THE GEOMETRY OF SUBGROUPS OF $PSp_4(2^n)$

BY

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1. Introduction

The determination of the subgroups of the linear and projective groups over finite fields in small dimension has been a significant problem for many years. The subgroups of the two dimensional groups have been fully developed in Mitchell [12], Dickson [5], and Huppert [10]. The three-dimensional subgroups were determined for odd characteristic by Mitchell [12] and for even characteristic by Hartley [9].

In 1914, H. H. Mitchell [14] determined the maximal subgroups of the four-dimensional symplectic groups over finite fields of odd characteristic. His methods were highly geometrical in nature. The purpose of this paper is to set forth the computational foundation and the geometry of centers and axes of involutions necessary for determining the maximal subgroups of the four-dimensional symplectic groups over finite fields of even characteristic. The keystone in the development is the outer automorphism of the symplectic group which is induced by a duality on the incidence structure of points and totally isotropic lines.

First, we show the existence of a duality on the incidence structure of points and totally isotropic lines. Then we classify the types of symplectic transformations according to their configurations of fixed points and fixed lines, summarizing the results in Table 1. In Section 4 we construct Table 2, which shows the possible types for the product of two involutions from the symplectic group. The Sylow 2-subgroup Theorem gives a geometric characterization of the Sylow 2-subgroups of a subgroup of the symplectic groups. The Center-Axis Theorem of Section 6 provides a major tool for determining further centers and axes of involutions in a symplectic subgroup, given various configurations of known centers and axes. Finally, the Duality Theorem states that the dual of a superprimitive subgroup of $PSp_4(2^n)$ is also superprimitive, that is, primitive and fixing no proper subgeometry and no totally isotropic reguli.

Some basic references for properties of the symplectic and orthogonal groups are [1], [2], [3], [5], [6], [7], and [10].

For the entire article, let V be a four-dimensional vector space over $F = GF(q)$, where $q = 2^n$, and f a nondegenerate, alternate, bilinear form on V. We will use the terminology and notation of [8] for the point and line geometry. Further,

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a *flag* is a pair (P, t), where P is a point and t a totally isotropic line containing P. Since the characteristic is two, the linear symplectic group $Sp_4(q)$, consisting of all linear transformations on V which preserve f, and the induced projective symplectic group $PSp_4(q)$ are naturally isomorphic and will be identified. Matrices representing symplectic transformations will be computed with respect to a *symplectic basis* $[x_1, \ldots, x_4]$, that is, an ordered basis such that

$$f(\sum a_i x_i, \sum b_j x_j) = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1.$$

We will use "I" to denote the identity transformation or any identity matrix of appropriate rank.

2. Types of symplectic transformations

L. E. Dickson [4] has computed the conjugacy classes in $PSp_4(q)$. Following Mitchell and Hartley, however, we are more interested in the *types of symplectic transformations*. Since the eigenvalues for any transformation in $PSp_4(q)$ all lie in $GF(q^4)$, we say that two transformations in $PSp_4(q)$ are *of the same type* if their configurations of fixed points and fixed lines are conjugate under $PSp_4(q^4)$. By considering the elements in $PSp_4(q)$ to be transformations in $PSp_4(q^4)$, we can apply the first ten cases of the column $p = 2$ from Dickson's table [4, p. 132] to obtain representatives for at most ten types. A slight computational modification is necessary since Dickson works with an ordered basis $[y_1, \ldots, y_4]$ such that

$$f(\sum a_i y_i, \sum b_j y_j) = a_1 b_2 + a_2 b_1 + a_3 b_4 + a_4 b_3.$$

Dickson's first ten cases in the column $p = 2$ correspond in order to the notation of Table 1 as follows: I, IIIb, IIIa, IIb, IIa, identity, CE, CSE, NCSE, and FF.

In Table 1, the computation of the configuration of fixed points and lines and the verification of the remarks is elementary and will be left to the reader. A comparison of the nature of the fixed configurations indicates that no two of the nine representatives can be of the same type. Note that the linear transformation representing a central elation is a transvection.

Remarks on Table 1.

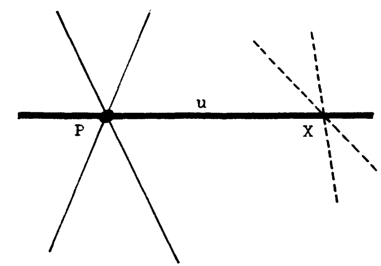
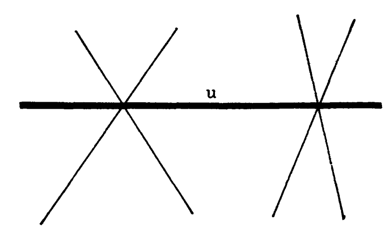
Type FF. The transformation T is called a *flag-fixer* with *center* P and *axis* u (or *at* (P, u)). The square of T is of type CSE at (P, u). The rank of $T - I$ is 3.

Type CSE. The transformation T is called a *centered skew elation* with *center* P and *axis* u (or *at* (P, u)). Through each point on u there is a planar pencil of fixed lines. All fixed lines through P are totally isotropic. The only totally isotropic fixed line through any X on u $(X \neq P)$ is u. The rank of $T - I$ is 2.

Type NCSE. The transformation T is called a *noncentered skew elation* with *axis* u (or *at* (P, u) for any point P on u). The fixed lines are exactly u plus the totally isotropic lines meeting u. The rank of $T - I$ is 2.

TABLE 1. *Types of nonidentity symplectic transformations*

A solid line denotes a totally isotropic fixed line, and a dotted line denotes a hyperbolic fixed line. A thick line (solid or dotted) denotes a line fixed pointwise.

Type	Representative T	Characteristic polynomial	Order	Fixed configuration
FF	$\begin{bmatrix} 1 & 1 & 0 & r \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$	$(X+1)^4$	4	
CSE	$\begin{bmatrix} 1 & 0 & 1 & 1 \\ & 1 & 0 & 1 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$	$(X+1)^4$	2	
NCSE	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ & 1 & 0 & 1 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$	$(X+1)^4$	2	

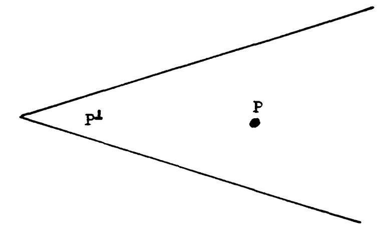
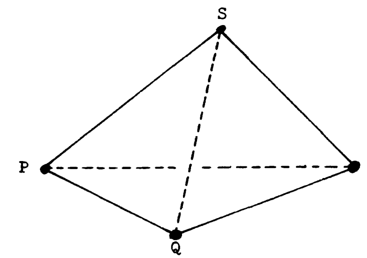
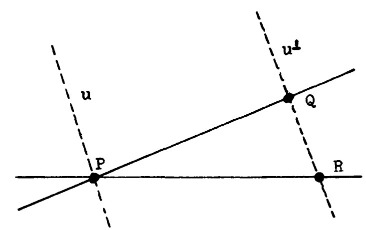
CE	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$	$(X + 1)^4$	2	
I	$\begin{bmatrix} k & & & \\ & m & & \\ & & 1/m & \\ & & & 1/k \end{bmatrix}$	$(X^2 + aX + 1)(X^2 + bX + 1)$	odd	
IIa	$\begin{bmatrix} 1 & 0 & 0 & 1 \\ & m & 0 & 0 \\ & & 1/m & 0 \\ & & & 1 \end{bmatrix}$	$(X + 1)^2(X^2 + bX + 1)$	$2 \cdot d$ d odd	

