ON CONJUGACY OF HOMOMORPHISMS OF TOPOLOGICAL GROUPS

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Let G and H be topological groups and let φ_1 and φ_2 be continuous homomorphisms of G into H. We say that φ_2 is conjugate to φ_1, if there exists an element $h \in H$ such that $h^{-1}\varphi_1(g)h = \varphi_2(g)$ for all $g \in G$. In [5], Iwasawa has shown among others that two homomorphisms belonging to the same connected component of the topological group of automorphisms of a compact group G are conjugate. Using his result, Hofmann and Mostert [4] have established the following result. Let G and H be compact topological groups. Then the set $\mathrm{Sur}(G, H)$ of all surjective morphisms of G onto H is a topological space under the uniform topology, on which the component H_0 of the identity of H acts continuously under the map $(\varphi, h) \rightarrow I_h \circ \varphi$, where I_h, for $h \in H_0$, denotes the inner automorphism of H, which is induced by h. Moreover, the connected components of the space $\mathrm{Sur}(G, H)$ are exactly the orbits of H_0 under the action.

In the present work we extend their theorem to the space of all homomorphisms (continuous) of compact groups, thus interpreting conjugacy of two homomorphisms of compact groups in terms of connected components of the space of homomorphisms. In §1, we collect some of known facts about the space $\mathrm{Hom}(G, H)$, the space of homomorphisms of G into H, and about the space of all compact subsets of a group. §2 carries the main results of this work. A special attention is to be directed to 2.8, which was proved first by Goto and Kimura in [3].

Notation. Throughout this work, the identity element of a group is denoted by 1 and the connected component of the identity of a group G by G_0. If G is a group and $g \in G$, then I_g denotes the inner automorphism of G which is induced by g. Finally, if φ_1 and φ_2 are homomorphisms of groups then we use $\varphi_1 \circ \varphi_2$ to denote the composite of them, whenever it makes a sense.

1. The spaces $\mathrm{Hom}(G, H)$ and $K(G)$

1.1. Let G and H be topological groups and $\mathrm{Hom}(G, H)$ the set of all continuous homomorphisms of G into H. Then the set $\mathrm{Hom}(G, H)$ becomes a topological space, when equipped with the uniform convergent topology on compact subsets. To describe neighborhoods of $\rho \in \mathrm{Hom}(G, H)$, let C be a compact subset of G and U a neighborhood of 1 in H. Then we define $W(\rho; C, U)$ to be the set of all $\psi \in \mathrm{Hom}(G, H)$ such that $\psi(g)\rho(g)^{-1} \in U$, for all $g \in C$. Then, when C and U run over all compact subsets of G and all

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neighborhoods of 1 in H, respectively, the $W(\rho; C, U)$'s form a neighborhood basis of ρ in Hom (G, H). If G is locally compact and H is a compact Lie group, then Hom (G, H) is also locally compact (See [2]) and, if H is the circle group, then the topology of the space is that of the character group of G.

1.2. Let G be a topological group and let $K(G)$ be the set of all compact subsets of G. After Gleason (See p. 200 of [1]), we introduce the following topology on $K(G)$. For $C_0 \in K(G)$ and U a neighborhood of 1, let (C_0, U) denote the set of all $C \in K(G)$ such that $C_0 \subseteq UC$ and $C \subseteq UC_0$. Then the sets (C_0, U)'s form a neighborhood system of C_0 in $K(G)$. If G is locally compact, then so is $K(G)$ and, in this case, Hom (G, G) acts continuously on $K(G)$ under $(\psi, C) \to \psi(C)$, $\psi \in$ Hom (G, G) and $C \in K(G)$.

The proof of the following two lemmas may be found in [4].

1.3. LEMMA. *Let G be a compact group and let $N(G)$ denote the subset of $K(G)$ consisting of all compact normal subgroup of G. Then $N(G)$ is totally disconnected.*

1.4. LEMMA *Let G be a compact group and H a Lie group. For $\psi \in$* Hom (G, H), *let* ker (ψ) *denote the kernel of ψ. Then $\psi \to$* ker (ψ) *is a continuous function of* Hom (G, H) *to* $N(G)$.

1.5. LEMMA. *Let G and H be compact groups and let* Aut (H) *be the topological group of all automorphisms of H under the uniform topology. Then* Aug (H) *acts continuously on the space* Hom (G, H) *under the map $(\theta, \psi) \to \theta \circ \psi$, for $\theta \in$* Aut (H) *and $\psi \in$* Hom (G, H).

