SOME FINITE DIFFEOMORPHISM GROUPS

BY
EDWARD C. TURNER

The group of isotopy classes of diffeomorphisms of a smooth compact manifold M (denoted by $\mathfrak{D}(M)$) is generally large and very hard to compute, even for simple manifolds. In this note we describe a class of manifolds M (compact, simply connected and without boundary) for which $\mathfrak{D}(M)$ is finite: the proof, however, gives no effective way to determine its order. The conditions defining the class depend only on the homotopy type of M minus a point, $M - m_0$, and included are manifolds whose homology requires arbitrarily many generators.

We say that M is *spherical* if $M = m_0$ is homotopy equivalent to a wedge of spheres $S^{k_1+1} \vee \cdots \vee S^{k_l+1}$. Poincaré duality implies that l is even and that if S^{k_i+1} appears in the wedge, so does S^{n-k_i-1}: thus $l = 2m$ and

$$M - m_0 \simeq S^{k_1+1} \vee \cdots \vee S^{k_m+1} \vee S^{(n-k_1-2)+1} \vee \cdots \vee S^{(n-k_m-2)+1}$$

and we say M is *spherical of type* $(k_1, \cdots, k_m; n)$. The $(m+1)$-tuple $(k_1, \cdots, k_m; n)$ is admissible if $1 < k_i + 1 < n/2$ (so $k_i < n - k_i - 2$), $1 \not\equiv n \not\equiv k_i \not\equiv 2 \pmod 4$ for all i, and none of the following can be expressed as linear combinations of $\{k_i, \cdots, k_m\}$ with non-negative integer coefficients: k_i (other than the trivial $k_i = k_i$), $n - 1$ and $n - 2$. (The condition $n \not\equiv k_i \not\equiv 2$ means there are no spheres of dimension 3 (mod 4) in the wedge.)

THEOREM. *If M is spherical of admissible type, then $\mathfrak{D}(M)$ is finite.*

Remarks and examples. (1) A spherical manifold of admissible type is 1-connected since $k_i \geqq 1$ for all i.

(2) If $m = 0$, M^n is homotopy sphere, whose diffeomorphism group is known to be finite.

(3) If $m = 1$, we need only that $n - 1$ and $n - 2$ not be multiples of k_1 and the mod 4 conditions. This includes many of the products $S^p \times S^q$ whose groups are known. Also included are many "homology tori" $(H_*(M) \simeq H_*(S^p \times S^q))$ for which the Hurewicz homomorphism is onto in both dimensions p and q (see Remark 6); e.g. p-sphere bundles over q-spheres which admit cross sections.

(4) There are many examples of admissible $(m+1)$-tuples: e.g., let $p = p_i \cdots p_m$ be a product of odd primes and $k_i = 4(p/p_i)$, $n \equiv 3 \pmod 4$; or $k_i = 3(p/p_i)$ and n be a multiple of 12.

(5) Every $(m+1)$-tuple corresponds to at least one spherical manifold:

$$(S^{k_1+1} \times S^{n-k_1-1}) \# \cdots \# (S^{k_m+1} \times S^{n-k_m-1})$$

Received November 13, 1972.

is spherical of type $(k_1, \cdots, k_m; n)$. In general, if M and M' are spherical of type $(k_1, \cdots, k_m; n)$ and $k_1', \cdots, k_{m'}'; n)$ respectively, then $M \# M'$ is spherical of type $(k_1, \cdots, k_m, k_1', \cdots, k_{m'}'; n)$.

(6) An algebraic characterization of spherical manifolds is that the homology be free and the Hurewicz homomorphism be onto in all dimensions less than n. There is then a natural map of a wedge of spheres into the manifold which induces an isomorphism on homology: by the Whitehead theorem, it is a homotopy equivalence.

Proof of the theorem. We proceed into two stages: let $\mathfrak{H}(M)$ be the group of (homotopy classes of) homotopy equivalences of M and $\mathfrak{D}^\pi(M)$ be the kernel of the forgetful map

$$\mathfrak{D}(M) \to \mathfrak{H}(M).$$

Then we show that $\mathfrak{H}(M)$ and $\mathfrak{D}^\pi(M)$ are finite by fitting them into exact sequences whose other terms are finite. We make use of Hilton's formula [1] for the homotopy groups of a wedge of spheres:

$$\pi_{p+1}(S^{k_1+1} \vee \cdots \vee S^{k_l+1}) = \oplus_q \, \pi_{p+1}(S^{q+1})$$

for q of the form $q = \sum a_j k_j$, a_j a non-negative integer. In particular, this group is finite unless there is such a $q = p$ or an odd q such that $p = 2q$. In what follows, $(k_1, \cdots, k_l)$, $l = 2m$, will come from the admissible $(m+1)$-tuple $(k_1, \cdots, k_m; n)$ by $k_{m+i} = n - k_i - 2$ for $1 \leqq i \leqq m$. Then in the admissible case, there are no solutions in non-negative integers a_j of

$$k_i = \textstyle\sum_{j \neq i} a_j k_j, \quad 1 \leqq i, j \leqq l,$$
$$n - 1 = \textstyle\sum_j a_j k_j, \quad 1 \leqq i, j \leqq l.$$

(To see this, start with the observation that since $n/2 < k_j$ if $m < j \leqq l$, at most one a_j can be nonzero in this range and can be at most 1.) It follows that for a spherical space M with admissible type, $\pi_n(M - m_0)$ and $\pi_{n-k_i-1}(M - m_0)$ are finite and $\pi_{k_i+1}(M - m_0)$ is the direct sum of a finite group and Z, corresponding to $\pi_{k_i+1}(S^{k_i+1})$.