TABLE 1. *Continued*

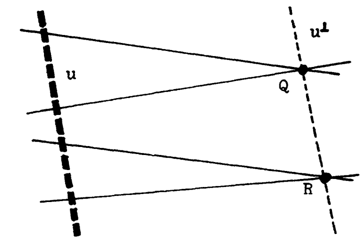
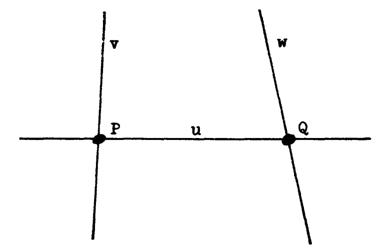
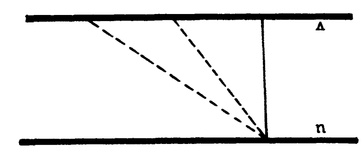
Type	Representative T	Characteristic polynomial	Order	Fixed configuration
IIb	$\begin{bmatrix} 1 & & & \\ & m & & \\ & & 1/m & \\ & & & 1 \end{bmatrix}$	$(X+1)^2(X^2+bX+1)$	odd	
IIIa	$\begin{bmatrix} k & k & & \\ & k & & \\ & & 1/k & 1/k \\ & & & 1/k \end{bmatrix}$	$(X^2+aX+1)^2$	$2\cdot d$ d odd	
IIIb	$\begin{bmatrix} k & & & \\ & k & & \\ & & 1/k & \\ & & & 1/k \end{bmatrix}$	$(X^2+aX+1)^2$	odd	

TABLE 2. *Products of involutions*

Let (P, k) and (Q, m) be flags. Let g and h be distance involutions in $PSp_4(q)$ at (P, k) and (Q, m), respectively. In the configuration diagrams, an open dot denotes the center of a CSE, a dashed line the axis of a CSE, a solid dot the center of a CE, and a solid line the axis of a NCSE. The letter d denotes an odd number.

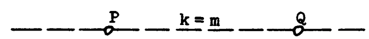
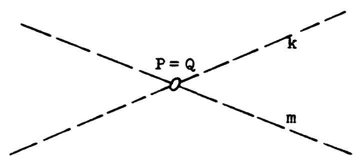
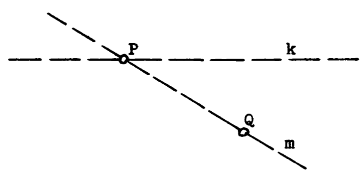
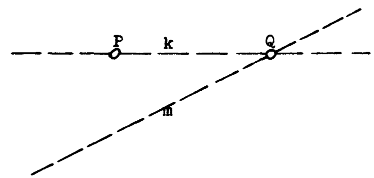
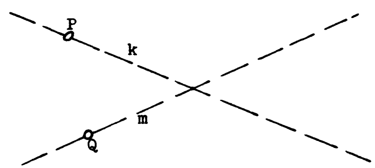
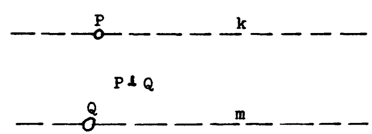
Case	g at (P, k)	h at (Q, m)	Configuration	gh	Order of gh	Center of gh	Axis of gh	Commute?
1	CSE	CSE		CSE NCSE CE	2 2 2	P — P	k k —	Yes Yes Yes
2	CSE	CSE		CSE CE	2 2	$\neq P, Q$ on k; $\neq P, Q$	k —	Yes Yes
3	CSE	CSE		CSE NCSE	2 2	P —	$\neq k, m$ through P; $\neq k, m$	Yes Yes
4	CSE	CSE		FF	4	P	m	No

TABLE 2. *Continued*

Case	g at (P, k)	h at (Q, m)	Configuration	gh	Order of gh	Center of gh	Axis of gh	Commute?
5	CSE	CSE		FF	4	Q	k	No
6	CSE	CSE		IIa IIb	$2 \cdot d$ odd	— —	— —	No No
7	CSE	CSE		IIIa IIIb	$2 \cdot d$ odd	— —	— —	No No

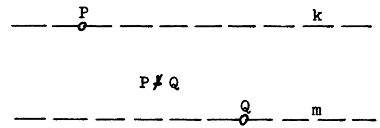
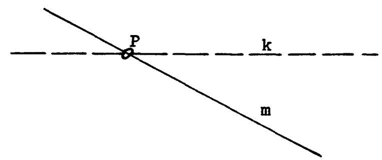
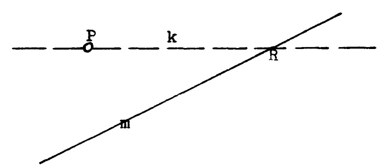
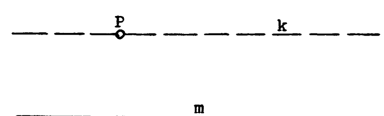
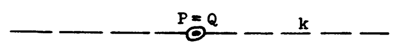
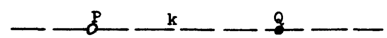
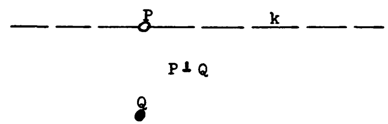
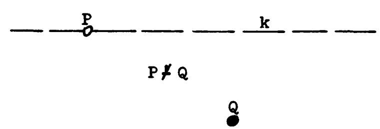
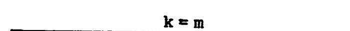
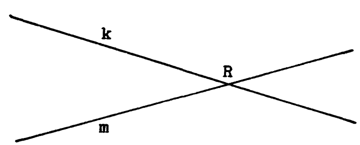
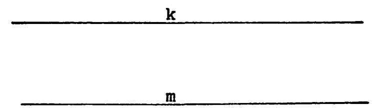
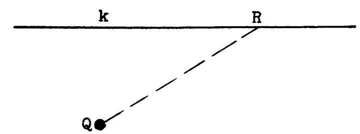
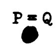
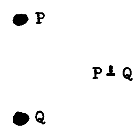
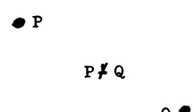
8	CSE	CSE		I IIIb	odd odd	— —	— —	No No
9	CSE	NCSE		CSE CE	2 2	P P	k —	Yes Yes
10	CSE	NCSE		CSE NCSE	2 2	P —	$\neq k, m$ through P; $\neq k, m$	Yes Yes
11	CSE	NCSE		FF	4	$R = k \cap m$	k	No
12	CSE	NCSE		IIIa	$2 \cdot d$	—	—	No

