\pi(E)}{\pi(F)} \tag{2}$$

for all E, F in Σ, $0 < \pi(F) < \infty$, and all x, y outside a fixed π-null set. This is a reasonable analogue of (1) for continuous state spaces. In this paper we show that this conjecture is false. Jain [7] has recently proved (2) under condition (C) for all E, F in Σ, $0 < \pi(F) < \infty$, and all x, y outside a π-null set which can depend upon E and F. Examples in [7] are given, due to Chung, showing that Jain's theorem could not be improved to yield convergence for *all* x, y in Ω. These examples partially suggested the idea of the counterexample given below.

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In the second part of this paper some positive results are given concerning the ratios in (2). In particular it is shown that an analogue of Orey's conjecture is valid for subsets of a fixed set of finite π measure. Elsewhere we shall examine some more detailed aspects of these ratios.

The construction of the counterexample follows: Let $X_1, X_2, \cdots$ be independent random variables each of which has the normal distribution with mean 0 and variance 1. The partial sums $\sum_{k=1}^{n} X_k = S_n$ determine a Markov process on $(R, \mathcal{B})$ where R is the real line and $\mathcal{B}$ is the class of Borel sets. If m is Lebesgue measure, results of [3] imply the validity of condition (C) (see also [5]). In this case π is Lebesgue measure.

Our aim now is to describe two Markov processes related to the S_n process. In the discussion to follow, there will be occasion to refer to conditional probabilities related to each of these three processes: S_n, T_n and U_n. The affix 1 will designate the T_n process and the affix 2 the U_n process. Lack of an affix refers to the process S_n. For instance,

$$L_2(x, E) = P(U_n \in E \quad \text{for some} \quad n \geq 1 \mid U_0 = x)$$

and

$$P(x, E) = P(S_1 \in E \mid S_0 = x).$$

Let $A = [-1, 1]$ be the closed interval of real numbers. Since A has positive, finite Lebesgue measure, $\sum_{k=1}^{\infty} P^k(x, A) = \infty$ for all x; moreover, for all x

$$\lim_{k\to\infty} P^k(x, A) = 0 \tag{3}$$

by [7, Theorem 2.5]. Consider, now, a countable set of replicas of A lying above A in an infinite stack. Call these $A_1, A_2, \cdots$; let the point of the j^{th} layer corresponding to $x \in A$ be called j_x and set $S_x = \bigcup_{j=1}^{\infty} j_x$. Endow each segment A_j with the topology of A, i.e., make the correspondence $x \leftrightarrow j_x$ into a homeomorphism, and designate the Borel field of A_j by $\mathcal{C}_j$. Let $A_0 = \bigcup_{j=1}^{\infty} A_j$, set $R_1 = R \cup A_0$ and $\mathcal{B}_1$ equal to the smallest σ-field in R_1 containing $\mathcal{B}$ and each $\mathcal{C}_j$. $\mathcal{B}_1$ is countably generated and contains the points of R_1. Extend Lebesgue measure m to a measure m_1 on $\mathcal{B}_1$ by placing $m_1(C) = 0$ for $C \subseteq A_0$. Let Bore measurable functions $a_j(x)$, $0 < a_j(x) \leq 1$ for $x \in A$, be chosen so that

$$b_n(x) = \tfrac{1}{2} \prod_{j=1}^{n} a_j(x) \geq \sup_{k \geqq n} P^k(x, A) \tag{4}$$

for all n sufficiently large, and so that in addition $b_n(x)$ converges monotonically to 0. This can be done since the right side of (4) converges monotonically to 0 by (3). The T_n process will now be defined on $(R_1, \mathcal{B}_1)$ by specifying one-step transition functions P_1. That the following functions are indeed transition functions (for fixed x, probabilities on R_1; for fixed sets, $\mathcal{B}_1$ measurable functions) is easily verified.