2. Main results

Throughout this section, we maintain the notation of §1. The technical Lemma 2.2 plays an important role in the proof of the Theorem 2.5.

2.1. LEMMA. *Let G be a compact Lie group and H any group. Let φ and ψ be continuous homomorphisms of G into H such that the kernels of φ and ψ coincide. If there exists an $h \in H$ such that $h^{-1}\varphi(G)h \subseteq \psi(G)$, then $h^{-1}\varphi(G)h = \psi(G)$.*

Proof. Let I_h denote the inner automorphism induced by h. By replacing φ by $I_h \circ \varphi$, it is sufficient to prove the assertion for $h = 1$. Since φ and ψ factor uniquely through $G/\ker(\varphi)$ and since by compactness bothinjections $G/\ker(\varphi) \to H$ are isomorphisms onto their images, we may further assume that $G \subset H$ and $\psi = 1_G$, the identity map on G. The map $\varphi : G \to \mathrm{Im}(\varphi)$ induces a monomorphism of Lie algebras $L(\varphi) : L(G) \to L(\mathrm{Im}(\varphi))$. Since $L(G)$ is finite-dimensional, $L(\mathrm{Im}(\varphi)) = L(G)$ and hence G_0 is identical with the identity component K of Im (φ). Now φ induces an injective morphism $\varphi : G/G_0 \to \mathrm{Im}(\varphi)/K$ of finite groups. Thus it must be a bijection. Hence $G = \mathrm{Im}(\varphi)$.

2.2. LEMMA. *Let G be a compact group and H any Lie group. If $\mathfrak{A}$ is a con-*

nected component of the space Hom (G, H) *and if* $\varphi, \psi \in \mathfrak{A}$, *then there exists* $h \in H_0$ *such that* $h^{-1}\varphi(G)h = \psi(G)$.

Proof. By 1.4, elements in $\mathfrak{A}$ have the same kernel. Let $\rho \in \mathfrak{A}$. Then $\rho(G)$ is a compact subgroup of the Lie group H. Hence, by a theorem of Montgomery and Zippin (See p. 216 of [6]), there exists a neighborhood U_ρ of 1 in H such that $\rho(G)$ is contained in U_ρ and that, if K is any compact subgroup of H contained in U_ρ, then there exists an $x \in H_0$ such that $x^{-1}Kx \subseteq \rho(G)$. Let $0 : G \to H$ denote the trivial map and consider the neighborhood $W(0; G, U_\rho)$ of ρ. By definition (See 1.1), $W(0; G, U_\rho)$ consists of $\psi \in$ Hom (G, H) with $\psi(G) \subseteq U_\rho$. Thus if $\psi \in \mathfrak{A} \cap W(0; G, U_\rho)$, then $x^{-1}\psi(G)x = \rho(G)$ for some $x \in H_0$ by 2.1.

Now the set $\mathfrak{A}$ is covered by $W(0; G, U_\rho)$, $\rho \in \mathfrak{A}$. Since $\mathfrak{A}$ is connected, we can find $\rho_0 = \varphi, \rho_1, \cdots, \rho_n, \rho_{n+1} = \psi$ in $\mathfrak{A}$ such that, with $U_i = U_{\rho_i}$, $0 \leq i \leq n+1$, we have

$$\mathfrak{A} \cap W(0; G, U_i) \cap W(0; G, U_{i+1}) \neq \emptyset, \qquad 0 \leq i \leq n.$$

Choose $\gamma_i \in \mathfrak{A} \cap W(0; G, U_i) \cap W(0; G, U_{i+1})$, $0 \leq i \leq n$. Then, by what we have observed in the previous paragraph, we can find $x_i, y_i \in H_0$ such that $x_i^{-1}\gamma_i(G)x_i = \rho_i(G)$ and $y_i^{-1}\gamma_i(G)y_i = \rho_{i+1}(G)$, $0 \leq i \leq n$. From the above, it is then quite easy to see that there exists an $h \in H_0$ with the desired property.