Table 2. *Continued*

Case	g at (P, k)	h at (Q, m)	Configuration	gh	Order of gh	Centre of gh	Axis of gh	Commute?
13	CSE	CE		CSE NCSE	2 2	P —	k k	Yes Yes
14	CSE	CE		CSE CE	2 2	$\neq P, Q$ on k; $\neq P, Q$	k —	Yes Yes
15	CSE	CE		FF	4	P	$\langle P, Q \rangle$	No
16	CSE	CE		IIa	$2 \cdot d$	—	—	No
17	NCSE	NCSE		NCSE	2	—	k	Yes
18	NCSE	NCSE		CSE	2	$R = k \cap m$	$\neq k, m$	Yes

19	NCSE	NCSE		IIIb	odd	—	—	No
20	NCSE	CE		CSE	2	Q	k	Yes
21	NCSE	CE		FF	4	$R = Q^{\perp} \cap k$	$<R, Q>$	No
22	CE	CE		CE	2	P	—	Yes
23	CE	CE		CSE	2	$\neq P, Q$	$<P, Q>$	Yes
24	CE	CE		IIb	odd	—	—	No

Type CE. The transformation T is called a *central elation* with *center* P (or *at* (P, u) for any totally isotropic line u through P). The fixed points are precisely those points in $P^\perp$. The fixed lines are precisely the lines either through P or in $P^\perp$. The rank of $T - I$ is 1.

Type I. The points P, Q, R, S form a tetrahedron. There are exactly four fixed points and six fixed lines. $a = k + 1/k$; $b = m + 1/m$; $k \neq m, 1/m$; $k, m \neq 1$; $a \neq b$; and $a, b \neq 0$.

Type IIa. The square of T is of type IIb, and T^d is of type CE. There are exactly three fixed points and four fixed lines. The rank of $T - I$ is 3. $b = m + 1/m$; $m \neq 1$; and $b \in F^*$.

Type IIb. The fixed points of T are Q and R together with the points on u. The fixed lines are u and $u^\perp$ together with the lines (totally isotropic) from Q and R to u. The rank of $T - I$ is 2. $b = m + 1/m$; $m \neq 1$; and $b \in F^*$.

Type IIIa. The square of T is of type IIIb, and T^d is of type NCSE. There are exactly two fixed points and three fixed lines (all totally isotropic). Both $T - kI$ and $T - (1/k)I$ have rank 3. $a = k + 1/k$; $k \neq 1$; and $a \in F^*$.

Type IIIb. The transformation T is called a *skew perspectivity* with *axes* u and v. The fixed points lie on a pair of skew, totally isotropic lines. The fixed lines are u and v together with the transversals to u and v. Both $T - kI$ and $T - (1/k)I$ have rank 2. $a = k + 1/k$; $k \neq 1$; and $a \in F^*$.

3. Duality

Associate to the projective symplectic space (V, f) the incidence structure $PT(V, f)$ whose points are the points in V, whose blocks are the totally isotropic lines in V, and whose incidence is given by incidence in V. The projective symplectic space can be recovered from $PT(V, f)$ since a hyperbolic line is the set of all points orthogonal to a given pair of nonorthogonal points, and since two points are orthogonal if and only if they are incident with a common totally isotropic line.

A *duality* δ on $PT(V, f)$ consists of an isomorphism ϕ from the points to the totally isotropic lines and an isomorphism θ from the totally isotropic lines to the points such that ϕ, ϕ^{-1}, θ, and θ^{-1} preserve incidence. We will show the existence of a special duality γ on $PT(V, f)$, construct an induced outer automorphism of $PSp_4(q)$, and examine the effect on the geometry of the projective symplectic space. The following development is based on lectures of J. E. McLaughlin. See [11] for properties of the exterior algebra.

Let $[x_1, \ldots, x_4]$ be a symplectic basis for (V, f). Consider the ordered basis

$$[x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4, x_3 \wedge x_4]$$

for the subspace $E^2(V)$ of elements of degree 2 in the exterior algebra of V. A vector z in $E^2(V)$ is *decomposable* provided $z = u \wedge v$ for some $u, v \in V$. Define the nontrivial linear functional $g\colon E^2(V) \to F$ by setting $g(x_i \wedge x_j) = f(x_i, x_j)$ for $1 \leq i < j \leq 4$ and extending by linearity. Let W denote the five-dimensional kernel of g. If $z = \sum_{i<j} p_{ij}(x_i \wedge x_j)$, then $g(z) = p_{14} + p_{23}$;

so z is in W if and only if $p_{14} = p_{23}$. Since a decomposable vector $u \wedge v$ is in W if and only if $u \perp v$, the totally isotropic lines $\langle u, v\rangle$ in V are in one-to-one correspondence with the decomposable points $\langle u \wedge v\rangle$ in W.

The function $Q\colon E^2(V) \to F$ given by

$$Q\left(\sum_{i<j} p_{ij}(x_i \wedge x_j)\right) = p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}$$

is a nondegenerate quadratic form of maximal index 3. The bilinear form e associated to the restriction $Q|_W$ is a degenerate, alternate form on W. The radical of (W, e) is the one-dimensional subspace $\langle w_0\rangle$, where

$$w_0 = (x_1 \wedge x_4) + (x_2 \wedge x_3).$$

Let $w_1 = x_1 \wedge x_2$, $w_2 = x_1 \wedge x_3$, $w_3 = x_2 \wedge x_4$, and $w_4 = x_3 \wedge x_4$. Then $[\bar{w}_1, \ldots, \bar{w}_4]$ is a symplectic basis for $(\bar{W}, \bar{e})$, where $^-\colon W \to W/\langle w_0\rangle$ is the natural homomorphism and $\bar{e}$ the induced form on $\bar{W}$. Each line in W through w_0 contains exactly one singular point for $Q|_W$, since the points on any of the lines $\langle w_0, w\rangle$ in W are $\langle w_0\rangle$ and $\langle aw_0 + w\rangle$ (for a in F), and since

$$Q(aw_0 + w) = 0 \quad \text{if and only if } a = \sqrt{Q(w)}.$$

Thus, the singular points of $Q|_W$ are in one-to-one correspondence with the $q^3 + q^2 + q + 1$ points of $\bar{W}$.

There is a bijection from the set of totally isotropic lines in V to the set of singular points for $Q|_W$ given by

$$\langle u, v\rangle \mapsto \langle u \wedge v\rangle$$

for any orthogonal vectors u and v in V, since direct calculation yields $Q(u \wedge v) = 0$. Composition with the correspondence $\langle u \wedge v\rangle \mapsto \langle (u \wedge v)^-\rangle$ of the previous paragraph yields a bijection θ from the set of totally isotropic lines of V to the points of $\bar{W}$, namely $\langle u, v\rangle \mapsto \langle (u \wedge v)^-\rangle$ for orthogonal vectors $u, v \in V$. Identify the two nondegenerate, four-dimensional symplectic spaces (V, f) and $(\bar{W}, \bar{e})$ by identifying their symplectic bases $[x_1, \ldots, x_4]$ and $[\bar{w}_1, \ldots, \bar{w}_4]$. Under this identification, if $u = \sum a_i x_i$ and $v = \sum b_j x_j$ are distinct orthogonal vectors in V, then $(u \wedge v)^-$ has coordinates

$$[a_1b_2 + a_2b_1, a_1b_3 + a_3b_1, a_2b_4 + a_4b_2, a_3b_4 + a_4b^3]^t.$$

Any point $\langle v\rangle$ in V is the intersection of two distinct, totally isotropic lines $\langle v, x\rangle$ and $\langle v, y\rangle$. Direct computation shows that the images $\langle (v \wedge x)^-\rangle$ and $\langle (v \wedge y)^-\rangle$ under θ are orthogonal points. If $\langle v, z\rangle$ is any totally isotropic line in V through $\langle v\rangle$, then $z = av + bx + cy$ for some $a, b, c \in F$, $v \wedge z = b(v \wedge x) + c(v \wedge y)$, and $\langle (v \wedge z)^-\rangle$ lies on the totally isotropic line $\langle (v \wedge x)^-, (v \wedge y)^-\rangle$ Thus, there is a bijection ϕ mapping each point $\langle v\rangle$ in V to the unique totally isotropic line spanned by the images under θ of all the totally isotropic lines in V through $\langle v\rangle$ It is clear that ϕ and θ provide a concrete duality γ on the incidence structure $PT(V, f)$.