$$\text{if} \quad x \in R, \quad P_1(x, E) = P(x, E), \qquad E \subseteq R \tag{5}$$

$$\text{(6)} \qquad \text{if } j_x \in S_x, \quad P_1(j_x, (j+1)_x) = a_j(x)$$

$$P_1(j_x, E) = (1 - a_j(x))P^{j+1}(x, E), \qquad E \subseteq R$$

The T_n process restricted to R behaves exactly as the S_n process. Clearly the measure m_1 is stationary and it is not difficult to check condition (C) for the T_n process. The U_n process, defined next, is of principal importance and exhibits the desired behavior. The U_n process is also defined on $(R_1, \mathcal{B}_1)$ with transition probabilities given by

$$\text{(7)} \qquad \text{if } x \in R - A, \quad P_2(x, E) = P_1(x, E) = P(x, E), \qquad E \subseteq R$$

$$\text{(8)} \qquad \text{if } x \in A, \quad P_2(x, E) = \tfrac{1}{2}P_1(x, E) = \tfrac{1}{2}P(x, E), \qquad E \subseteq R$$

$$P_2(x, 1_x) = \tfrac{1}{2}$$

$$\text{(9)} \qquad \text{if } j_x \in S_x, \quad P_2(j_x, E) = P_1(j_x, E), \qquad E \subseteq R_1.$$

LEMMA 0. *Let $[a, b]$ be a closed, bounded interval and let E be a fixed set with $m(E) > 0$. Then $\inf_{x \in [a,b]} P(x, E) > 0$.*

Proof. For the S_n process,

$$P(x, E) = \frac{1}{\sqrt{2\pi}} \int_E \exp\left[-\frac{(u - x)^2}{2}\right] du,$$

a continuous function of x for fixed E. Thus the image of $[a, b]$ is compact and so closed so that the lemma is true unless $P(x_0, E) = 0$ for some x_0, which is impossible.

LEMMA 1. *The U_n process satisfies condition* (C) *relative to the measure m_1.*

Proof. If $x \in R - A$, $L_2(x, A) = 1$ by (7). Also,

$$\inf_{x \in A} L_2(x, A) \geq \inf_{x \in A} P_2(x, A) = \inf_{x \in A} \tfrac{1}{2}P(x, A) > 0$$

by Lemma 0. Hence, $\inf_{x \in R} L_2(x, A) > 0$. Thus $Q_2(x, R) = Q_2(x, A)$ for every $x \in R_1$ by [2, Prop. 7]. If $k_x \in S_x$,

$$L_2(k_x, R) \geq 1 - \textstyle\prod_{j=k+1}^{\infty} a_j(x) = 1$$

by (4) and (6) and because $b_n(x) \downarrow 0$. Therefore, $L_2(x, R) = 1$ for $x \in R_1$. Again by [2], $1 = Q_2(x, R_1) = Q_2(x, R)$ for all $x \in R_1$ and $Q_2(x, A) = 1$ for all $x \in R_1$ by the above. Let $m_1(E) > 0$. Without loss of generality assume $E \subseteq R$, $m(E) > 0$. By Lemma 0,

$$\inf_{x \in A} L_2(x, E) \geq \inf_{x \in A} P_2(x, E) > 0,$$

and once more [2] and the above show $Q_2(x, E) = 1$ for all $x \in R_1$. The proof is concluded.

LEMMA 2. *For the transition probabilities of the T_n process*

$$P_1^k(j_x, E) \leq P_1^{j+k}(x, E) = P^{j+k}(x, E)$$

for all $j \geq 1$, $k \geq 1$, *all* $x \in A$ *and* $E \subseteq R$.

Proof. The lemma is true when $k = 1$ by (6). The proof is by induction on k. Assume the lemma true for k. Then

$$\begin{aligned} P_1^{k+1}&(j_x, E) \\ &= \int_R P_1(j_x, dy)P_1^k(y, E) + a_j(x)P_1^k((j+1)_x, E) \\ &= (1 - a_j(x)) \int_R P^{j+1}(x, dy)P^k(y, E) + a_j(x)P_1^k((j+1)_x, E) \\ &\leq (1 - a_j(x))P^{j+k+1}(x, E) + a_j(x)P^{j+k+1}(x, E) \\ &= P^{j+k+1}(x, E), \qquad\qquad E \subseteq R. \end{aligned}$$

The proof is complete.