Step 1. *$\mathfrak{H}(M)$ is finite.* $\mathfrak{H}(M)$ is a natural subgroup of the semi-group $[M, M]$ of homotopy classes of maps of M to itself which fits into the exact sequence corresponding to the coexact sequence of spaces.

$$M - m_0 \to M \to M/M - m_0 \cong S^n$$

namely

$$\pi_n(M) \to [M, M] \to [M - m_0, M].$$

Since M is 1-connected, $[M - m_0, M] \simeq [M - m_0, M - m_0]$ which is just

$$\oplus_{i=1}^l \pi_{k_i+1}(M - m_0)$$

(as a set, not as a group). Using Hilton's calculation and the assumption of admissibility, this sum is finite except for the image of

$$\oplus_{i=1}^l \pi_{k_i+1}(S^{k_i+1}) \simeq \mathbf{Z}^l.$$

However, any homotopy equivalence will have to have ± 1 in each direct summand—this can easily be seen by considering the induced automorphism on homology groups and recalling that the k_i are distinct. Therefore the image of $\mathcal{H}(M)$ in $[M - m_0, M]$ is finite.

To see that $\pi_n(M)$ is also finite we consider the homotopy exact sequence of the pair $(M, M - m_0)$:

$$\pi_n(M - m_0) \to \pi_n(M) \xrightarrow{a} \pi_n(M, M - m_0).$$

$\pi_n(M - m_0)$ is finite by Hilton's result, so it suffices to show that a is the zero map. If not, then from the diagram

$$\begin{array}{ccc} \pi_n(M) & \xrightarrow{a} & \pi_n(M, M - m_0) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle \cong} \\ H_n(M) & \longrightarrow & H_n(M, M - m_0) \end{array}$$

there is a map $f\colon S^n \to M^n$ such that $f_*([S^n]) = k[M^n]$ for some nonzero k. (The isomorphism in the diagram follows from homotopy excision [3; p 484] and the fact that M is 3-connected.) Then

$$f_*(f^*(c) \cap [S^n]) = c \cap k[M] = k(c \cap [M]).$$

Since $f^*(c) = 0$ (it lies in a zero group) if $i \neq n$, and by Poincaré duality, every element of $H_i(M)$ is of the form $c \cap [M^n]$, $H_i(M)$ is pure torsion: as this is impossible for spherical M, a is zero. Therefore $\pi_n(M)$ is finite.

Although the sequence

$$\pi_n(M) \xrightarrow{\alpha} [M, M] \xrightarrow{\beta} [M - m_0, M]$$

is not an exact sequence of groups, for each $f \in [M - m_0, M]$, the set of $g \in [M, M]$ such that $\beta(g) = f$ is in $(1 - 1)$ correspondence with $\pi_n(M)$. Restricting to

$$\alpha^{-1}(\mathcal{H}(M)) \to \mathcal{H}(M) \to \beta(\mathcal{H}(M)),$$

we conclude that $\mathcal{H}(M)$ is finite.

Step 2. *$\mathfrak{D}^\pi(M)$ is finite.* In [4], it was shown that $\mathfrak{D}^\pi(M)$ is a homomorphic image of $\mathcal{S}[M \times I]$ (the homotopy smoothings of $M \times I$ is in [5]) which fits into an exact sequence:

$$bP_{n+2} \to \mathcal{S}[M \times I] \to [\Sigma M, G/0].$$

Since bP_{n+2} is always finite, it suffices to show that for spherical M of admissible type, $[\Sigma M, G/0]$ is finite. From the coexact sequence

$$\Sigma(M - m_0) \to \Sigma(M) \to S^{n+1}$$

we have

$$\pi_{n+1}(G/0) \to [\Sigma M, G/0] \to [\Sigma(M - m_0), G/0]$$

and the last term is just

$$\oplus \pi_l(G/0).$$

where l runs over the values $k_i + 2$ and $n - k_i$. Since $\pi_l(G/0)$ is finite unless $l = 0 \pmod 4$ and the conditions of admissibility preclude this possibility for l of the form $k_i + 2$, $n - k_i$ or $n + 1$, $[\Sigma M, G/0]$ is finite.

Bibliography

1. P. J. Hilton, *On the homotopy groups of the union of spheres*, J. Lond. Math. Soc., vol. 30 (1955), pp. 154–171.
2. A. Hafliger, *Differential embeddings of S^n in S^{n+q} for $q > 2$*, Ann. of Math. (2), vol. 83 (1966), pp. 402–436.
3. E. Spanier, *Algebraic topology*, McGraw Hill, New York, 1966.
4. E. C. Turner, *Diffeomorphisms Homotopic to* 1, Trans. Amer. Math. Soc., vol. 188 (1974), pp. 1–10.
5. C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, New York, 1970.

State University of New York at Albany
Albany, New York