PROJECTIVE CLASSES OF COTORSION GROUPS[1]

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1. Introduction

Associated with any class $\mathcal{C}$ of abelian groups[2] is the class $\mathcal{E}(\mathcal{C})$ of short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ for which for every $G \in \mathcal{C}$, Hom $(G, B) \to$ Hom (G, C) is an epimorphism. Associated with any class $\mathfrak{D}$ of short exact sequences is the class $\mathcal{P} = \mathcal{P}(\mathfrak{D})$ of all groups G for which the map Hom (G, B) $\to$ Hom (G, C) is an epimorphism for every sequence $A \rightarrowtail B \twoheadrightarrow C \in \mathfrak{D}$. These classes satisfy $\mathcal{E}(\mathcal{C}) = \mathcal{E}(\mathcal{P}(\mathcal{E}(\mathcal{C})))$ and $\mathcal{P}(\mathfrak{D}) = \mathfrak{D}(\mathcal{E}(\mathcal{P}(\mathfrak{D})))$ for any classes $\mathcal{C}$ and $\mathfrak{D}$ of groups and short exact sequences, respectively. The class $\bar{\mathcal{C}} = \mathcal{P}(\mathcal{E}(\mathcal{C}))$ is the *projective closure* of $\mathcal{C}$. In this paper we take several classes $\mathcal{C}$ of cotorsion groups, and describe the projective closure $\bar{\mathcal{C}}$. Many of the results in this paper are generalizations of theorems in [6], where the projective closure of the class of torsion complete p-groups was described, and criteria for $\bar{\mathcal{C}}$ to contain certain divisible groups was given.

A group G is *cotorsion* if it cannot be properly extended by any torsion free group. This is equivalent to the equation Ext $(Q, G) = 0$, where Q is the group of rationals. Cotorsion groups were introduced by Harrison [2], Nunke [5] and Fuchs [1]. We summarize some of the properties of these groups in the next few paragraphs. Details can be found in the above papers or in any recent text in abelian groups.

Any reduced group R can be embedded as a subgroup of a reduced cotorsion group with the quotient torsion free divisible. This can be done in the following way. Applying the Hom and Ext functors to the short exact sequence $Z \rightarrowtail Q \twoheadrightarrow Q/Z$ gets the short exact sequence

$$R \cong \text{Hom}\ (Z, R) \rightarrowtail \text{Ext}\ (Q/Z, R) \twoheadrightarrow \text{Ext}\ (Q, R).$$

The group Ext $(Q/Z, R)$ is the *cotorsion completion* of R. One can see that a reduced group R is cotorsion if and only if the map

$$R \to \text{Ext}\ (Q/Z, R)$$

is an isomorphism. For any group R, Ext $(Q/Z, R)$ is reduced and cotorsion, and Ext (Q, R) is torsion free divisible. If T is a reduced torsion group, T is the torsion subgroup of its cotorsion completion Ext $(Q/Z, T)$.

The primary decomposition $Q/Z \cong \sum_p Z(p^\infty)$, leads to the decomposition $G \cong \prod_p \text{Ext}\ (Z(p^\infty), G)$ for reduced cotorsion groups G. The groups Ext $(Z(p^\infty), G)$ are modules over the ring of p-adic integers. It is often

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[2] Throughout this paper the word *group* will mean *abelian group*.

convenient to specialize to p-adic cotorsion groups, and this decomposition allows us to do so.

A group is *algebraically compact* if it is a direct summand whenever it is embedded as a pure subgroup. A reduced torsion group is *torsion complete* if it is a direct summand whenever it is embedded as a pure subgroup of a torsion group. The reduced algebraically compact groups are precisely the reduced cotorsion groups whose torsion subgroups are torsion complete. They are also precisely the cotorsion groups with no elements of infinite height. The class of algebraically compact groups is the same as the class of direct products of primary cyclic groups and groups of type $Z(p^\infty)$, where p ranges over the set of primes.

A cotorsion group G is adjusted if it is reduced and has no nonzero torsion free summands, or equivalently if G is reduced and G/tG is divisible, where tG denotes the maximum torsion subgroup of G. If G is a reduced cotorsion group, let $A/tG = d(G/tG)$, the maximum divisible subgroup of G/tG. Then A is an adjusted cotorsion group, and $G = A \oplus R$ with R torsion free cotorsion. We take advantage of this decomposition and study adjusted and torsion free reduced cotorsion groups separately. The adjusted cotorsion groups we investigate have nice torsion subgroups, e.g., torsion complete, direct sums of cyclic, or direct sums of torsion complete subgroups.

In the next section we show the projective closures of these classes of reduced cotorsion groups contain certain divisible groups. In the later sections we show the projective closure of each class is the smallest class containing the original class, all free groups, the divisible groups determined in Section 2, and which is closed under direct summands and arbitrary direct sums. It is easy to show that the smallest class with these properties is the following. Let $\mathcal{C}$ be any class of groups, and $\mathfrak{D}$ a class of divisible groups closed under sums and summands. The smallest class containing $\mathcal{C}$, $\mathfrak{D}$ and free groups which is closed under sums and summands is the class of all groups of the form $G = C \oplus D \oplus F$ with C a summand of a direct sum of groups in $\mathcal{C}$, $D \in \mathfrak{D}$ and F free. For each class $\mathcal{C}$ we look at, we proceed by proving there are *enough projectives* of this form, i.e., for each group G there is a short exact sequence

$$K \rightarrowtail P \twoheadrightarrow G \in \mathcal{E}(\mathcal{C})$$

with $P = C \oplus D \oplus F$ with C a direct sum of groups in $\mathcal{C}$, D in the appropriate class of divisible groups and F free. Such a sequence is called a *$\mathcal{C}$-projective presentation of G*. If $G = \sum_{\alpha\in A} G_\alpha$ and

$$K_\alpha \rightarrowtail P_\alpha \twoheadrightarrow G_\alpha \in \mathcal{E}(\mathcal{C}) \quad \text{for each } \alpha \in A,$$

then it is easy to see that the sequence $K \rightarrowtail \sum_{\alpha\in A} P_\alpha \twoheadrightarrow G$ obtained by taking the direct sum of the maps $P_\alpha \to G$, again belongs to $\mathcal{E}(\mathcal{C})$. For this reason, it will suffice to find $\mathcal{C}$-projective presentations for $Z(p^\infty)$ for each prime p, for Q, and for all reduced groups.

We single out one special sequence that will be referred to a number of times.

1.1 LEMMA. *For each prime p, there is an exact sequence*

$$E(p) : P(p) \rightarrowtail D \twoheadrightarrow Z(p^\infty)$$

with $P(p)$ the group of p-adic integers and D torsion free divisible.

Proof. Tensoring $P = P(p)$ with the sequence $Z \rightarrowtail Q \twoheadrightarrow Q/Z$, where Z is the group of integers and Q the group of rationals, gives the short exact sequence $P \otimes Z \rightarrowtail P \otimes Q \twoheadrightarrow P \otimes Q/Z$. Now $P \otimes Z \cong P$, and $P \otimes Q$ is torsion free divisible. For primes $q \neq p$, $P \otimes Z(p^\infty) = 0$, so

$$P \otimes Q/Z \cong P \otimes Z(p^\infty).$$

The group $P \otimes Z(p^\infty)$ is a divisible p-group, and using the fact that P/pP is cyclic of order p, we see that it has p-rank one. Thus $P \otimes Z(p^\infty) \cong Z(p^\infty)$.