2.3 LEMMA. *Let* G, H *and* $\mathfrak{A}$ *be as in* 2.2 *and let* $\rho \in \mathfrak{A}$ *be fixed. Putting* $\rho(G) = K$, *let* N *denote the normalizer of* K *in* H_0 *and let* $\mathrm{Int}_N(K)$ *denote the closure of the subgroup of* Aut (K), *which consists of the restrictions to* K *of the inner automorphisms of* H *which are induced by elements of* N. *Then*

(i) *the quotient space* Aut $(K)/\mathrm{Int}_N(K)$ *is discrete, and*

(ii) *if we choose* $h \in H_0$ *with* $h^{-1}\psi(G)h = K$ *for* $\psi \in \mathfrak{A}$, *then*

$$I_h \circ \psi \circ \rho^{-1} \in \mathrm{Aut}\,(K).$$

Proof. (i) follows from a theorem of Iwasawa (See p. 509 of [5]) and (ii) is clear from 2.1, since the kernels of ρ and ψ coincide.

2.4. LEMMA. *Let everything be the same as in* 2.3 *and let* λ *be the quotient map of* Aut (K) *onto* Aut $(K)/\mathrm{Int}_N(K)$. *We define*

$$\pi : \mathfrak{A} \to \mathrm{Aut}\,(K)/\mathrm{Int}_N\,(K)$$

as follows. For each $\psi \in \mathfrak{A}$, *choose* $h \in H_0$ *so that* $h^{-1}\psi(G)h = K$. (*By* 2.2, *such an* h *always exists.*) *Then from* (ii), 2.3 $I_h \circ \psi \circ \rho^{-1} \in$ Aut (K). *Then let* $\pi(\psi) = \lambda(I_h \circ \psi \circ \rho^{-1})$. *Then*

(i) π *is well defined, and*

(ii) π *is continuous.*

Proof. Suppose that $h_1 \in H_0$ is so chosen that $h_1^{-1}\psi(G)h_1 = K$ holds. Then $h^{-1}h_1$ belongs to N and it is then clear that $\lambda(I_{h_1} \circ \psi \circ \rho^{-1}) = \lambda(I_h \circ \psi \circ \rho^{-1})$.

Therefore π is well defined. To prove (ii), let $\psi \in \mathfrak{A}$ and let ψ_i, $i \in I$ be a net in $\mathfrak{A}$ converging to ψ. Then, again by a theorem of Montgomery and Zippin [6], we can find a neighborhood U of 1 in H containing $\psi(G)$ such that, if C is any compact subgroup of H contained in U, then there exists an $x \in H_0$ sufficiently close to 1 so that $h^{-1}Ch \subseteq \psi(G)$. Since the net ψ_i converges to ψ, we assume that this net is contained in the neighborhood $W(0;\, G,\, U)$ of ψ. Then ψ_i's have all the common kernel and hence, for each $i \in I$, there exists $h_i \in H_0$ such that $h_i^{-1}\psi_i(G)h_i = \psi(G)$. Since such h_i's may be found arbitrarily close to 1, we may also assume that h_i's are all contained in a compact neighborhood of 1. Thus the net h_i, $i \in I$ has a convergent subnet, say $h_{i(j)}$, $j \in J$. Let $h = \lim_j h_{i(j)}$. Then $h \in H_0$. It is also clear that the automorphisms $I_{h_{i(j)}} \circ \psi_{i(j)} \circ \rho^{-1}$ converge to the automorphism $I_h \circ \psi \circ \rho^{-1}$ and thus we conclude that $\pi(\psi_{i(j)})$ converge to $\pi(\psi)$, which proves the continuity of π at ψ.

2.5. Theorem. *Let G be a compact group and H any Lie group. If φ and ψ belong to the same connected component of the space* Hom (G, H), *then there exists an element $h \in H$ such that $h^{-1}\varphi(g)h = \psi(g)$ for al $g \in G$.*

Proof. Let $\mathfrak{A}$ be the connected component of Hom (G, H), to which φ and ψ belong. Fixing an element $\rho \in \mathfrak{A}$ as in 2.4 and using the notation of 2.4, we see that $\pi(\mathfrak{A})$ must be a singleton in the space Aut $(K)/\mathrm{Int}_N(K)$ because of the continuity of π. Thus there exist h_1 and h_2 in H_0 such that

$$I_{h_1} \circ \psi \circ \rho^{-1} = I_{h_2} \circ \varphi \circ \rho^{-1} \mod \mathrm{Int}_N(K).$$

Therefore there exists an element $h \in H$ such that $h^{-1}\varphi(g)h = \psi(g)$, for all $g \in G$.