LEMMA 3. *For the transition probabilities of the U_n process*

$$P_2^k(x, E) \leqq P_1^k(x, E)$$

for all $k \geq 1$, $x \in R_1$ *and* $E \subseteq R$.

Proof. The lemma is again by induction on k. For $k = 1$, the truth of the lemma follows from (7)–(9). Assume its truth for k. If $x \in R - A$

$$\begin{aligned} P_2^{k+1}(x, E) &= \int_{R_1} P_2(x, dy)P_2^k(y, E) = \int_R P_1(x, dy)P_2^k(y, E) \\ &\leq \int_R P_1(x, dy)P_1^k(y, E) = P_1^{k+1}(x, E). \end{aligned}$$

If $x \in A$

$$\begin{aligned} P_2^{k+1}(x, E) &= \int_R P_2(x, dy)P_2^k(y, E) + P_2(x, 1_x)P_2^k(1_x, E) \\ &\leq \tfrac{1}{2} \int_R P_1(x, dy)P_1^k(y, E) + \tfrac{1}{2} P_1^k(1_x, E) \\ &\leq \tfrac{1}{2}P_1^{k+1}(x, E) + \tfrac{1}{2}P_1^{k+1}(x, E) = P_1^{k+1}(x, E). \end{aligned} \tag{10}$$

The last inequality in (10) follows from Lemma 2.

Finally, if $j_x \in S_x$

$$\begin{aligned} P_2^{k+1}(j_x, E) &= \int_{R_1} P_2(j_x, dy)P_2^k(y, E) = \int_{R_1} P_1(j_x, dy)P_2^k(y, E) \\ &\leq \int_{R_1} P_1(j_x, dy)P_1^k(y, E) = P_1^{k+1}(j_x, E), \end{aligned}$$

concluding the proof.

Let π_2 be the stationary measure for the U_n process. $\pi_2(S_x) = 0$ for each x by results of [4], since otherwise $Q_2(z, S_x) = 1$ for some x and all $z \in R_1$, which is clearly impossible from the definition of the process. Also, $\pi_2(A) > 0$ since $m_1(A) > 0$, by [4]. If Orey's question were answered affirmatively

$$(11) \qquad \lim_{n\to\infty} \frac{\sum_{k=1}^{n} P_2^k(x, S_x)}{\sum_{k=1}^{n} P_2^k(x, A)} = \frac{\pi_2(S_x)}{\pi_2(A)} = 0$$

for all x outside a fixed π_2-null set, and for all S_x with $x \in A$, provided $\pi_2(A) < \infty$, which we assume for the moment. However,

$$P_2^k(x, S_x) = \tfrac{1}{2}\textstyle\prod_{j=1}^{k-1} a_j(x) \geq \sup_{n \geq k-1} P^n(x, A) \geq P^k(x, A)$$

for all k sufficiently large, according to (4).

This is sufficient to imply, for all $x \in A$,

$$(12) \qquad \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} P_2^k(x, S_x)}{\sum_{k=1}^{n} P^k(x, A)} \geq 1.$$

Lemma 3 yields

$$(13) \qquad \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} P_2^k(x, S_x)}{\sum_{k=1}^{n} P_2^k(x, A)} \geq \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} P_2^k(x, S_x)}{\sum_{k=1}^{n} P^k(x, A)}.$$

(12) and (13) together contradict (11) for an x set of positive π measure. If $\pi(A) = \infty$, let $A' \subset A$ be chosen with $0 < \pi(A') < \infty$. Substituting A' for A in (12) makes the ratio larger and (13) is still valid. This contradicts (11) for A replaced by A' and again a contradiction results. This completes the discussion of the counter-example.