1.2 LEMMA. *If $\mathcal{C}$ is a class of cotorsion groups, and $\mathcal{C}$ contains no group with a non-zero p-primary subgroup, then $E(p) \in \mathcal{E}(\mathcal{C})$.*

Proof. Let $G \in \mathcal{C}$, and let P be the group of p-adic integers. Then

$$P \cong \text{Ext}\,(Z(p^\infty)), Z)$$

and

$$\text{Ext}\,(G, P) \cong \text{Ext}\,(G, \text{Ext}\,(Z(p^\infty), Z)) \cong \text{Ext}\,(\text{Tor}\,(G, Z(p^\infty)), Z)$$
$$\cong \text{Ext}\,(G_p, Z) = 0,$$

since $G_p = 0$. Then $\text{Hom}\,(G, D) \to \text{Hom}\,(G, Z(p^\infty)) \to \text{Ext}\,(G, P) = 0$ exact implies $E(p) \in \mathcal{E}(\mathcal{C})$.

A *$\mathcal{C}$-projective resolution* of a group G is an exact sequence

$$\cdots \longrightarrow P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow$$
$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} G$$

with $\text{Ker} f_n \rightarrowtail P_n \twoheadrightarrow \text{Im} f_n \in \mathcal{E}(\mathcal{C})$ for each n, and $P_n \in \mathcal{C}$ for each n. The group G has *$\mathcal{C}$-projective dimension* $\leq n$ if G has a $\mathcal{C}$-projective resolution with $P_m = 0$ for all $m > n$, and G has *infinite $\mathcal{C}$-projective dimension* if G has a $\mathcal{C}$-projective resolution but does not have $\mathcal{C}$-projective dimension $\leq n$ for any n. The groups of $\mathcal{C}$-projective dimension zero are precisely the groups in the projective closure of $\mathcal{C}$. We find, for each class $\mathcal{C}$, some groups which have $\mathcal{C}$-projective dimension one. We give an example in Section 5 of a group of $\mathcal{C}$-projective dimension two.

The relative Ext functors determined by a projective class $\mathcal{C}$ will be denoted $\text{Pext}^n_{\mathcal{C}}\,(A, B)$. These functors lead to exact sequences as follows. If

$$A \rightarrowtail B \to C \in \mathcal{E}(\mathcal{C})$$

and G is any group, there are exact sequences

$$\begin{aligned}0 &\to \operatorname{Hom}(G, A) \to \operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C)\\ &\to \operatorname{Pext}^1_{\mathcal{C}}(G, A) \to \operatorname{Pext}^1_{\mathcal{C}}(G, B) \to \operatorname{Pext}^1_{\mathcal{C}}(G, C)\\ &\to \operatorname{Pext}^2_{\mathcal{C}}(G, A) \to \cdots \to \operatorname{Pext}^n_{\mathcal{C}}(G, C) \to \operatorname{Pext}^{n+1}_{\mathcal{C}}(G, A) \to \cdots\end{aligned}$$

and

$$\begin{aligned}0 &\to \operatorname{Hom}(C, G) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G)\\ &\to \operatorname{Pext}^1_{\mathcal{C}}(C, B) \to \operatorname{Pext}^1_{\mathcal{C}}(B, G) \to \operatorname{Pext}^1_{\mathcal{C}}(A, G)\\ &\to \operatorname{Pext}^2_{\mathcal{C}}(C, G) \to \cdots \to \operatorname{Pext}^n_{\mathcal{C}}(A, G) \to \operatorname{Pext}^{n+1}_{\mathcal{C}}(C, G) \to \cdots.\end{aligned}$$

A group A has $\mathcal{C}$-projective dimension n if and only if there exists a group B with $\operatorname{Pext}^n_{\mathcal{C}}(A, B) \neq 0$ while $\operatorname{Pext}^m_{\mathcal{C}}(A, G) = 0$ for all groups G, and all $m > n$.

2. Divisible groups in the projective closure

The projective closure $\bar{\mathcal{C}}$ of a class $\mathcal{C}$ of groups is closed under arbitrary direct sums and direct summands, and $\bar{\mathcal{C}}$ contains all free groups. If $\mathcal{C}$ is a set, $\bar{\mathcal{C}}$ is the smallest class satisfying these requirements [6]. This is not the case in general, however. In [6], it was shown that if $\mathcal{C}$ is the class of all torsion complete p-primary groups, $\bar{\mathcal{C}}$ consists of all groups of the form $F \oplus C \oplus D$ with F free, C a direct sum of torsion complete p-primary groups and D a divisible p-primary group. We will find similar characterizations for certain classes of cotorsion groups. We need the following theorem. By an ∞-group we mean a torsion free group.

2.1 THEOREM. *Let $\mathcal{C}$ be a class of abelian groups and p a prime or ∞. If for each cardinal β there exists a group G in $\mathcal{C}$ with a subgroup H such that $|H| > \beta$, $2^{|H|} < 2^{|G|}$, and G/H contains a divisible p-group of rank $|G|$, then the projective closure of $\mathcal{C}$ contains the divisible p-groups.*

This is a slightly stronger statement than made in Theorem 2 of [6], and requires only a slight modification in the proof. Namely, replace the assumption "$G/H \cong \sum C$" by "G/H contains a subgroup isomorphic to $\sum_{|G|} C$".

2.2 COROLLARY. *If a class of groups contains all p-adic torsion free algebraically compact groups, then the projective closure of that class contains all torsion free divisible groups.*

Proof. Given a cardinal β, let α be a cardinal such that $\alpha > \beta$ and $\alpha^{\aleph_0} = 2^{\alpha}$. (See [6].) Let H be a free group of rank α and let $G = \operatorname{Ext}(Z(p^\infty), H)$. Then G is a p-adic torsion free algebraically compact group. Letting $Z \subset R \subset Q$ with $R/Z \cong Z(p^\infty)$, we have short exact sequences,

$$H \rightarrowtail G \twoheadrightarrow \operatorname{Ext}(R, H) \quad \text{and}$$

$$\operatorname{Hom}(R, \textstyle\sum_\alpha Q) \rightarrowtail \operatorname{Hom}(R, \sum_\alpha Q/Z) \twoheadrightarrow \operatorname{Ext}(R, H).$$

The quotient $G/H \cong \text{Ext}(R, H)$ is divisible, since R is torsion free. We need to show that this group has a torsion free subgroup of cardinal $|G|$, and that $2^{|G|} > 2^{|H|}$. The monomorphism

$$\text{Hom}(Z(p^\infty), \textstyle\sum_\alpha Q/Z) \rightarrowtail \text{Hom}(R, \textstyle\sum_\alpha Q/Z)$$

shows that $|\text{Hom}(R, \sum_\alpha Q/Z)| = 2^\alpha$. Since $\text{Hom}(R, \sum_\alpha Q) \cong \sum_\alpha Q$ has cardinal α, we conclude that $|G| = |\text{Ext}(R, H)| = 2^\alpha$ and hence that $2^{|H|} < 2^{|G|}$. Also, the torsion subgroup of $\text{Hom}(R, \sum_\alpha Q/Z)$ is simply $\bigcup_{n=1}^\infty \text{Hom}(R, (\sum_\alpha Q/Z)[n])$, so

$$|\text{tHom}(R, \textstyle\sum_\alpha Q/Z)| = \alpha.$$

This, together with $|\text{Hom}(R, \sum_\alpha Q)| = \alpha$, implies $|\text{tExt}(R, H)| = \alpha$, and hence that $\text{Ext}(R, H)$ has a torsion free divisible subgroup of cardinal 2^α.