Remark. If H is compact, then we can choose h from H_0 because $\mathrm{Int}_N(K)$ consists of I_x restricted to K for $x \in N$.

From the above remark, it is now easy to formulate the following theorem:

2.6. Theorem. *Let G and H be compact groups and let H_0 be the identity component of H. Then H_0 acts continuously on the space* Hom (G, H) *under the map $(h, \varphi) \to I_h \circ \varphi$, $h \in H_0$ and $\varphi \in$* Hom (G, H). *Moreover, the connected components of the space* Hom (G, H) *are exactly the orbits of H_0.*

Proof. The first part of the assertion is clear from the property of uniform topology given on Hom (G, H). To prove the second part, let $\mathfrak{A}$ be a connected component containing $\psi \in$ Hom (G, H) and let φ be an arbitrary element in $\mathfrak{A}$. Let $N_H(1)$ be a neighborhood system of 1 in H. Then it is well known that each $U \in N_H(1)$ contains a compact normal subgroup K_U such that H/K_U is a Lie group. Let $\alpha_U : H \to H/K_U$ denote the quotient map, $U \in N_G(1)$. Then each α_U induces a continuous function

$$\bar{\alpha}_U : \mathrm{Hom}\,(G, H) \to \mathrm{Hom}\,(G, H/K_U)$$

via $\varphi \to \alpha_U \circ \varphi$, for $\varphi \in$ Hom (G, H). Thus $\bar{\alpha}_U(\varphi)$ and $\bar{\alpha}_U(\psi)$ are both in the same connected component of Hom $(G, H/K_U)$. Hence, by 2.5, there exists

$h_U \in H_0$ such that $h_U^{-1}\varphi(g)h_U = \psi(g)$, for all $g \in G$ (mod K_U). Since H is compact, the net h_U, $U \in N_G(1)$ has a convergent subnet, say $h_{U(j)}$, $j \in J$. Thus $h = \lim h_{U(j)}$ exists and belongs to H_0. Then it is clear that φ and ψ are conjugate with respect to this h. Therefore φ is contained in the orbit on ψ. Since each orbit of H_0 is connected, it follows that $\mathfrak{A}$ is identical with the orbit on ψ, which finishes our proof.

2.7. COROLLARY. *Let φ and ψ be continuous homomorphisms of a compact group G into a connected compact group H. Then φ and ψ are conjugate if and only if they are in the same connected component of the space* Hom (G, H).

2.8. COROLLARY (Goto and Kimura). *Let G be a compact group and $i : G \to G$ the identity map. Then the connected component of* Hom (G, G) *containing i is exactly the component of the identity of the group* Aut (G).

2.9. COROLLARY. *Let G be a connected compact group and H a finite-dimensional connected locally compact group. Then any two continuous homomorphisms are conjugate, if they belong to the same connected component of the space* Hom (G, H).

Proof. Since H is finite-dimensional, there exists a compact, central, and totally disconnected subgroup K of H such that H/K is a Lie group. Let $\pi : H \to H/K$ be the quotient map and let $\pi' =$ Hom (G, π) be the induced continuous map from Hom (G, H) into Hom $(G, H/K)$. If φ and ψ belong to the same connected component, then $\pi'(\varphi)$ and $\pi'(\psi)$ both belong to a connected component. Then by 2.6, there exists an element $h \in$ H such that

$$\pi(h)^{-1}\pi'(\varphi)(g)\pi(h) = \pi'(\psi)(g),$$

for all $g \in G$. Hence, it follows that $h^{-1}\varphi(g)h\psi(g)^{-1} \in K$ for all $g \in G$. Define $\alpha : G \to K$ by $\alpha(g) = h^{-1}\varphi(g)h\psi(g)^{-1}$, $g \in G$; then $\alpha(1) = 1$ and the continuity of α and the total disconnectedness of K implies that $\alpha(G) = 1$. Thus, $h^{-1}\varphi(g)h = \psi(g)$, for all $g \in G$, follows.

Remark. We note here that 2.5 and 2.6 can not be extended to non-compact groups G as character groups of non-compact abelian groups (such as vector groups) indicate. However, it would be interesting to see if 2.5 still holds for any locally compact H. Of course, it is necessary, in this case, to replace conjugacy by automorphisms belonging to the identity component of the automorphism group of H, since this component is in general, much larger than the inner automorphism group.

Added in proof. We have recently obtained a result which generalizes 2.6 to any locally compact group H. See our forthcoming paper in this journal.

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