POSITIVELY CURVED INTEGRAL SUBMANIFOLDS OF A CONTACT DISTRIBUTION

BY

D. E. BLAIR[1] AND K. OGIUE[2]

1. Statement of results

In [1] we studied some properties of maximal integral manifolds of a contact distribution of a Sasakian space form. Let $M^{2n+1}(c)$ denote a $(2n+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c. The purpose of this paper is to prove the following.

THEOREM. *Let M be an n-dimensional compact integral manifold of the contact distribution of $M^{2n+1}(c)$ which is minimally immersed. If the sectional curvature of M is greater than*

$$\frac{(n-2)(c+3)}{4(2n-1)},$$

then M is totally geodesic.

In [1] we showed that the 5-dimensional unit sphere S^5 with its usual Sasakian structure admits S^2 as a totally geodesic integral surface of its contact distribution and the flat 2-dimensional torus as a minimal nontotally geodesic integral surface. Thus in dimension five the number in the theorem is best possible.

2. Basic lemmas

We use the same notation and terminologies as in [1] unless otherwise stated. It was proved in [1] that the second fundamental form of the immersion satisfies

$$(1)\qquad \tfrac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum_{i,j,k,l,m}(h^m_{ij}h^m_{kl}R_{lijk} + h^m_{ij}h^m_{il}R_{lkjk}) + \frac{1}{2}\sum_{i,j}\operatorname{tr}(A_iA_j - A_jA_i)^2 + \frac{c-1}{4}\|\sigma\|^2,$$

where R_{ijkl} are the components of the curvature tensor of M.

On the one hand, using the equation of Gauss we obtain

$$(2)\qquad \sum_{i,j,k,l,m}(h^m_{ij}h^m_{kl}R_{lijk} + h^m_{ij}h^m_{il}R_{lkjk}) = \frac{n(c+3)}{4}\|\sigma\|^2 + \frac{1}{2}\sum_{i,j}\operatorname{tr}(A_iA_j - A_jA_i)^2 - \sum_{i,j}(\operatorname{tr}A_iA_j)^2.$$

Received April 11, 1975.

[1] Partially supported by a National Science Foundation grant.

[2] Partially supported by a National Science Foundation grant and the Matsunaga Science Foundation.

On the other hand, Yau's idea in [4] can be applied as follows. For each m, let $h_1^m, \ldots, h_n^m$ be the eigenvalues of A_m. Then we have

$$\sum_{i,j,k,l} (h_{ij}^m h_{kl}^m R_{lijk} + h_{ij}^m h_{il}^m R_{lkjk}) = \sum_{i,k} \{h_i^m h_k^m R_{kiik} + (h_i^m)^2 R_{ikik}\}$$

$$= \frac{1}{2} \sum_{i,k} (h_i^m - h_k^m)^2 R_{ikik}.$$

Therefore if the sectional curvature of M is greater than δ, then we have

$$\sum_{i,j,k,l} (h_{ij}^m h_{kl}^m R_{lijk} + h_{ij}^m h_{il}^m R_{lkjk}) \geq \frac{1}{2} \sum_{i,k} (h_i^m - h_k^m)^2 \delta$$

$$= n\delta \sum_i (h_i^m)^2$$

$$= n\delta \operatorname{tr} A_m^2,$$

since M is minimal so that $\sum_i h_i^m = 0$. This implies that

$$\sum_{i,j,k,l,m} (h_{ij}^m h_{kl}^m R_{lijk} + h_{ij}^m h_{il}^m R_{lkjk}) \geq n\delta \|\sigma\|^2. \tag{3}$$

From (1), (2), and (3) we have:

LEMMA 1. *If the sectional curvature of M is greater than δ, then*

$$\tfrac{1}{2}\Delta\|\sigma\|^2 \geq \|\nabla'\sigma\|^2 + (1 + a)n\delta\|\sigma\|^2 - \frac{na(c + 3) - (c - 1)}{4} \|\sigma\|^2$$

$$+ \frac{1 - a}{2} \sum_{i,j} \operatorname{tr} (A_i A_j - A_j A_i)^2 + a \sum_{i,j} (\operatorname{tr} A_i A_j)^2$$

for all $a \geq -1$. (Equality holds for $a = -1$.)

The following lemma is due to Ikawa, Kon, and Yamaguchi [3].