2.3 Corollary. *If a class of groups contains all adjusted algebraically compact p-adic groups, then the projective closure of that class contains the p-primary divisible groups and the torsion free divisible groups.*

Proof. Given a cardinal β, let α be a cardinal such that $\alpha > \beta$ and $\alpha^{\aleph_0} = 2^\alpha$. Let $B = \sum_{n=1}^\infty Z(p^n)$ and $H = \sum_\alpha B$. Let $\bar{H}$ be the torsion completion of H, and $G = \text{Ext}(Z(p^\infty), \bar{H})$, the cotorsion completion of $\bar{H}$. Then G is an adjusted algebraically compact group. Now $|\bar{H}| = |H|^{\aleph_0} = 2^\alpha$. The group $G/H \cong \bar{H}/H \oplus G/\bar{H}$ is divisible, so we have $|\bar{H}| \leq |G| \leq |H|^{\aleph_0}$, implying $|G| = 2^\alpha$. In particular, $2^{|H|} < 2^{|G|}$. Now $\bar{H}/H$ is a p-primary divisible group of cardinal $|\bar{H}| = 2^\alpha$, so we infer that the projective closure of any class of groups which contains all adjusted algebraically compact p-adic groups contains all p-primary divisible groups. We claim that the torsion free divisible group $G/\bar{H}$ also has cardinal 2^α. We can write $H = \sum_{i=1}^\infty H_i$ with each H_i a direct sum of 2^α cyclic groups of order p^i. Then $\bar{H}$ is the torsion subgroup of the direct product $\prod_{i=1}^\infty H_i = A$. Now A is an algebraically compact p-adic group, so $\text{Ext}(Z(p^\infty), A) \cong A$. Applying the functor $\text{Ext}(Z(p^\infty),\ )$ to the sequence $\bar{H} \rightarrowtail A \twoheadrightarrow A/\bar{H}$ we get

$$\bar{H} \subseteq G \subseteq A = \textstyle\prod_{i=1}^\infty H_i.$$

In fact, $G/\bar{H} = d(A/\bar{H})$, the maximum divisible subgroup of $A/\bar{H}$. Thus it will suffice to find 2^α sequences $x = \{h_i\} \in \prod_{i=1}^\infty H_i$ which are divisible mod $\bar{H}$ and not equivalent mod $\bar{H}$. Write $H_i = \sum_{\gamma \in I} \langle h_{i,\gamma} \rangle$, where I is a set of cardinal 2^α. Let

$$x_\gamma = (h_{1,\gamma}, ph_{2,\gamma}, ph_{3,\gamma}, p^2h_{4,\gamma}, p^2h_{5,\gamma}, p^3h_{6,\gamma}, \cdots).$$

Then $x_\gamma + \bar{H} \in d(A/\bar{H})$ for each $\gamma \in I$, and $x_\gamma \notin \bar{H}$. Suppose $\sum_{\gamma \in J} n_\gamma x_\gamma \in \bar{H}$ for some finite set J. Then $p^n(\sum_{\gamma \in J} n_\gamma x_\gamma) = 0$ for some positive integer n. Coordinatewise, this says that $p^n \sum_{\gamma \in J} n_\gamma p^{[i/2]} h_{i,\gamma} = 0$ for each i. Since the $h_{i,\gamma}$'s are independent, we have $p^n n_\gamma p^{[i/2]} h_{i,\gamma} = 0$ for all i, γ. Let p^m

be the largest power of p dividing any n_γ, and let $i > 2m + 2n$. Then

$$p^n n_\gamma p^{[i/2]} h_{i,\gamma} \neq 0,$$

since $p^i \nmid p^n n_\gamma p^{[i/2]}$. Thus the set of cosets $\{x_\gamma + \bar{H}\}_{\gamma \in I}$ is independent. We conclude that $|G/\bar{H}| \geq |I| = 2^\alpha$ and thus that $|G/\bar{H}| = 2^\alpha$. The corollary follows.

2.4 COROLLARY. *If a class of groups contains all adjusted cotorsion groups whose torsion subgroups are a direct sum of cyclic p-groups, then the projective closure of the class contains all torsion free divisible groups.*

Proof. Given a cardinal β, let α be a cardinal such that $\beta < \alpha$ and $\alpha^{\aleph_0} = 2^\alpha$. Let $B = \sum_{n=1}^{\infty} Z(p^n)$, $H = \sum_\alpha B$, and $G = \operatorname{Ext}(Z(p^\infty), H)$. Then G is an adjusted cotorsion group whose torsion subgroup, being isomorphic to H, is a direct sum of cyclic p-groups. Also $|G| = |H|^{\aleph_0}$, so that $2^{|H|} < 2^{|G|}$. The quotient $G/H \cong \operatorname{Ext}(Q, H)$ is torsion free and divisible. The corollary follows.

3. Cotorsion groups

The first class we consider is the class of all cotorsion groups.

3.1 THEOREM. *The projective closure of the class of all reduced cotorsion groups contains enough projectives. A group G is in the projective closure of this class if and only if $G \cong C \oplus D \oplus F$ with C a direct summand of a direct sum of reduced cotorsion groups, D divisible, and F free.*[3]

Proof. We have shown that the projective closure contains all divisible groups, so the class of all groups of the form $C \oplus D$ with C reduced cotorsion and D divisible generates the projective closure. This class is closed under homomorphic images. The theorem now follows from Theorem 2.4 of [6].

If G contains no unbounded cotorsion subgroups, e.g., if G/tG is countable, then the sequence $K \rightarrowtail F \oplus \sum_n G[n] \twoheadrightarrow G$ gotten by adding an epimorphism $F \to G$ from a free group F and the inclusion maps

$$G[n] = \{x \in G \mid nx = 0\} \to G,$$

is a $\mathcal{C}$-projective resolution if $\mathcal{C}$ is the class of all reduced cotorsion groups. The kernel K, being a subgroup of a direct sum of cyclic groups, is a direct sum of cyclic groups. Thus if G contains no unbounded cotorsion subgroups, G has $\mathcal{C}$-projective dimension zero or one.

Since $\mathcal{C}$ contains all of the cyclic groups and all of the divisible groups, if $A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}(\mathcal{C})$, it is a pure exact sequence and the sequence $dA \rightarrowtail dB \twoheadrightarrow dC$ of divisible subgroups is splitting exact [6]. If C contains no unbounded

[3] It has been suggested that these groups are reminiscent of the cohomology groups of Nunke and Rotman, J. London Math. Soc., vol. 37 (1962), pp. 301–306, which are of the form $C \oplus D \oplus F$ where C is cotorsion, D is divisible and F is a subgroup of $\prod Z$.

cotorsion subgroups, these two conditions are sufficient to imply $A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}(\mathcal{C})$.

Restricting consideration to the class $\mathcal{C}_p$ of reduced p-adic cotorsion groups we obtain similar results.