In this section we consider the general Markov process $\{X_n\}$, $n \geq 1$ defined on (Ω, Σ) with stationary probabilities $P^k(x, E)$ as introduced in the first paragraph of this paper. Condition (C) is assumed satisfied. A measurable function f on Ω is called *excessive at* x (see e.g. [6]) if

$$\int P(x, dy) f(y) \leq f(x).$$

For ease of typography, set

$$(14) \qquad A_n(x, y, E, F) = \frac{\sum_{k=1}^{n} P^k(x, E)}{\sum_{k=1}^{n} P^k(y, F)}.$$

In case $x = y$ in (14) write $A_n(x, E, F)$; if $E = F$, write (14) as $A_n(x, y, E)$.

The complement of a set E is E'. A set E of full π measure means that E' is π-null.

LEMMA 4. *Let E and F be any fixed sets, $0 < \pi(F)$, and let x_0 be any fixed point of Ω. Then* $\liminf_{n\to\infty} A_n(x, x_0, E, F) = f(x)$ *is excessive at all $x \in \Omega$.*

Proof. Apply Fatou's lemma and use the divergence of $\sum_{k=1}^{\infty} P^k(x_0, F)$ on

$$\int P(x, dy)A_n(y, x_0, E, F) = \frac{\sum_{k=2}^{n+1} P^k(x, E)}{\sum_{k=1}^{n} P^k(x_0, F)}$$

to obtain

$$\int P(x, dy)f(y) \leq f(x).$$

LEMMA 5. *Let F be a fixed set, $0 < \pi(F) < \infty$. Let E be any set. Then the function* $\liminf_{n\to\infty} A_n(x, E, F) = g(x, E)$ *is excessive for all x outside a fixed π-null set (independent of E).*

Proof. For all x, y inside a set Ω_0 of full π measure, $\lim_{n\to\infty} A_n(x, y, F) = 1$ by [7, Theorem 3.4]. Without loss of generality, assume Ω_0 stochastically closed, i.e., $P(x, \Omega_0) = 1$ for every $x \in \Omega_0$. For, defining

$$A_0 = \Omega_0' \quad \text{and} \quad A_n = \{x: P(x, A_{n-1}) > 0\},$$

$\Omega_0 - \bigcup_{n=0}^{\infty} A_n$ is stochastically closed and π-full. Thus, the process may be restricted to Ω_0 assumed stochastically closed and Lemma 4 is valid for this restriction. If E_0 is the restriction of E to Ω_0, $P^k(x, E_0) = P^k(x, E)$ for all k and all $x \in \Omega_0$. Now, if x_0 is any fixed point of Ω_0 and x is any point of Ω_0

$$\begin{aligned} &\liminf_{n\to\infty} A_n(x, E, F) \\ &\qquad = \liminf_{n\to\infty} A_n(x, x_0, E, F)/\lim_{n\to\infty} A_n(x, x_0, F) \\ &\qquad = \liminf_{n\to\infty} A_n(x, x_0, E, F). \end{aligned} \tag{15}$$

(15) and Lemma 4 complete the proof.

COROLLARY. *Under the assumptions of Lemma 5,*

$$\liminf_{n\to\infty} A_n(x, E, F) \geq \pi(E)/\pi(F)$$

for all x outside a fixed π-null set.

Proof. From Lemma 5, if r is a positive integer and

$$\liminf_{n\to\infty} A_n(x, E, F) = g(x, E)$$

then

$$\int P^r(x, dy)g(y, E) = \int_{\Omega_0} P^r(x, dy)g(y, E) \leq g(x, E)$$

for $x \in \Omega_0$. There is a set $\Omega_1 \subseteq \Omega_0$ of full π measure so that

$$\lim_{n\to\infty} A_n(y, E, F) = \pi(E)/\pi(F)$$

by [7]. Therefore

$$P^r(x, \Omega_1)\frac{\pi(E)}{\pi(F)} = \int_{\Omega_1} P^r(x, dy)g(y, E) \leq \int P^r(x, dy)g(y, E) \leq g(x, E).$$

r is arbitrary and $\lim_{r\to\infty} P^r(x, \Omega_1) = 1$ for every $x \in \Omega$, yielding the conclusion.