LEMMA 2. $\|\nabla'\sigma\|^2 \geq \|\sigma\|^2$.

Proof. If we denote by h_{ijk}^α the components of $\nabla'\sigma$, then we have

$$\begin{aligned} h_{ijk}^0 &= G((\nabla'_{X_k}\sigma)(X_i, X_j), \xi) \\ &= G(\nabla^\perp_{X_k}(\sigma(X_i, X_j)), \xi) \\ &= -G(\sigma(X_i, X_j), \nabla^\perp_{X_k}\xi) \\ &= -G(\sigma(X_i, X_j), \tilde{\nabla}_{X_k}\xi) \\ &= G(\sigma(X_i, X_j), \phi X_k) \\ &= h_{ij}^k, \end{aligned}$$

from which it follows that

$$\begin{aligned}\|\nabla'\sigma\|^2 &= \sum_{i,j,k,\alpha} h^\alpha_{ijk}h^\alpha_{ijk} \\ &= \sum_{i,j,k,m} h^m_{ijk}h^m_{ijk} + \sum_{i,j,k} h^0_{ijk}h^0_{ijk} \\ &= \sum_{i,j,k,m} h^m_{ijk}h^m_{ijk} + \|\sigma\|^2 \\ &\geq \|\sigma\|^2, \qquad \text{Q.E.D.}\end{aligned}$$

Finally the following lemma is purely algebraic.

LEMMA 3.

$$\frac{1}{n}\|\sigma\|^4 \leq \sum_{i,j} (\operatorname{tr} A_iA_j)^2 \leq \|\sigma\|^4.$$

3. Proof of the theorem

Since the symmetric (n, n)-matrix $(\operatorname{tr} A_iA_j)$ is covariant for an orthogonal change of bases, for a suitable choice of basis we may assume that

$$\operatorname{tr} A_iA_j = 0 \quad \text{for } i \neq j. \tag{4}$$

An algebraic lemma of Chern, doCarmo, and Kobayashi (Lemma 1 in [2]) implies that

$$\begin{aligned}\sum_{i,j} \operatorname{tr}(A_iA_j - A_jA_i)^2 &\geq -2\sum_{i\neq j}(\operatorname{tr} A_i^2)(\operatorname{tr} A_j^2) \\ &= -2\|\sigma\|^4 + 2\sum_i (\operatorname{tr} A_i^2)^2.\end{aligned}\tag{5}$$

From Lemma 1, Lemma 2, (4), and (5), it follows that

$$\begin{aligned}\tfrac{1}{2}\Delta\|\sigma\|^2 \geq (1 + a)n\delta\|\sigma\|^2 - \frac{(na - 1)(c + 3)}{4}\|\sigma\|^2 \\ - (1 - a)\|\sigma\|^4 + \sum_i (\operatorname{tr} A_i^2)^2\end{aligned}$$

for all $a \in [-1, 1]$. This, together with Lemma 3 and (4), implies that

$$\tfrac{1}{2}\Delta\|\sigma\|^2 \geq (1 + a)n\delta\|\sigma\|^2 - \frac{(na - 1)(c + 3)}{4}\|\sigma\|^2 + \left\{\frac{1}{n} - (1 - a)\right\}\|\sigma\|^4.$$

In particular, putting $a = 1 - (1/n)$, we obtain

$$\tfrac{1}{2}\Delta\|\sigma\|^2 \geq \left\{(2n - 1)\delta - \frac{(n - 2)(c + 3)}{4}\right\}\|\sigma\|^2,$$

from which the theorem follows.

REFERENCES

1. D. E. BLAIR AND K. OGIUE, *Geometry of integral submanifolds of a contact distribution*, Illinois J. Math., vol. 19 (1975), pp. 269–276.
2. S. S. CHERN, M. P. DOCARMO, AND S. KOBAYASHI, "Minimal submanifolds of a sphere with second fundamental form of constant length," in *Functional analysis and related fields*, Springer, 1970, pp. 59–75.
3. T. IKAWA, M. KON, AND S. YAMAGUCHI, *On C-totally real submanifolds*, J. Diff. Geom., to appear.
4. S. T. YAU, *Submanifolds with constant mean curvature II*, to appear.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN
UNIVERSITÉ DE STRASBOURG
STRASBOURG, FRANCE
TOKYO METROPOLITAN UNIVERSITY
TOKYO, JAPAN