3.2 THEOREM. *The projective closure of the class of all reduced p-adic cotorsion groups contains enough projectives. A group G is in the projective closure of this class if and only if* $G \cong C \oplus D \oplus F$ *with C a direct summand of a direct sum of reduced p-adic cotorsion groups, D divisible with tD a p-group, and F free.*

Proof. If R is reduced, C a reduced p-adic cotorsion group, and $f : C \to R$ a homomorphism, then $\operatorname{Im} f$ is a reduced p-adic cotorsion group. Thus if R is any reduced group, let $\mathcal{S} = \{S \subseteq R \mid S \in \mathcal{C}_p\}$, and let $F \to R$ be an epimorphism with F free. Summing this map with the inclusion maps $S \to R$ gives the short exact sequence

$$K \rightarrowtail F \oplus \textstyle\sum_{S \in \mathcal{S}} S \twoheadrightarrow R \in \mathcal{E}(\mathcal{C}).$$

If $q \neq p$, we have the sequence $E(q)$ of Lemma 1.1 in $\mathcal{E}(\mathcal{C})$, and it is of the desired form for $Z(q^{\infty})$. By Corollary 2.3, the projective closure of this class contains all torsion free divisible groups and all divisible p-groups. The theorem now follows from Theorem 2.4 of [6].

If G contains no unbounded p-adic cotorsion subgroups, there is a $\mathcal{C}_p$-projective resolution

$$K \rightarrowtail F \oplus \textstyle\sum_n G[p^n] \twoheadrightarrow G$$

of length one. If $A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}(\mathcal{C}_p)$, it is p-pure, the sequences

$$dA_p \rightarrowtail dB_p \twoheadrightarrow dC_p \quad \text{and} \quad A \cap dB \rightarrowtail dB \twoheadrightarrow dC$$

are exact, and $A \cap dB$ is cotorsion. If C contains no unbounded p-adic cotorsion subgroup, these conditions are sufficient to imply

$$A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}(\mathcal{C}_p).$$

4. Torsion free cotorsion groups

The next theorem describes the projective closure of the class of reduced torsion free p-adic cotorsion groups. We need the following lemma.

4.1 LEMMA. *Let G be a reduced group, C a reduced torsion free cotorsion p-adic group, and* $f : C \to G$ *a homomorphism. Then C has a direct summand* C_0 *with* $|C_0| \leq |G|^{\aleph_0}$ *such that f factors through* C_0.

Proof. Let B be a p-basic subgroup of C, i.e., B is a p-pure subgroup of C, B is a direct sum of cyclic groups and C/B is p-divisible. Then C/B is divisible, since C is q-divisible for all primes $q \neq p$. Write $B = \sum_{\alpha \in I} \langle b_\alpha \rangle$. The relation $\alpha \sim \beta$ if $f(b_\alpha) = f(b_\beta)$, is an equivalence relation on I. Let J be a set of representatives, one from each equivalence class, and $\gamma : I \to J$ the function such that $f(b_\alpha) = f(b_{\gamma(\alpha)})$. Let $B_0 = \sum_{\alpha \in J} \langle b_\alpha \rangle$ and define $g : B \to B_0$ by $g(b_\alpha) = b_{\gamma(\alpha)}$. Then $fg(b_\alpha) = f(b_{\gamma(\alpha)}) = f(b_\alpha)$ for all $\alpha \in I$, and we have a commutative diagram

$$\begin{tikzcd} & B \arrow[dl, "g"'] \arrow[d, "f"] \\ B_0 \arrow[r, "f"] & G. \end{tikzcd}$$

Now C/B has no non-zero p-component since B is p-pure in C and C is torsion free. It follows that $\mathrm{Ext}(Z(p^\infty), B) \cong \mathrm{Ext}(Z(p^\infty), C) \cong C$. The map $g : B \to B_0$ gives a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"] & \mathrm{Ext}\,(Z(p^\infty), B) \cong C \arrow[d, "\bar{g}"] \\ B_0 \arrow[r] & \mathrm{Ext}\,(Z(p^\infty), B_0) = C_0. \end{tikzcd}$$

Since g is a splitting map, C_0 is a summand of C. Now f and $f\bar{g}$ agree on B. Thus $\mathrm{Im}(f - f\bar{g})$ is divisible. Then, G being reduced, we have $f = f\bar{g}$. This proves the lemma.

4.2 THEOREM. *The projective closure of the class of all reduced torsion free p-adic cotorsion groups contains enough projectives. A group G is in the projective closure of this class if and only if $G \cong C \oplus D \oplus F$ with C a direct sum of reduced torsion free cotorsion p-adic groups, D a torsion free divisible group, and F free.*

Proof. Let $\mathcal{C}$ be the class of reduced torsion free cotorsion p-adic groups. It follows from Corollary 2.2 that groups $G \cong C \oplus D \oplus F$ with C a direct sum of reduced torsion free cotorsion p-adic groups, D torsion free divisible, and F free, are in the projective closure of $\mathcal{C}$. We will show that for every group A there is a short exact sequence $K \rightarrowtail G \twoheadrightarrow A \in \mathcal{E}(\mathcal{C})$ with G as above, i.e., there are enough projectives of this form. It follows from Lemmas 1.1 and 1.2 that there are enough projectives for all divisible groups. Let R be any reduced group, and $F \to R$ an epimorphism with F free. Let

$$\mathcal{S} = \{C \mid C \in \mathcal{C} \text{ and } |C| \leq |R|^{\aleph_0}\}.$$

Lemma 4.1 implies that the sequence $K \rightarrowtail F \oplus \sum_{C \in \mathcal{S}} (\sum_{\mathrm{Hom}(C,R)} C) \twoheadrightarrow R \in \mathcal{E}(\mathcal{C})$. From this it follows that $\bar{\mathcal{C}}$ consists of all groups $G = C \oplus D \oplus F$ with C a direct summand of a direct sum of reduced torsion free cotorsion p-adic groups, D divisible, and F free. A theorem of R. B. Warfield, Jr. [9] says that a direct summand of a direct sum of algebraically compact groups is again a direct sum of algebraically compact groups. It is clear that each summand in this sum must again be torsion free and p-adic. The theorem follows.

4.3 *Theorem. The projective closure of the class of all reduced torsion free* *cotorsion groups contains enough projectives. A group G is in the projective closure of this class if and only if $G \cong C \oplus D \oplus F$ with C a direct sum of reduced torsion free cotorsion groups, D a torsion free divisible group and F free.*

Proof. We know all groups of the form described are in the projective closure. We need to show there are enough projectives of this form. Let C be the class of all torsion free cotorsion groups. For each prime p, the sequence $P(p) \rightarrowtail D \twoheadrightarrow Z(p^\infty)$ of Lemma 1.1 belongs to $E(C)$, and it follows easily that there are enough projectives for all divisible groups. Let R be any reduced group, C torsion free cotorsion, and $F : C \to R$ a homomorphism. Now $C = \prod_p C^{(p)}$ with $C^{(p)}$ a torsion free cotorsion p-adic group. By Lemma 4.1, for each prime p, there is a direct summand $C_0^{(p)}$ of $C^{(p)}$ with $|C_0^{(p)}| \leq |R|^{\aleph_0}$, and a commutative diagram

$$\begin{tikzcd} & C^{(p)} \arrow[dl, "g_p"'] \arrow[d, "f"] \\ C_0^{(p)} \arrow[r, "f"] & R. \end{tikzcd}$$

Let $C_0 = \prod_p C_0^{(p)}$. Then C_0 is a direct summand of C and $|C_0| \leq (|R|^{\aleph_0})^{\aleph_0} = |R|^{\aleph_0}$. Let $g : C \to C_0$ be the direct product of the maps g_p. Then $\operatorname{Ker}(fg - f) \supseteq \sum_p C^{(p)}$. But the quotient $\prod_p C^{(p)} / \sum_p C^{(p)}$ is divisible. Since $fg - f$ factors through this quotient, and R is reduced, we must have $fg = f$. Now we see that if $F \to R$ is an epimorphism with F free, and $\mathcal{S} = \{C \mid C \in \mathcal{C} \text{ and } |C| \leq |R|^{\aleph_0}\}$, the sequence

$$K \rightarrowtail \textstyle\sum_{C \in \mathcal{S}} (\sum_{\operatorname{Hom}(C,R)} C) \oplus F \twoheadrightarrow R$$

belongs to $\mathcal{E}(\mathcal{C})$. Applying Warfield's theorem [9] again, which implies a direct summand of a direct sum of torsion free cotorsion groups is again a direct sum of torsion free cotorsion groups, the theorem is proved.

