INVARIANT SUBSPACES OF LINEAR TRANSFORMATIONS

BY

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A (not necessarily bounded) linear transformation on a Banach space is *intransitive* if it has a proper closed invariant subspace; otherwise it is *transitive*. The general invariant subspace problem asks whether a separable infinite-dimensional Banach space can possess bounded transitive linear transformations.

Shields [5, Theorem 1] constructed a transitive (not necessarily bounded) linear transformation on a separable infinite-dimensional Hilbert space. Later Halmos [4, p. 898] asked whether a bounded transitive linear transformation can have an intransitive square (or inverse). This paper extends Shields' techniques to answer similar questions for (not necessarily bounded) linear transformations on a separable infinite-dimensional Banach space.

The first theorem (Theorem A) shows that a linear transformation L can be found such that every nonconstant polynomial in L and L^{-1} is transitive.

The second theorem (Theorem B) shows that it is possible to find a transitive linear transformation whose square does not have dense range. Such a transformation can never be bounded.

The third theorem (Theorem C) shows that it is possible to find a transitive linear transformation with both an intransitive square and an intransitive inverse.

In Theorems A and C certain classes of polynomials are excluded. This is necessary in Theorem A because scalar operators are never transitive and in Theorem C because if $mL + b$ is intransitive and $m \neq 0$, then L is intransitive.

THEOREM A. *If Y is a separable infinite-dimensional Banach space, then there is a bijective (not necessarily bounded) linear transformation L on Y such that for every pair p, q of polynomials, not both constant, $p(L) + q(L^{-1})$ is transitive.*

THEOREM B. *If Y is a separable infinite-dimensional Banach space, M is a closed infinite-dimensional subspace of Y, and p is a nonconstant polynomial, then there is a (not necessarily bounded) linear transformation L on Y such that:*

(i) $p(L)(Y) \subset M$;

(ii) *if q is any polynomial with $1 \leq \deg q < \deg p$, then $q(L)$ is transitive*;

(iii) $p(L)|_M$ *is transitive.*

THEOREM C. *If Y is a separable infinite-dimensional Banach space, then there is a bijective (not necessarily bounded) linear transformation L on Y such that L*

Received December 11, 1974.

is transitive, but such that $p(L) + q(L^{-1})$ *is intransitive for every pair* p, q *of polynomials with* $p(x) + q(1/x)$ *not of the form* $mx + b$ $(m \neq 0)$.

Since the proofs of these theorems are mostly algebraic, we shall state and prove them in a completely algebraic setting (Theorems A′, B′, C′). To connect the "algebraic" theorems to their "Banach space" counterparts we need a fact (Lemma 5) about closed subspaces of a separable infinite-dimensional Banach space.

Throughout, X will be a fixed vector space over a field F, and $\mathscr{M}$ will be an infinite collection of infinite-dimensional proper subspaces of X. (In the Banach space application the role of $\mathscr{M}$ will be played by the collection of all proper closed infinite-dimensional subspaces.) Write $\mathscr{M} = \{M_\alpha : \alpha \in A\}$, where A is the smallest ordinal with card A = card $\mathscr{M}$ (where "card" denotes cardinality). We shall always assume that card $A > \aleph_0$, and that, for each α in A, dim $M_\alpha \geq$ card A (where "dim" denotes linear dimension).

A linear transformation on X is *$\mathscr{M}$-intransitive* if it has a nonzero invariant subspace that is either finite-dimensional or in $\mathscr{M}$; otherwise it is *$\mathscr{M}$-transitive.*

The symbols Z and Z^+ will denote the sets of integers and nonnegative integers, respectively. If $S \subset X$, then $\vee S$ denotes the linear span of S, and S' denotes the set-theoretic complement of S.

THEOREM A′. *There is a bijective linear transformation* L *on* X *such that* $p(L) + q(L^{-1})$ *is* $\mathscr{M}$*-transitive for every pair* p, q *of polynomials such that* $p(x) + q(1/x)$ *is not constant.*

Proof. We begin by defining a function $f\colon A \times (Z \cup \{\infty\}) \to X$ so that:

(1) $f(A \times (Z \cup \{\infty\}))$ is linearly independent;
(2) for all (β, n) in $A \times Z$, $f(\beta, n) \in M_\beta$ if and only if $n \neq 0$.

Let α be any ordinal in A, and suppose that $f(\beta, n)$ has been defined for all $\beta < \alpha$ and n in $Z \cup \{\infty\}$ so that (1), (2) hold when A is replaced by

$$\{\beta \in A : \beta < \alpha\}.$$

Since

$$\dim \vee f(\alpha \times (Z \cup \{\infty\})) \leq \max(\operatorname{card} \alpha, \aleph_0) < \operatorname{card} A \leq \dim M_\alpha \leq \dim X,$$

we can choose vectors $f(\alpha, n)$ for each n in $Z \cup \{\infty\}$ so that (1), (2) hold when A is replaced by $\{\beta \in A : \beta \leq \alpha\}$. Using transfinite induction we can extend f to all of $A \times (Z \cup \{\infty\})$ so that (1), (2) hold.