We will now construct the induced outer automorphism of $Sp_4(q)$ We can identify $E^2(V)$ with the 4×4 alternate matrices over F by using the vector space isomorphism which maps $\sum_{i<j} p_{ij}(x_i \wedge x_j)$ to $[p_{ij}]$, where $p_{ii} = 0$ for $i = 1, \dots, 4$ and $p_{ji} = p_{ij}$ for $i < j$. Let $H = [h_{ij}]$ be the matrix for h in $Sp_4(q)$ with respect to the symplectic basis $[x_1, \dots, x_4]$. Associate to h the linear transformation $\hat{h}$ on $E^2(V)$ given by $\hat{h}(u \wedge v) = h(u) \wedge h(v)$. Direct computation shows that $\hat{h}(P) = HPH^t$ for P in $E^2(V)$.

If $u \perp v$, then $h(u) \perp h(v)$ and $\hat{h}(u \wedge v)$ is in W. Computation using the matrix for w_0 shows that $\hat{h}(w_0) = w_0$. Thus, $\hat{h}$ stabilizes W.

For P in $E^2(V)$ we have $Q(P) = Pf(P)$, where Pf denotes the Pfaffian [1, pp. 140–142]. Since $Pf(HPH^t) = Pf(P)$ for H in $Sp_4(q)$, the image $\hat{h}$ lies in the orthogonal group $GO(W, Q|_W)$ and induces an element (which we also call $\hat{h}$) in $Sp(\overline{W}, \bar{e})$. This yields a map from $Sp_4(q)$ to $Sp(\overline{W})$ given by $h \mapsto \hat{h}$, where $\hat{h}(u \wedge v)^- = (h(u) \wedge h(v))^-$. Since each point in $\overline{W}$ is the image under the duality γ of some totally isotropic line $\langle u, v\rangle$, direct computation shows that $\hat{h} = \gamma h \gamma^{-1}$. Direct computation also verifies that the map $h \mapsto h^\gamma = \hat{h}$ is an automorphism γ of $Sp_4(q)$ induced by conjugation by the duality γ on $PT(V, f)$.

Let δ be any duality on $PT(V, f)$. Then $\delta\gamma^{-1}$ is an automorphism of $PT(V, f)$ and induces a collineation (also called $\delta\gamma^{-1}$) on the projective space. Thus, $\delta\gamma^{-1}$ is induced by a semi-linear automorphism αg for some $\alpha \in \operatorname{Aut} F$ and some $g \in GL_4(q)$. Since $\delta\gamma^{-1}$ and α preserve orthogonality, so does g, and we may suppose, without loss of generality, that g is in $Sp_4(q)$. We conclude that any duality δ on $PT(V, f)$ may be written as $\delta = \alpha g \gamma$, where α is in Aut F and g in $Sp_4(q)$, and induces by conjugation an automorphism of $Sp_4(q)$.

PROPOSITION 1. *Let δ be any duality on $PT(V, f)$.*

(a) *Two distinct points are orthogonal if and only if their images under δ (or under δ^{-1}) are intersecting totally isotropic lines.*

(b) *The images under δ (or under δ^{-1}) of the points on a polar pair are the rulers and directrices of a totally isotropic regulus. The images under δ (or under δ^{-1}) of the (totally isotropic) transversals to the polar pair are the points of the regulus.*

Proof. The first part is a trivial consequence of the definitions of duality.

Let $\{k, k^\perp\}$ be a polar pair and P and S distinct points on $k^\perp$. Then the images of P and S under δ are skew, totally isotropic lines. Since the $q + 1$ points on k are orthogonal to both P and S, their images are the totally isotropic transversals to the images of P and S and form the rulers of a totally isotropic regulus whose directrices are the images of the points on $k^\perp$.

By analyzing the duals of the fixed configurations for the various types of symplectic transformations, we can determine the effect of the automorphism induced by a duality, which must be an outer automorphism since the types are not preserved. For example, the fixed points for a central elation T consist of all the points orthogonal to the center P. Hence the fixed totally isotropic lines of the image T^δ consist of all the totally isotropic lines meeting the totally isotropic line $\delta(P)$. The only possibility is that T^δ is a noncentered skew elation with axis $\delta(P)$. It is similarly easy to verify that δ preserves types FF, CSE, and I, interchanges type CE with NCSE, interchanges type IIa with IIIa, and interchanges type IIb with IIIb.

4. Products of involutions

The set S of matrices

$$\begin{bmatrix} 1 & a & b & c \\ & 1 & d & b + ad \\ & & 1 & a \\ & & & 1 \end{bmatrix}$$

such that a, b, c, and d are in F represents a subgroup of $PSp_4(q)$ of order q^4 and hence is a Sylow 2-subgroup of $PSp_4(q)$, whose order is $q^4(q^2 - 1)(q^4 - 1)$, where $q = 2^n$ [10]. This particular Sylow 2-subgroup fixes the flag $(\langle x_1\rangle, \langle x_1, x_2\rangle)$ and, as is easily verified, consists of the following nonidentity elements:

(a) For a and d nonzero,

$$\begin{bmatrix} 1 & a & b & c \\ & 1 & d & b + ad \\ & & 1 & a \\ & & & 1 \end{bmatrix}$$

is a flag-fixer with center $\langle x_1\rangle$ and axis $\langle x_1, x_2\rangle$;

(b)

$$\begin{bmatrix} 1 & 0 & b & c \\ & 1 & d & b \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

is a central elation (centered skew elation) with center $\langle(\sqrt{c})x_1 + (\sqrt{d})x_2\rangle$ (and axis $\langle x_1, x_2\rangle$) if $b^2 + cd = 0$ (if $b^2 + cd \neq 0$).

(c)

$$\begin{bmatrix} 1 & a & b & c \\ & 1 & 0 & b \\ & & 1 & a \\ & & & 1 \end{bmatrix}$$

is a noncentered skew elation (centered skew elation) with axis $\langle x_1, bx_2 + ax_3\rangle$ (and center $\langle x_1\rangle$) if $c = 0$ (if $c \neq 0$).