One can observe from Lemmas 1.1 and 1.2 that if $\mathcal{C}$ is the class of reduced torsion free cotorsion p-adic groups, or the class of all reduced torsion free cotorsion groups, then torsion divisible groups have projective dimension one.

The following easy lemma enables us to describe some of the other groups of $\mathcal{C}$-projective dimension one.

4.4 Lemma. *If $\mathcal{C}$ is any class of torsion free groups and $A \rightarrowtail B \twoheadrightarrow C$ is exact with A cotorsion, then $A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}(\mathcal{C})$.*

Proof. $\operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C) \to \operatorname{Ext}(G, A) = 0$ is exact for all torsion free G, in particular for all $G \in \mathcal{C}$.

Let $\mathcal{C}$ be the class of all torsion free reduced cotorsion groups. If G is any reduced cotorsion group, let $K \rightarrowtail F \twoheadrightarrow G$ be exact with F free. Then there is a short exact sequence

$$\operatorname{Ext}(Q/Z, K) \rightarrowtail \operatorname{Ext}(Q/Z, F) \twoheadrightarrow \operatorname{Ext}(Q/Z, G) \approx G.$$

Both $\operatorname{Ext}(Q / Z, K)$ and $\operatorname{Ext}(Q / Z, F)$ are torsion free cotorsion. Applying 4.4, this sequence is a $\mathcal{C}$-projective resolution of G. Thus all cotorsion groups have $\mathcal{C}$-projective dimension less than or equal to one.

If G is a group which contains no nonzero cotorsion subgroup, then a free resolution of G is a $\mathcal{C}$-projective resolution of G, so G has dimension less than or equal to one.

Combining these two observations gets the groups G whose set of cotorsion subgroups generates a cotorsion subgroup to be of dimension less than or equal to one. Simply take a $\mathcal{C}$-projective resolution of length one of the maximum cotorsion subgroup of G and add it to a free resolution of length one of G. The result is a C-projective resolution of length one of G. The above observations are all contained in the following statement.

4.5 PROPOSITION. *If $\mathcal{C}$ is the class of all reduced torsion free cotorsion groups, and if the set of all cotorsion subgroups of G generates a cotorsion subgroup, then G has $\mathcal{C}$-projective dimension less than or equal to one.*

We are able to show an important class of torsion groups, namely totally projective groups, have $\mathcal{C}$-projective dimension one. First we prove this property for cyclic p-groups.

4.6 LEMMA. *The cyclic groups have $\mathcal{C}$-projective dimension one.*

Proof. Let $P(p)$ be the group of p-adic integers. Then

$$P(p)/p^nP(p) \simeq Z(p^n)$$

gives a $\mathcal{C}$-projective resolution

$$p^nP(p) \rightarrowtail P(p) \twoheadrightarrow Z(p^n)$$

of $Z(p^n)$, since $p^nP(p) \approx P(p)$ is torsion free cotorsion.

The class $\mathcal{P}$ of reduced totally projective groups is the smallest class of groups having the following properties.

(1) $Z(p^n) \in \mathcal{P}$ for all primes p and positive integers n.
(2) $\mathcal{P}$ is closed under arbitrary direct sums and direct summands.
(3) If G is a group and α an ordinal, then $G \in \mathcal{P}$ if and only if both $p^\alpha G \in \mathcal{P}$ and $G/p^\alpha G \in \mathcal{P}$.

This class of groups also has the following property.

(4) If $G \in \mathcal{P}$ is of limit length α, then $G = \sum_{\beta<\alpha} G_\beta$ where $G_\beta \in \mathcal{P}$ and G_β has length $\leq \beta < \alpha$.

Totally projective groups were introduced by R. J. Nunke [5]. We use only the above properties to prove these groups have $\mathcal{C}$-projective dimension one.

4.7 PROPOSITION. *If $\mathcal{C}$ is the class of all torsion free cotorsion groups and G is a nonzero totally projective group, then G has $\mathcal{C}$-projective dimension one.*

Proof. We may as well assume G is a p-group since $G = \sum_p G_p$. We induct on the length of G. If G has length one, G is a direct sum of cyclic groups of order p, whence G has $\mathcal{C}$-projective dimension one by 4.6 and the fact that the class of groups of $\mathcal{C}$-projective dimension less than or equal to one is closed under direct sums. Now suppose the length γ of G is greater than one. If γ is not a limit ordinal, $\gamma = \alpha + 1$ for some ordinal α. Now $p^\alpha G$ is a direct sum of cyclic groups of order p, and hence cotorsion. Thus the exact sequence

$$p^\alpha G \rightarrowtail G \twoheadrightarrow G/p^\alpha G \in \mathcal{E}(\mathcal{C}).$$

By our induction hypothesis, $G/p^\alpha G$ has $\mathcal{C}$-projective dimension one. We get the exact sequence

$$0 = \operatorname{Pext}^2(G/p^\alpha G, X) \to \operatorname{Pext}^2(G, X) \to \operatorname{Pext}^2(p^\alpha G, X) = 0$$

for all groups X. This gets the $\mathcal{C}$-projective dimension of G less than or equal to one, and hence equal to one.

If G has limit length γ, write $G = \sum_{\beta<\gamma} G_\beta$ with G_β of length $< \gamma$. Then by our induction hypotheses, each G_β has $\mathcal{C}$-projective dimension less than or equal to one and thus G has $\mathcal{C}$-projective dimension one.

For torsion groups in general, we can show the $\mathcal{C}$-projective dimension does not exceed two. Let G be a reduced torsion group. Then the exact sequence

$$K \rightarrowtail \textstyle\sum_{n=1}^\infty G[n] \rightarrowtail G$$

is in $\mathcal{E}(\mathcal{C})$ since any homomorphism from a cotorsion group to a reduced torsion group G must have a bounded image. Thus the homomorphism can be factored through the inclusion $G[n] \to G$ for some positive integer n. Now both $\sum_{n=1}^\infty G[n]$ and K are direct sums of cyclic groups so they have $\mathcal{C}$-projective dimension one. Thus we have the following exact sequence for $m \geq 3$ and for all X,

$$0 = \operatorname{Pext}_{\mathcal{C}}^{m-1}(K, X) \to \operatorname{Pext}_{\mathcal{C}}^m(G, X) \to \operatorname{Pext}_{\mathcal{C}}^m(\textstyle\sum_{n=1}^\infty G[n], X) = 0,$$

implying $\operatorname{Pext}_{\mathcal{C}}^m(G, X) = 0$. Thus the $\mathcal{C}$-projective dimension of G is less than or equal to two.