THEOREM 1. *Let B be any fixed set, $0 < \pi(B) < \infty$. Let R and S be subsets of B with $0 < \pi(S)$. Then*

$$\lim_{n\to\infty} A_n(x, y, R, S) = \pi(R)/\pi(S) \tag{16}$$

for all x, y outside a fixed π-null set (independent of R and S).

Proof. Let F be any fixed set, $0 < \pi(F) < \infty$. By the preceding corollary,

$$\liminf_{n\to\infty} A_n(x, E, F) \geq \pi(E)/\pi(F)$$

for all x in the fixed π-full set Ω_0, for any set E. For fixed B, on a π-full set $\tilde{\Omega}_0$,

$$\lim_{n\to\infty} A_n(x, B, F) = \pi(B)/\pi(F)$$

by [7]. For any set $R \subseteq B$, and any $x \in \Omega_0 \cap \tilde{\Omega}_0$,

$$\begin{aligned} \pi(B)/\pi(F) &= \lim A_n(x, B, F) \\ &\geq \limsup_{n\to\infty} A_n(x, R, F) + \liminf_{n\to\infty} A_n(x, B - R, F) \\ &\geq \pi(R)/\pi(F) + \pi(B - R)/\pi(F) = \pi(B)/\pi(F) \end{aligned} \tag{17}$$

by the corollary, and so all inequalities in (17) reduce to equalities. This means

$$\limsup_{n\to\infty} A_n(x, R, F) \leq \pi(R)/\pi(F)$$

and by the corollary

$$\lim_{n\to\infty} A_n(x, R, F) = \pi(R)/\pi(F).$$

This is true for any $R \subseteq B$ and any x restricted to a π-full set depending only upon F and B. If $0 < \pi(S)$ and $S \subseteq B$,

$$A_n(x, R, F)/A_n(x, S, F) = A_n(x, R, S)$$

and taking limits proves (16) when $x = y$. We have

$$A_n(x, y, S)A_n(y, x, F) = A_n(x, S, F)A_n(y, F, S). \tag{18}$$

For x, y in $\Omega_0 \cap \tilde{\Omega}_0$, (18) yields $\lim_{n\to\infty} A_n(x, y, S) = 1$, provided $\pi(S) > 0$. Then

$$A_n(x, y, S)A_n(x, R, S) = A_n(x, y, R, S), \tag{19}$$

and for x, y in $\Omega_0 \cap \tilde{\Omega}_0$, taking limits in (19) gives the desired result.

COROLLARY. *Let Ω be a topological space and Σ the Borel field. Suppose further that the topology $\mathfrak{J}$ on Ω has a countable base consisting of sets of finite π measure. Then, if R and S are any sets contained in any compact set,* $0 < \pi(S)$,

$$\lim_{n\to\infty} A_n(x, y, R, S) = \pi(R)/\pi(S)$$

for all x, y outside a fixed π-null set.

Proof. For each set B_n in a countable base, apply the construction in Theorem 1 so that there is a π-full set $\Omega_1 \subseteq \Omega_0$ with

$$\lim_{k\to\infty} A_k(x, B_n, F) = \pi(B_n)/\pi(F)$$

for each base element B_n and all x in Ω_1. If R and S are contained in compact K, then $R \cup S \subseteq \bigcup_{n=1}^{N} B_n$. Clearly

$$\lim_{n\to\infty} A_n(x, R, F) = \pi(R)/\pi(F) \quad \text{and} \quad \lim_{n\to\infty} A_n(x, S, F) = \pi(S)/\pi(F)$$

for $x \in \Omega_1$. The proof is completed along the lines of Theorem 1.

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HUNTER COLLEGE
NEW YORK, NEW YORK