Given involutions g and h and the configuration of their centers and axes, it will be useful to know what the possibilities are for the type of the product gh, which is the same as the type for hg. The entries in Table 2 can be verified by direct computation, using the fact that in dual cases, such as cases 2 and 3, only one case needs to be verified directly. By way of example, we will verify cases 2 and 8.

For case 2, let $\langle x_1\rangle = P$ and $\langle x_2\rangle = Q$. Then

$$\begin{bmatrix} 1 & 0 & a & b \\ & 1 & 0 & a \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x & 0 \\ & 1 & y & x \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+x & b \\ & 1 & y & a+x \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}$$

where $a, b, x, y \neq 0$. If $a^2 + x^2 + by \neq 0$, then gh is a centered skew elation with axis $\langle x_1, x_2\rangle$ and center $\langle(\sqrt{b})x_1 + (\sqrt{y})x_2\rangle$, which is different from both $\langle x_1\rangle$ and $\langle x_2\rangle$. If $a^2 + x^2 + by = 0$, then gh is a central elation with center $\langle(\sqrt{b})x_1 + (\sqrt{y})x_2\rangle$, which is on $\langle x_1, x_2\rangle$, but different from $\langle x_1\rangle$ and $\langle x_2\rangle$.

For case 8, let $\langle x_1\rangle = P$, $\langle x_1, x_2\rangle = k$, $\langle x_4\rangle = Q$, and $\langle x_3, x_4\rangle = m$. Then

$$\begin{bmatrix} 1 & 0 & a & b \\ & 1 & 0 & a \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ x & 0 & 1 & \\ y & x & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+ax+by & bx & a & b \\ ay & 1+ax & 0 & a \\ x & 0 & 1 & 0 \\ y & x & 0 & 1 \end{bmatrix}$$

where $a, b, x, y \neq 0$. The determinant of $gh - I$ is $(ax)^2$, which is nonzero. So 1 is not an eigenvalue for gh, and gh must be of type I, IIIa, or IIIb. The following proposition rules out type IIIa.

PROPOSITION 2. *If g and h are two centered skew elations in $PSp_4(q)$ whose axes are skew and such that gh is of type* IIIa, *then their centers are orthogonal, and the line joining their centers is the axis of the noncentered skew elation $(gh)^d$ for some odd integer d.*

Proof. Let g and h be centered skew elations in $PSp_4(q)$ at (P, k) and (Q, m), respectively, such that k and m are skew and gh is of type IIIa. So gh has exactly two fixed points in $V_4(q^4)$, and $(gh)^d$ is a noncentered skew elation for some odd integer d.

If X is a fixed point of gh in $V_4(q^4)$, then X lies on a totally isotropic line u fixed by both g and h. Indeed, since k and m are skew, X is not fixed by both g and h, and hence is fixed by neither. If $X' = h(X)$, then h interchanges X and X', as does g. So g and h fix $u = \langle X, X'\rangle$. Thus, $(gh)^d$ fixes u, which must then be totally isotropic. Since the totally isotropic fixed lines of a centered skew elation all contain the center, u must equal $\langle P, Q\rangle$, and P and Q are orthogonal.

Further, if Y is the second fixed point of gh, then Y also lies on $\langle P, Q\rangle$. Since X and Y are distinct fixed points of gh, they are fixed points of the noncentered skew elation $(gh)^d$ and hence span the axis of $(gh)^d$.

COROLLARY 2.1. *If g and h are two centered skew elations in $PSp_4(q)$ whose axes are skew, then one of the following is true:*

(i) $\langle g, h\rangle$ *is a dihedral group of order* $2 \cdot d$, *where* d *is odd, or*

(ii) *the centers of* g *and* h *are orthogonal, and for some odd integer* d, $(gh)^d$ *is a noncentered skew elation with axis the line joining the centers of* g *and* h.

COROLLARY 2.2. *If* g *and* h *are two centered skew elations in* $PSp_4(q)$ *whose centers are nonorthogonal, then one of the following is true*:

(i) $\langle g, h\rangle$ *is a dihedral group of order* $2 \cdot d$, *where* d *is odd, or*

(ii) *the axes of* g *and* h *intersect in a point* R, *and for some odd integer* d, $(gh)^d$ *is a central elation with center* R.

Corollary 2.1 follows from Proposition 2 and Table 2, and Corollary 2.2 is the dual of Corollary 2.1.

PROPOSITION 3. *If* g *is a centered skew elation and* h *a noncentered skew elation in* $PSp_4(q)$ *whose axes are skew, then the product* gh *is of type* IIIa, *and the axis of the noncentered skew elation* $(gh)^d$ *(for some odd* d*) is the unique totally isotropic line from the center of* g *to the axis of* h.

Proof. Let g be a centered skew elation at (P, k) and h a noncentered skew elation with axis m such that k and m are skew. Let $R = m \cap P^{\perp}$. By Table 2, the product gh is of type IIIa and fixes exactly two points X and Y in $V_4(q^4)$. As in the proof of Proposition 2, g and h each fix the line $u = \langle X, X'\rangle$, where $X' = h(X)$. Since the fixed lines of a noncentered skew elation meet the axis and are all totally isotropic, u meets m and is totally isotropic. Since the totally isotropic fixed lines of a centered skew elation all contain the center, u contains P. Thus, $u = \langle P, R\rangle$. Similarly, Y lies on $\langle P, R\rangle$. Since X and Y are fixed by $(gh)^d$, the axis of $(gh)^d$ is $\langle X, Y\rangle = \langle P, R\rangle$.

5. Sylow 2-subgroups

If G is a subgroup of $PSp_4(q)$ and (P, k) a flag in V, then define the subsets $\mathscr{X} = \mathscr{X}(P, k, G)$ for $\mathscr{X} = \mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots, \mathscr{I}$ as follows:

$\mathscr{A}$ = {all FF's in G at (P, k)},
$\mathscr{B}$ = {all CSE's in G at (P, k)},
$\mathscr{C}$ = {all CSE's in G at (Q, k) for all $Q \neq P$ on k},
$\mathscr{D}$ = {all CSE's in G at (P, m) for all $m \neq k$ through P},
$\mathscr{E}$ = {all NCSE's in G with axis k},
$\mathscr{F}$ = {all NCSE's in G with axis $m \neq k$ through P},
$\mathscr{G}$ = {all CE's in G with center P},
$\mathscr{H}$ = {all CE's in G with center $Q \neq P$ on k}, and
$\mathscr{I}$ = {1}.

If $G = PSp_4(q)$, we may write $\mathscr{X}(P, k)$ for $\mathscr{X}(P, k, G)$.

It is trivial to verify that the sets $\mathscr{A}, \mathscr{B}, \ldots$, and $\mathscr{I}$ are closed under taking inverses, and that for y in $PSp_4(q)$ the set $\mathscr{X}(P, k, G)^y$ of y-conjugates of elements in $\mathscr{X}(P, k, G)$ is equal to the set $\mathscr{X}(y(P), y(k), G^y)$. Further, if H is also a subgroup of $PSp_4(q)$, then $H \cap \mathscr{X}(P, k, G) = \mathscr{X}(P, k, H \cap G)$ for $\mathscr{X} = \mathscr{A}, \mathscr{B}, \ldots, \mathscr{I}$.