In conclusion we observe that if there is a torsion group G of $\mathcal{C}$-projective dimension two, there is also such a torsion free group. We may assume G is reduced. Let $K \rightarrowtail F \twoheadrightarrow G$ be a free resolution of G. Then

$$\operatorname{Ext}(Q/Z, K) \rightarrowtail \operatorname{Ext}(Q/Z, F) \twoheadrightarrow \operatorname{Ext}(Q/Z, G)$$

is a short exact sequence. Since G is reduced, $G \subseteq \operatorname{Ext}(Q/Z, G)$. Let H be the inverse image of G in $\operatorname{Ext}(Q/Z, F)$. Then H is torsion free. Since $\operatorname{Ext}(Q/Z, K)$ is cotorsion, the sequence

$$\operatorname{Ext}(Q/Z, K) \rightarrowtail H \twoheadrightarrow G$$

belongs to $\mathcal{E}(\mathcal{C})$. Since $\operatorname{Ext}(Q/Z, K)$ is torsion free cotorsion, it has $\mathcal{C}$-projective dimension zero, i.e., $\mathrm{Pext}^n_{\mathfrak{C}}(\mathrm{Ext}(Q/Z, K), X) = 0$ for all $n \geq 1$ and for all X. It follows that $\mathrm{Pext}^n_{\mathfrak{C}}(H, X) \cong \mathrm{Pext}^n_{\mathfrak{C}}(G, X)$ for all $n \geq 1$ and for all X. In particular if G has $\mathfrak{C}$-projective dimension two, so does H.

The corresponding statements hold for the class of torsion free reduced p-adic cotorsion groups. Replace the word cotorsion by p-adic cotorsion throughout, and replace torsion group by p-group.

5. Adjusted cotorsion groups

The classes of adjusted cotorsion groups we look at will all be subclasses of the class $\mathfrak{C}$ of all adjusted cotorsion groups whose torsion subgroups are direct sums of torsion complete groups. We prove a lemma similar to Lemma 4.1 for this class of groups.

5.1 Lemma. *Let C be an adjusted cotorsion group whose torsion subgroup is a direct sum of torsion complete groups, and let R be a reduced group. If $f : C \to R$ is a homomorphism, f factors through a direct summand C_0 of C with $|C_0| \leq 2^\gamma$ where $\gamma = |R|^{\aleph_0}$. If tC can be written as a direct sum of no more than $|R|^{\aleph_0}$ torsion complete groups plus a direct sum of cyclic groups, there is such a C_0 with $|C_0| \leq |R|^{\aleph_0}$.*

Proof. Let the torsion subgroup of C be $T = \sum_{\alpha \in A} T_\alpha$ with each T_α either a torsion complete p-group or a direct sum of cyclic p-groups. Let f' denote the restriction of f to T. Suppose we can factor f' through a summand of T, say $T = S \oplus P$ and

$$T \xrightarrow{f'} R = T \xrightarrow{g} S \xrightarrow{f'} R.$$

Then the group $C \cong \mathrm{Ext}(Q/Z, T)$ decomposes as $\mathrm{Ext}(Q/Z, S) \oplus \mathrm{Ext}(Q/Z, P)$, and g induces a map $\bar{g} : C \to \mathrm{Ext}(Q/Z, S)$ satisfying $f\bar{g} = f$. Moreover,

$$|\mathrm{Ext}(Q/Z, S)| \leq |S|^{\aleph_0}$$

by [8], since $\mathrm{Ext}(Q/Z, S)/S$ is divisible. Thus it will suffice to prove we can factor f' through a summand S of T with $|S| \leq |R|^{\aleph_0}$ or $|S| \leq 2^\gamma$ where $\gamma = |R|^{\aleph_0}$. Consider f restricted to T_α. It is proved in [6] that T_α has a direct summand S_α with $|S_\alpha| \leq |R|^{\aleph_0}$ such that f factors through S_α. Thus f restricted to T factors through the direct summand $S = \sum_{\alpha \in A} S_\alpha$ of T. If $|A| \leq |R|^{\aleph_0}$, then $|S| \leq |R|^{\aleph_0}$ and

$$|\mathrm{Ext}(Q/Z, S)| \leq |S|^{\aleph_0} \leq |R|^{\aleph_0}.$$

The only problem remaining is when the number of unbounded torsion complete summands of T exceeds $\gamma = |R|^{\aleph_0}$. The relation $\alpha \sim \beta$ if there exists an isomorphism

$$g_{\alpha\beta} : S_\alpha \to S_\beta \quad \text{with } fg_{\alpha\beta} = f$$

is an equivalence relation on A. Now

$$|\mathrm{Hom}(S_\alpha, R)| \leq |R|^{|S_\alpha|} \leq |R|^\gamma = 2^\gamma,$$

and the number of nonisomorphic groups of cardinal $\leq\gamma$ is $\leq \gamma^\gamma = 2^\gamma$. We conclude there are no more than 2^γ equivalence classes. Let $J \subset A$ be a set of representatives. Then there is a commutative diagram

$$\begin{tikzcd} S = \sum_{\alpha\in A} S_\alpha \arrow[d] & T = \sum_{\alpha\in A} T_\alpha \arrow[l] \arrow[d, "f"] \\ S_0 = \sum_{\alpha\in J} S_\alpha \arrow[r, "f"] & R \end{tikzcd}$$

Now S_0 is a direct summand of T, and $|S_0| \leq 2^\gamma$ where $\gamma = |R|^{\aleph_0}$. The lemma follows.

We now look at three different classes to which the above lemma will apply, first the class of adjusted algebraically compact groups, then the class of adjusted cotorsion groups whose torsion subgroups are direct sums of torsion complete groups, and finally the class of adjusted cotorsion groups whose torsion subgroups are direct sums of cyclic groups. An adjusted cotorsion group is algebraically compact if and only if its torsion subgroup is torsion complete. Corollary 2.3 and Lemmas 5.1 and 1.1 apply immediately to get the next four theorems. We also apply the theorem of R. B. Warfield, Jr. [9] which implies a summand of a direct sum of adjusted algebraically compact groups is again a direct sum of adjusted algebraically compact groups.

5.2 THEOREM. *The projective closure of the class of adjusted algebraically compact p-adic groups contains enough projectives. A group G is in the projective closure of this class if and only if $G = C \oplus D \oplus F$ with C a direct sum of adjusted algebraically compact groups, D divisible, F free, and tG is a p-primary group.*

5.3 THEOREM. *The projective closure of the class of adjusted algebraically compact groups contains enough projectives. A group G is in the projective closure of this class if and only if $G = C \oplus D \oplus F$ with C a direct sum of adjusted algebraically compact groups, D divisible, and F free.*

5.4 THEOREM. *The projective closure of the class $\mathcal{C}$ of all adjusted cotorsion p-adic groups whose torsion subgroups are direct sums of torsion complete groups contains enough projectives. A group belongs to the projective closure of this class if and only if it is of the form $C \oplus D \oplus F$* with *C a summand of a direct sum of groups in $\mathcal{C}$, D is divisible with tD a p-group, and F free.*

5.5 THEOREM. *The projective closure of the class $\mathcal{C}$ of all adjusted cotorsion groups whose torsion subgroups are direct sums of torsion complete groups contains enough projectives. A group belongs to the projective closure of $\mathcal{C}$ if and only if it is of the form $C \oplus D \oplus F$ with C a summand of a direct sum of groups in $\mathcal{C}$, D divisible and F free.*

5.6 THEOREM. *The projective closure of the class $\mathcal{C}$ of all adjusted cotorsion* *groups whose torsion subgroups are direct sums of cyclic p-groups contains enough projectives. A group G belongs to $\bar{\mathcal{C}}$ if and only if $G = C \oplus D \oplus F$ with C a summand of a direct sum of groups in $\mathcal{C}$, D torsion free divisible and F free.*

Proof. We need to show that all torsion divisible groups are the proper image of a projective. If $q \neq p$, the sequence

$$E(q) : P(q) \rightarrowtail D \twoheadrightarrow Z(q^\infty)$$

of Lemma 1.1 is proper, and D is torsion free divisible. We will show there is a proper short exact sequence $K \rightarrowtail C \oplus D \twoheadrightarrow Z(p^\infty)$ with $C \in \mathcal{C}$ and D torsion free divisible. Let $B = \sum_{n=1}^\infty B_n$ with $B_n = \langle b_n \rangle \cong Z(p^n)$ for each n. There is a homomorphism

$$f : B \rightarrow Z(p^\infty)$$

such that $f : B_n \rightarrow Z(p^\infty)$ is a monomorphism for each n. Let $\hat{f} : \hat{B} \rightarrow Z(p^\infty)$ be an extension of f to the cotorsion completion $\hat{B} = \mathrm{Ext}(Q/Z, B)$ of B. Let

$$P(p) \rightarrowtail D \xrightarrow{g} Z(p^\infty)$$

be the sequence of Lemma 1.1, and let $h = g \oplus \hat{f} : D \oplus \hat{B} \rightarrow Z(p^\infty)$. We claim the sequence

$$K \rightarrowtail D \oplus \hat{B} \xrightarrow{h} Z(p^\infty)$$

is proper. First we show it is a p-pure exact sequence. Let $C = \langle c \rangle$ be a cyclic p-group, and $k : C \rightarrow Z(p^\infty)$ a homomorphism. Let $p^n = o(h(c))$. Then $k(c) = mf(b_n) = f(mb_n)$ for some m relatively prime to p, and $o(mb_n) = p^n \leq o(c)$. There is a homomorphism $j : C \rightarrow D \oplus \hat{B}$ defined by $j(c) = mb_n$. Clearly $hj = k$. It follows that the sequence

$$K \rightarrowtail D \oplus \hat{B} \xrightarrow{g} Z(p^\infty)$$

is p-pure. Now let G be any adjusted cotorsion group such that tG is a direct sum of cyclic p-groups, and let $k : G \rightarrow Z(p^\infty)$ be a homomorphism. There is a homomorphism

$$j : tG \rightarrow D \oplus \hat{B}$$

such that $hj = k$, since tG is a direct sum of cyclic p-groups. There is an extension $\hat{j} : G \rightarrow D \oplus \hat{B}$ of this map. Now $s = k - h\hat{j}$ contains tG in its kernel. Let $\pi : G \rightarrow G/tG$ be the natural map. Then $s = r\pi$ for some $r : G/tG \rightarrow Z(p^\infty)$. Now $P(p) \rightarrowtail D \twoheadrightarrow Z(p^\infty)$ gives the exact sequence

$$\mathrm{Hom}(G/tG, D) \rightarrow \mathrm{Hom}(G/tG, Z(p^\infty)) \rightarrow \mathrm{Ext}(G/tG, P(p)) = 0.$$

Thus there is a map $t : G/tG \rightarrow D$ such that $gt = r$. Thus $s = gt\pi = k - h\hat{j}$, and $k = gt\pi + h\hat{j}$. Let $i : D \rightarrow D \oplus \hat{B}$ be the inclusion. Then $gt = hit$.

Thus we have $k = h(it\pi + \hat{\jmath})$, or the commutative diagram

$$\begin{tikzcd} & G \arrow[dl, "it\pi + \hat{\jmath}"'] \arrow[d, "k"] \\ D \oplus B \arrow[r, "h"] & Z(p^\infty). \end{tikzcd}$$

It follows that the sequence

$$K \rightarrowtail D \oplus \hat{B} \xrightarrow{h} \twoheadrightarrow Z(p^\infty)$$

is a proper exact sequence in the relative homological algebra induced by $\mathcal{C}$. The theorem follows.

5.7 THEOREM. *The projective closure of the class C of all adjusted cotorsion groups whose torsion subgroups are direct sums of cyclic groups contains enough projectives. A group G belongs to $\bar{\mathcal{C}}$ if and only if $G = C \oplus D \oplus F$ with C a summand of a direct sum of groups in $\bar{\mathcal{C}}$, D torsion free divisible, and F free.*

Proof. Let $\mathcal{C}_p$ be the class of all p-adic groups in $\mathcal{C}$. There is a short exact sequence $K \rightarrowtail G \twoheadrightarrow Z(p^\infty) \in \mathcal{E}(\mathcal{C}_p)$ by 5.6, with $G = D \oplus A$, D torsion free divisible, and $A \in \mathcal{C}_p$. Now if $C \in \mathcal{C}$,

$$C = C^{(p)} \oplus \prod\nolimits_{q \neq p} C^{(q)}.$$

Write $B = \prod_{q \neq p} C^{(q)}$. If $f : C^{(p)} \to Z(p^\infty)$, f factors through $G \to Z(p^\infty)$ by 5.6. If $f : B \to Z(p^\infty)$, f factors through $B \to B/tB$ since $f(tB) = 0$, say

$$\left(C^{(p)} \xrightarrow{f} Z(p^\infty)\right) = \left(B \xrightarrow{\pi} B/tB \xrightarrow{g} Z(p^\infty)\right).$$

But B/tB is torsion free divisible. Thus $B/tB \in \bar{\mathcal{C}}_p$ implies

$$B/tB \xrightarrow{g} Z(p^\infty)$$

factors through $G \to Z(p^\infty)$. This gives a commutative diagram

$$\begin{tikzcd} & B/tB \arrow[d, "h"'] & B \arrow[l, "\pi"'] \arrow[d, "f"] \\ K \arrow[r] & G \arrow[r, two heads, "g"] & Z(p^\infty) \end{tikzcd}$$

and it follows that $K \rightarrowtail G \twoheadrightarrow Z(p^\infty) \in \mathcal{E}(\mathcal{C})$. The theorem now follows from 2.4 and 5.1.