Let G be a subgroup of $PSp_4(q)$. A point P is called a *central center* (for G) provided G has a central elation with center P and a *skew center* (for G) provided G has a centered skew elation with center P; in either case P is a *center*. A totally isotropic line k is called a *centered axis* (for G) provided G has a centered skew elation with axis k and a *noncentered axis* (for G) provided G has a noncentered skew elation with axis k; in either case k is an *axis*. A flag (P, k) is a *skew flag* (for G) provided G has a centered skew elation at (P, k). A skew flag (P, k) for G is *special* provided either G has a flag-fixer at (P, k) or G has no flag-fixers. If G is transitive on its skew flags, then all skew flags are special and we drop the label.

SYLOW 2-SUBGROUP THEOREM. *Let G be a subgroup of $PSp_4(q)$ containing centered skew elations. If (P, k) is a special skew flag for G, then $\bigcup \mathscr{X}(P, k, G)$ is a Sylow 2-subgroup of G. Conversely, if S is a Sylow 2-subgroup of G, then $S = \bigcup \mathscr{X}(P, k, G)$ for some special skew flag (P, k) for G.*

Proof. Whenever $\bigcup \mathscr{X}(P, k, G)$ appears, the union is assumed to be over $\mathscr{X} = \mathscr{A}, \mathscr{B}, \ldots, \mathscr{I}$. The theorem is trivial if $G = PSp_4(q)$.

Let (P, k) be a special skew flag for G and $S = \bigcup \mathscr{X}(P, k, G)$. It is clear that S is a 2-subgroup of $PSp_4(q)$. Suppose S is not a Sylow 2-subgroup of G. Then there is a 2-element h in G, but not in S, such that $\langle S, h\rangle$ is a 2-group.

Suppose G contains flag-fixers. Hence $\mathscr{A}(P, k, G) \neq \emptyset$, since (P, k) is special. The 2-subgroup $\langle S, h\rangle$ of G lies in a Sylow 2-subgroup $S' = \bigcup \mathscr{X}(Q, m)$ of $PSp_4(q)$ for some flag (Q, m). Since the flag-fixers in S fix only (P, k) and those in S' fix only (Q, m), we conclude that $(Q, m) = (P, k)$, and so h lies in S, contrary to assumption.

Suppose G contains no flag-fixers. Since (P, k) is a skew flag for G, there is a centered skew elation g in $\mathscr{B}(P, k, G)$. Since $\langle S, h\rangle$ is a 2-group, gh is a 2-element, but not a flag-fixer. This limits the possible configurations for centers and axes of the pair g, h. It is easily verified that each of the possible cases from Table 2 leads to a contradiction of the assumption that h is not in S.

Thus, $S = \bigcup \mathscr{X}(P, k, G)$ is a Sylow 2-subgroup of G.

Conversely, given a special skew flag (P, k) and S as above, any Sylow 2-subgroup of G can be expressed as a conjugate $S^y = \bigcup \mathscr{X}(Q, m, G)$ for some y in G, where $(Q, m) = y(P, k)$ is clearly special.

Let G be a subgroup of $PSp_4(q)$ which contains centered skew elations and is transitive on its skew flags. So for each $\mathscr{X} = \mathscr{A}, \mathscr{B}, \ldots, \mathscr{H}$, the order of $\mathscr{X}(P, k, G)$ is independent of the particular skew flag (P, k). We can catalog such subgroups G according to the pattern of emptiness or nonemptiness of the sets $\mathscr{A}, \mathscr{B}, \ldots, \mathscr{H}$. The *pattern for G* (or for a Sylow 2-subgroup of G) is labeled according to the following scheme:

(a) The basic label is according to the configuration of skew flags for a Sylow 2-subgroup of G:

(1) if $\mathscr{C}$ and $\mathscr{D}$ are both empty,
(2) if $\mathscr{C}$ is nonempty and $\mathscr{D}$ empty,
(3) if $\mathscr{C}$ is empty and $\mathscr{D}$ nonempty, and
(4) if $\mathscr{C}$ and $\mathscr{D}$ are both nonempty.

(b) To the basic label is added:
F if G contains flag-fixers,
C if G contains central elations, and
N if G contains noncentered skew elations.

If $\mathscr{C}$ and $\mathscr{D}$ are both nonempty, then Table 2 shows that G contains flag-fixers. Thus, patterns (4), (4C), (4N), and (4CN) are impossible.

Consider now a subgroup G of $PSp_4(q)$ which contains no centered skew elations. Trivially, G is transitive on its skew flags and contains no flag-fixers. If L has odd order, we label with pattern (0). If G has even order, then Table 2 indicates that G can have either central elations (pattern (0C)) or noncentered skew elations (pattern (0N)), but not both. If G has pattern (0C), then a Sylow 2-subgroup of G is elementary abelian and consists of all the central elations in G with a given center. A similar remark holds if G has pattern (0N).

The outer automorphism induced by a duality on $PT(V, f)$ effects the pattern scheme by fixing labels (0), (1), (4), and F, interchanging (2) with (3), and interchanging C with N.

6. Center-axis geometry

The Center-Axis Theorem below gives some techniques for studying the geometric configuration of centers and axes for a subgroup of $PSp_4(q)$. A subgroup G of $PSp_4(q)$ is *irreducible* provided G does not fix any point, line, or plane in V, and *primitive* provided (a) G is irreducible, (b) G does not fix a pair of skew lines in V, and (c) G does not fix a tetrahedron in V.

CENTER-AXIS THEOREM. *Let G be a subgroup of $PSp_4(q)$ and δ a duality on the incidence structure $PT(V, f)$ of points and totally isotropic lines.*

(i) *If a skew center P lies on a centered axis m, then (P, m) is a skew flag.*

(ii) *Two distinct, intersecting axes, at least one of which is a noncentered axis, meet in a skew center.*

(ii*) *The line joining two distinct, orthogonal centers, at least one of which is a central center, is a centered axis.*

(iii) *If a centered axis u and a noncentered axis v are skew, then each totally isotropic transversal to u and v meeting u in a skew center is a noncentered axis.*

(iii*) *If a skew center Q is not orthogonal to a central center P, then every centered axis through Q meets $P^{\perp}$ in a central center.*

(iv) *A noncentered axis meets the polar of a nonincident center in a skew center.*

(iv*) *The unique totally isotropic line from a central center P to an axis not containing P is a centered axis.*

(v) *If G is primitive, then all central elations in G are conjugate, and G is transitive on its central centers and on its skew flags.*

(v*) *If G is primitive, then all noncentered skew elations in G are conjugate, and G is transitive on its noncentered axes.*

(vi) *Suppose G is primitive and contains both centered skew elations and noncentered skew elations. If P and Q are distinct, orthogonal skew centers, then $\langle P, Q\rangle$ is an axis. Further, if G does not have pattern* (2FN) *or* (2FCN), *then $\langle P, Q\rangle$ is a noncentered axis. If G does have pattern* (2FN) *or* (2FCN), *then $\langle P, Q\rangle$ is either a noncentered axis or the unique centered axis through P* (*or through Q*).