We now turn to the question of dimension. Suppose $\mathcal{C}$ is any class of p-adic adjusted cotorsion groups, and let $q \neq p$. By 1.1 there is a short exact sequence

$$E : P(q) \rightarrowtail D \twoheadrightarrow Z(q^\infty)$$

with $P(q)$ the group of q-adic integers and D torsion free divisible. Now $\operatorname{Ext}(C, P(q)) = 0$ for C any p-adic group (both $\operatorname{Ext}(C/tC, P(q)) = 0$ and $\operatorname{Ext}(tC, P(q)) = 0$), so the exact sequence

$$\operatorname{Hom}(C, D) \to \operatorname{Hom}(C, Z(q^\infty)) \to \operatorname{Ext}(C, P(q)) = 0$$

implies $E \in \mathcal{E}(\mathcal{C})$. Any free resolution $K \rightarrowtail F \twoheadrightarrow P(q)$ belongs to $\mathcal{E}(\mathcal{C})$ also since $\operatorname{Hom}(C, P(q)) = 0$ for all $C \in \mathcal{C}$. Composing these two sequences gives a $\mathcal{C}$-projective resolution

$$K \rightarrowtail F \to D \twoheadrightarrow Z(q^\infty)$$

if $D \in \bar{\mathcal{C}}$. Since $\bar{\mathcal{C}}$ contains torsion free divisible groups for every class $\mathcal{C}$ of this section, this shows $Z(q^\infty)$ has $\mathcal{C}$-projective dimension two or less if $\mathcal{C}$ is the class of all p-adic adjusted cotorsion groups whose torsion subgroups are torsion complete, direct sums of torsion complete, or direct sums of cyclic groups. Suppose that for any of these projective classes

$$C_1 \oplus D_1 \oplus F_1 \rightarrowtail C_0 \oplus D_0 \oplus F_0 \twoheadrightarrow Z(q^\infty)$$

is a $\mathcal{C}$-projective resolution of $Z(q^\infty)$, with C_i a summand of a direct sum of adjusted cotorsion groups, tC_i a p-group, D_i torsion free divisible and F_i free. Then $tC_0 \subseteq C_1$ and

$$(C_1/tC_0) \oplus D_1 \oplus F_1 \rightarrowtail (C_0/tC_0) \oplus D_0 \oplus F_0 \twoheadrightarrow Z(q^\infty)$$

is also a proper projective resolution of $Z(q^\infty)$, since C_0/tC_0 is torsion free divisible. Thus we may assume $C_0 = 0$. It follows that $tC_1 = 0$ and hence $C_1 = 0$. Now we have a proper projective resolution of $Z(q^\infty)$ of the form

$$D_1 \oplus F_1 \rightarrowtail D_0 \oplus F_0 \twoheadrightarrow Z(q^\infty)$$

with D_i torsion free divisible and F_i free. Now

$$(D_1 \oplus F_1) \cap D_0 = D_1 \oplus (F_1 \cap D_0)$$

is cotorsion since $Q \in \bar{\mathcal{C}}$ (see [6]). Thus $F_1 \cap D_0 = 0$. Then

$$D_1 \rightarrowtail D_0 \twoheadrightarrow Z(q^\infty)$$

is exact and hence $D_0 \approx D_1 \oplus Z(q^\infty)$. But D_0 was torsion free. It follows that $Z(q^\infty)$ has dimension exactly two in any of these relative homological algebras.

This accounts for all divisible groups except in the last two cases, where $\mathcal{C}$ is the class of all adjusted cotorsion groups whose torsion subgroups are direct sums of cyclic p-groups or direct sums of cyclic groups. The $\mathcal{C}$-projective dimension of $Z(p^\infty)$ for either class $\mathcal{C}$ can be found by examining the proper projective resolution constructed in the proof of Theorem 5.6. For $Z(p^\infty)$, the exact sequence

$$(*) \qquad K \rightarrowtail D \oplus B \twoheadrightarrow Z(p^\infty)$$

gives the exact sequence

$$\mathrm{Hom}\,(Q, D \oplus \hat{B}) \longrightarrow \mathrm{Hom}\,(Q, Z(p^\infty))$$

$$\xrightarrow{\phi} \mathrm{Ext}\,(Q, K) \longrightarrow \mathrm{Ext}\,(Q, D \oplus \hat{B})$$

Since the sequence $(^*)$ is proper, $\phi = 0$. On the other hand, D divisible and $\hat{B}$ cotorsion imply $\mathrm{Ext}\,(Q, D \oplus \hat{B}) = 0$. Thus $\mathrm{Ext}\,(Q, K) = 0$. We conclude that K is cotorsion. Now $dK \subseteq d(D \oplus \hat{B}) = D$ and $K \cap D = P(p)$ which is reduced, so K is reduced. Consider the exact sequence

$$J \rightarrowtail B \xrightarrow{f} Z(p^\infty).$$

Now $B = t(D \oplus \hat{B})$, and $(D \oplus \hat{B})/J \approx D \oplus (\hat{B}/J)$ is divisible since both $\hat{B}/B$ and B/J are divisible. Since K/J is a pure subgroup of $(D \oplus \hat{B})/J$, K/J and hence K/tK are divisible. Now $tK \subseteq B$ implies tK is a direct sum of cyclic p-groups. It follows that K is an adjusted cotorsion group with torsion subgroup a direct sum of cyclic p-groups, i.e., $K \in \mathcal{C}$. Thus $Z(p^\infty)$ has $\mathcal{C}$-projective dimension one for both of these classes. Now let $\mathcal{C}$ be any class of adjusted p-adic cotorsion groups. Every reduced group G with no nonzero p-component has $\mathcal{C}$-projective dimension one or less. For if $f : C \to G$ is a homomorphism with $G[p] = 0$, G reduced, C adjusted cotorsion and tC a p-group, then $\mathrm{Ker}\, f \supseteq tC$ implies $\mathrm{Im}\, f$ is divisible and hence 0. It follows that a free resolution of such a group is a $\mathcal{C}$-projective resolution.

If $\mathcal{C}$ is any of the classes of adjusted cotorsion groups considered, all reduced torsion free groups have $\mathcal{C}$-projective dimension one or less. More generally, all groups G with G/tG reduced have $\mathcal{C}$-projective dimension one or less. For such groups, there is a $\mathcal{C}$-projective resolution of the form

$$K \rightarrowtail F \oplus \textstyle\sum_{n=1}^{\infty} G[n] \twoheadrightarrow G \quad \text{or} \quad K \rightarrowtail F \oplus \textstyle\sum_{n=1}^{\infty} G[p^n] \twoheadrightarrow G$$

with F free. In particular, all unbounded reduced torsion groups have dimension one or less.

Bibliography

1. L. Fuchs, *Notes on abelian groups II*, Acta. Math. Acad. Sci. Hung., vol. 11 (1960), pp. 117–125.
2. D. K. Harrison, *Infinite abelian groups ond homological methods*, Ann. of Math., vol. 69 (1959), pp. 366–391.
3. P. Hill, *The isomorphic refinement theorem for direct sums of closed groups*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 913–919.
4. J. Irwin, F. Richman, and E. A. Walker, *Countable direct sums of torsion complete groups*, Proc. Amer. Math. Soc., vol. 17 (1966), pp. 763–766.
5. R. J. Nunke, *Modules of extensions over Dedekind rings*, Illinois J. Math., vol. 3 (1959), pp. 222–241.
6. F. Richman, C. Walker, and E. A. Walker, "Projective classes of abelian groups" in *Studies on abelian groups*, Springer, Berlin, 1968, pp. 335–343.
7. C. Walker, *Relative homological algebra and abelian groups*, Illinois J. Math., vol. 10 (1966), pp. 186–209.

8. E. A. WALKER, *Quotient groups of reduced groups*, Proc. Amer. Math. Soc., vol. 12 (1961), pp. 91–92.
9. R. B. WARFIELD, JR., *An isomorphic refinement theorem for abelian groups*, Pacific J. Math., vol. 34 (1970), pp. 237–255.

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