(vi*) *Suppose that G contains centered skew elations and central elations, and that the dual G^δ of G is primitive. If k and m are distinct, intersecting centered axes, then $k \cap m$ is a center. Further, if G does not have pattern* (3FC) *or* (3FCN), *then $k \cap m$ is a central center. If G does have pattern* (3FC) *or* (3FCN), *then $k \cap m$ is either a central center or the unique skew center on k* (*or on m*).

Proof. It suffices to prove only one part of each dual pair of statements.

For part (i), let g and h be centered skew elations in G at (P, k) and (Q, m), respectively, such that (P, m) is a flag. Using Table 2 to analyze each case, it is trivial to show that (P, m) is a skew flag.

Part (ii) is a trivial consequence of Table 2.

Part (iii) is a corollary of Proposition 3.

For part (iv), let P be a center and m a noncentered axis not containing P. If P is a central center, then Table 2 yields the result. So suppose P is a skew center. Let k be a centered axis containing P and let $R = m \cap P^\perp$. If $k = \langle P, R\rangle$, then part (ii) implies that R is a skew center. If $k \neq \langle P, R\rangle$, then k and m are skew, and parts (iii) and (ii) imply that k is a skew center.

For part (v), let G be a primitive subgroup of $PSp_4(q)$ and M a G-orbit of central centers. Suppose there is a central center P not in M. Then M is contained in the polar of P, contrary to G being primitive, since by Table 2 central elations with nonorthogonal centers generate a dihedral group in which all the involutions are conjugate. Thus, G is transitive on its central centers.

To show that all central elations in G are conjugate, it suffices to show the conjugacy of any two central elations g and h in G with a common center P. Since G is primitive, there is a central elation y in G whose center R is not in the polar of P. Again g and y are conjugate in the dihedral group they generate, as are h and y. Hence g and h are conjugate in G.

The proof of the last claim in part (v) is made easier by the following lemma.

LEMMA. *Let G be a subgroup of $PSp_4(q)$. If (P, k), (P, m), (R, u), and (R, v) are distinct skew flags with P not orthogonal to R, then there is an element g in G which maps (P, k) to (P, m).*

Proof of Lemma. Let g, h, x, and y be centered skew elations in G at the distinct flags (P, k), (P, m), (R, u), and (R, v), respectively. Applying Corollary 2.2 to each of the pairs $\{g, x\}$ and $\{h, x\}$, we may assume, without loss of generality, that k meets u in a central center S, or m meets u in a central center S', otherwise (P, k) and (P, m) are conjugate through a chain of dihedral groups generated by the given centered skew elations. Both cases cannot occur since $P \not\perp R$. Similarly, we may assume that k meets v in a central center T', or m meets v in a central center T, but not both. Further, neither k nor m can meet both u and v. Two cases are possible, and by symmetry we may assume the case occurs in which k meets u in the central center S and m meets v in the central center T. If s and t are central elations in G with the nonorthogonal centers S and T, respectively, then by Tables 1 and 2, the dihedral group $\langle s, t\rangle$ fixes P and contains an element z mapping S to T. Thus, z sends (P, k) to (P, m).

Returning to part (v), we now prove that the primitive group G is transitive on its skew flags. Suppose G has centered skew elations, otherwise the result is trivial, and let S be a Sylow 2-subgroup of G. By the Sylow 2-Subgroup Theorem, there is a special skew flag (P, k) for G such that $S = \bigcup \mathscr{X}(P, k, G)$. Clearly, $\mathscr{B}(P, k, G) \neq \emptyset$.

Let (Q, m) be any skew flag and h a centered skew elation in G at (Q, m). We will show that (Q, m) is conjugate in G to (P, k). Since Sylow's Theorem implies that a conjugate of h lies in S, we may assume that h itself is in $\mathscr{B}(P, k, G)$, $\mathscr{C}(P, k, G)$, or $\mathscr{D}(P, k, G)$. In the first case $(Q, m) = (P, k)$ and we are done.

Suppose h is in $\mathscr{D}(P, k, G)$. Then $P = Q$ and $k \neq m$. Since G is primitive, the G-orbit of P contains a skew center R not orthogonal to P. Since P lies on two distinct centered axes, namely k and m, so does its G-conjugate R. The lemma implies that (P, k) and (Q, m) are conjugate.

Suppose, finally, that h is in $\mathscr{C}(P, k, G)$. Then $P \neq Q$ and $k = m$. The G-orbit $G(k)$ containing the totally isotropic line k must also contain a line r not meeting k; if not, then k is the only line in the orbit $G(k)$ which intersects every other line in $G(k)$ in a point, contrary to G being primitive. Since k contains two distinct skew centers (P and Q), so does its G-conjugate r. Then we may apply the lemma to the dual group G^{δ}, since r and k skew implies that $\delta(r)$ and $\delta(k)$ are nonorthogonal, to conclude that $(\delta(k), \delta(P))$ and $(\delta(k), \delta(Q))$ are conjugate in G^{δ}. Thus, (P, k) and (Q, k) are conjugate in G, and the proof of part (v) is completed.

For part (vi), let $k = \langle P, Q\rangle$. The hypothesis, together with part (v) and the Sylow 2-Subgroup Theorem, implies that each skew center lies on a noncentered axis. Let P and Q be distinct, orthogonal skew centers. If there is a centered axis (different from k) through P, then part (iii) implies that the unique totally isotropic line (namely k) joining P to any noncentered axis (different from k) through Q must be a noncentered axis. Suppose there is no centered axis through P different from $\langle P, Q\rangle$. Then (P, k) and (Q, k) are skew flags, $\mathscr{B}(P, k, G) \neq \emptyset$, $\mathscr{C}(P, k, G) \neq \emptyset$, and $\mathscr{D}(P, k, G) = \emptyset$. If g is a centered skew elation in G at (P, k) and h a noncentered skew elation in G with axis different from k and containing Q, then Table 2 shows that gh is a flag-fixer at (Q, k), and so $\mathscr{A}(P, k, G) \neq \emptyset$. Since G is assumed to have noncentered skew elations, we conclude that either k is a noncentered axis, or G has pattern (2FN) or (2FCN).

7. Subgeometries

Let $F' = GF(q')$ be a subfield of $F = GF(q)$, where $q = 2^n$. A *subgeometry* of V over F' is a subset $U = \{\sum a_i x_i \mid a_i \in F'\}$, where $[x_1, \ldots, x_4]$ is a symplectic basis for (V, f). Note that a set of vectors in U is independent over F' if and only if it is independent over F. The vectors in U are called *rational vectors*. The proper F-subspaces of V spanned by rational vectors are *rational points, lines,* and *planes*. Let $PT(U)$ denote the substructure of $PT(V, f)$ consisting of rational points and rational totally isotropic lines. It is easy to verify the following: (i) two distinct rational points span a rational line; (ii) the intersection of rational subspaces is rational; and (iii) the polar of a rational subspace is rational.

Let $[x_1, \ldots, x_4]$ be a symplectic basis, γ the special duality on $PT(V, f)$ given in Section 3, and U' the subgeometry $\{\sum a_i x_i \mid a_i \in F'\}$. Since $Sp_4(q)$ is transitive on symplectic bases, and hence on subgeometries over F', and since conjugation by γ induces an automorphism on $Sp_4(q)$, it is easy to verify that γ maps $PT(U)$ for any subgeometry U over F' to $PT(U^*)$ for some subgeometry U^* over F'. We conclude that any duality δ on $PT(V, f)$ does the same, since $\delta = \alpha g \gamma$ for some α in Aut F and some g in $Sp_4(q)$.

PROPOSITION 4. *Let U be a subgeometry of (V, f) over F' and G a subgroup of $PSp_4(q)$. Then G fixes the set of rational points (for U) if and only if G fixes U.*

Proof. The reverse implication is trivial.

Suppose G fixes the set of rational points (for U). Let $[u_1, \ldots, u_4]$ be a symplectic basis for V such that $U = \{\sum a_i x_i \mid a_i \in F'\}$, and let g be in G. Then $g(u_i) = k_i v_i$ for some k_i in F^* and v_i in U, for $i = 1, \ldots, 4$, and $[v_1, \ldots, v_4]$ is a basis (over F') for U. Since x equal to $u_1 + u_2 + u_3 + u_4$ spans a rational vector, $g(x) = ry$ for some r in F^* and some $y = \sum a_i v_i$ in U. Computation shows that $k_i = ra_i$ for $i = 1, \ldots, 4$. Using the nonsingular matrix $B = [b_{ij}]$ over F' such that $v_i = \sum b_{ij} u_j$, we compute that the transformation g has matrix M equal to $[ra_i b_{ij}]$ with respect to the symplectic basis $[u_1, \ldots, u_4]$. Since M has determinant 1, and hence $1/r^4 = \det [a_i b_{ij}]$, we conclude that r is in F'. Thus, g fixes U.

COROLLARY 4.1. *Let G be a subgroup of $PSp_4(q)$ and δ a duality on $PT(V, f)$. If G stabilizes a subgeometry over F', then the dual G^δ stabilizes a subgeometry over F'.*

8. Superprimitive subgroups

PROPOSITION 5. *Let Q be a quadratic form on (V, f) with quadric K of singular points. The stabilizer in $PSp_4(q)$ of K is the orthogonal group $GO(Q)$.*

Proof. Let T be an element of $PSp_4(q)$ which stabilizes K. If u is a singular vector, then $Q(Tu) = 0 = Q(u)$. Let v be any nonsingular vector and $a = Q(v)$. Since K does not lie entirely in $\langle v\rangle^\perp$, there is a vector r such that $r \not\perp v$ and $f(v, r) = a$. Since $Q(v + r) = 0$, we conclude that $Q(T(v + r))$ is also zero.

Hence $Q(T(v)) = a = Q(v)$ and T is in $GO(Q)$.

As a result of Proposition 5, the stabilizer in $PSp_4(q)$ of a totally isotropic regulus is the maximal index orthogonal group whose quadric is that regulus [8, Theorem 1]. Furthermore, Proposition 1 implies that for any duality δ on $PT(V, f)$, the dual $GO(Q)^\delta$ of a maximal index orthogonal group is the stabilizer in $PSp_4(q)$ of a pair of polar hyperbolic lines.

PROPOSITION 6. *Let δ be a duality on $PT(V, f)$ and G a primitive subgroup of $PSp_4(q)$. Then one of the following is true:*

(i) *the dual G^δ of G is primitive, or*

(ii) *G is a subgroup of the orthogonal group $GO(Q)$ for some maximal index quadratic form Q on (V, f).*

Proof. If G^δ fixes a point, a totally isotropic line, or a pair of totally isotropic lines, then G fixes a totally isotropic line, a point, or a pair of points, respectively, contrary to G being primitive. If G^δ fixes a polar pair, then G fixes a totally isotropic regulus, and Proposition 5 implies case (ii). If G^δ stabilizes a pair of distinct nonpolar hyperbolic lines, then Theorem 2 in [8] implies that G^δ also fixes the unique totally isotropic line associated to the pair, and hence G fixes a point.

Suppose G^δ acts transitively on the vertices of a tetrahedron T. Since the vertices of a tetrahedron span V, there are at most four totally isotropic sides to T. So T can have no totally isotropic sides, otherwise G would act on a set of four or fewer points, contrary to G being primitive. Each of the three pairs of opposite sides to T must be a pair of distinct, nonpolar hyperbolic lines, to which Theorem 2 in [8] associates a unique totally isotropic line. Hence G acts on a set of three points, contrary to G being primitive. Thus, G^δ fixes no tetrahedron, and we have examined all the cases.

If we define a subgroup G of $PSp_4(q)$ to be *superprimitive* provided G is primitive, (d) G does not fix any subgeometry over a proper subfield of $GF(q)$, and (e) G does not fix any totally isotropic regulus, then we obtain the Duality Theorem.

DUALITY THEOREM. *Let δ be a duality on the incidence structure $PT(V, f)$ and G a subgroup of $PSp_4(q)$. Then G is superprimitive if and only if its dual G^δ is superprimitive.*

Proof. Since δ^{-1} is also a duality on $PT(V, f)$, it suffices to prove only one of the implications. Suppose G is superprimitive. By Proposition 6, the group G^δ is primitive, since G fixes no totally isotropic regulus. Corollary 4.1, applied using δ^{-1}, shows that G^δ fixes no subgeometry over a proper subfield of F.

Finally, since G is primitive, G fixes no polar pair, and so G^δ fixes no totally isotropic regulus. Thus, G^δ is superprimitive.

References

1. E. Artin, *Geometric algebra*, Interscience, New York, 1957.
2. R. Baer, *Linear algebra and projective geometry*, Academic Press, New York, 1952.
3. P. Dembowski, *Finite geometries*, Springer-Verlag, Berlin, 1968.
4. L. E. Dickson, *Canonical forms of quaternary abelian substitutions in an arbitrary Galois field*, Trans. Amer. Math. Soc., vol. 2 (1901), pp. 103–138.
5. ———, *Linear groups, with an exposition of Galois field theory*, reprint of 1901 ed., Dover, New York, 1958.
6. J. Dieudonné, *La géometrié des groupes classiques*, 2nd ed., Springer-Verlag, Berlin, 1963.
7. ———, *Sur les groupes classiques*, 3rd ed., Hermann, Paris, 1967.
8. D. E. Flesner, *Finite Symplectic Geometry in Dimension Four and Characteristic Two*, Illinois J. Math., vol. 19 (1975), pp. 41–47 (this issue).
9. R. W. Hartley, *Determination of the ternary collineation groups whose coefficients lie in the* $GF(2^n)$, Ann. of Math., vol. 27 (1925), pp. 140–158.
10. B. Huppert, *Endliche Gruppen*, vol. I, Springer-Verlag, Berlin, 1967.
11. S. MacLane and G. Birkhoff, *Algebra*, Macmillan, New York, 1967.
12. H. H. Mitchell, *Determination of the ordinary and modular ternary linear groups*, Trans. Amer. Math. Soc., vol. 12 (1911), pp. 207–242.
13. ———, *Determination of the finite quaternary linear groups*, Trans. Amer. Math. Soc., vol. 14 (1913), pp. 123–142.
14. ———, *The subgroups of the quaternary abelian linear group*, Trans. Amer. Math. Soc., vol. 15 (1914), pp. 379–396.

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