We now extend $f(A \times Z)$ to a Hamel basis for X by adding a set of the form $\{g(b, n) : (b, n) \in B \times Z\}$. To be able to do this we only need an infinite linearly independent set which is independent from $f(A \times Z)$. However, $f(A \times \{\infty\})$ is such a set.

We now define a linear transformation L on X by defining $L(f(\alpha, n)) = f(\alpha, n + 1)$ and $L(g(b, n)) = g(b, n + 1)$ for each α in A, b in B, n in Z, and then extending L to all of X by linearity.

Suppose that p, q are polynomials such that $p(x) + q(1/x)$ is not constant. If $\deg p = n \geq 1$, then

$$[p(L) + q(L^{-1})]f(\alpha, -n) \notin M_\alpha$$

while $f(\alpha, -n) \in M_\alpha$ for each α in A. On the other hand if $\deg q = m \geq 1$, then

$$[p(L) + q(L^{-1})]f(\alpha, m) \notin M_\alpha$$

while $f(\alpha, m) \in M_\alpha$ for each α in A. Hence $p(L) + q(L^{-1})$ leaves no M_α invariant. Since the application of $p(L) + q(L^{-1})$ to a nonzero linear combination of basis vectors yields a linear combination involving at least one new basis vector, then $p(L) + q(L^{-1})$ has no finite-dimensional invariant subspaces. Thus $p(L) + q(L^{-1})$ is $\mathscr{M}$-transitive.

We next state a lemma whose proof is an elementary exercise in induction.

LEMMA 1. *If V is a vector space with a Hamel basis $\{e_k: k \in Z^+\}$, and if p is any polynomial with positive degree n, then there is a unique Hamel basis $\{f_k: k \in Z^+\}$ such that:*

(i) $f_k = e_k$ *if* $0 \leq k \leq n - 1$;

(ii) *if L is the linear transformation on V defined by $L(f_k) = f_{k+1}$ for each k in Z^+, then $p(L)f_k = e_{k+n}$ for each k in Z^+.*

THEOREM B′. *If p is a polynomial of positive degree and if M is a subspace of X with* $\dim M = \dim X$, *then there is a linear transformation L on X such that:*

(i) $p(L)(X) \subset M$;

(ii) *$q(L)$ is $\mathscr{M}$-transitive for every polynomial q with* $1 \leq \deg q < \deg p$;

(iii) *if for some α in A, $M \cap M_\alpha \neq M$ and* $\dim (M \cap M_\alpha) \geq \operatorname{card} A$, *then $p(L)$ does not leave $M \cap M_\alpha$ invariant.*

Proof. Let $n = \deg p$ and $E = \{0, 1, \ldots, n - 1\}$. We can use the transfinite induction technique of the proof of Theorem A′ to construct a function $f: A \times E \times (Z^+ \cup \{\infty\}) \to X$ so that:

(1) $f(A \times E \times (Z^+ \cup \{\infty\}))$ is linearly independent;

(2) $f(\beta, i, i) \notin M_\beta$ if $\beta \in A$, $1 \leq i \leq n - 1$;

(3) $f(\beta, i, j) \in M_\beta$ if $\beta \in A, j \in E$, $1 \leq i \leq n - 1$, $i \neq j$;

(4) $f(\beta, i, j) \in M$ if $\beta \in A$, $1 \leq i \leq n - 1 < j < \infty$;

(5) $f(\beta, 1, 0) \in M \cap M_\beta$, $f(\beta, 1, n) \notin M \cap M_\beta$ if $\beta \in A$, $M \cap M_\beta \neq M$, $\dim (M \cap M_\beta) \geq \operatorname{card} A$;

(6) $f(\beta, i, \infty) \in M$ if $\beta \in A$, $i \in E$.

We now extend $f(A \times E \times Z^+)$ to a Hamel basis for X by adding a set of the form $\{g(b, i): (b, i) \in B \times Z^+\}$ such that $g(b, i) \in M$ whenever $b \in B$, $i \geq n$. To be able to do this we need that

$$\dim (\vee [M \cap f(A \times E \times Z^+)']) = \dim X.$$

Since $\dim M = \dim X$, we only need that

$$\dim (\vee [M \cap f(A \times E \times Z^+)']) \geq \operatorname{card} A.$$

We therefore need a linearly independent subset of $\vee\ [M \cap f(A \times E \times Z^+)']$ whose cardinality is card A. However, $f(A \times E \times \{\infty\})$ is such a set.

Using Lemma 1 we can construct functions

$$f_1: A \times E \times Z^+ \to X \quad \text{and} \quad g_1: B \times Z^+ \to X$$

so that:

(7) $f_1(A \times E \times Z^+) \cap g_1(B \times Z^+) = \emptyset$ and $f_1(A \times E \times Z^+) \cup g_1(B \times Z^+)$ is a Hamel basis for X;

(8) $f_1(\beta, i, j) = f(\beta, i, j)$ if $\beta \in A$, $i \in E$, $0 \leq j \leq n - 1$;

(9) if L is the linear transformation on X defined by

$$L(f_1(\beta, i, j)) = f_1(\beta, i, j + 1) \quad \text{and} \quad L(g_1(b, j)) = g_1(b, j + 1)$$

for each b in B, β in A, i in E, j in Z^+, then

$$p(L)f_1(\beta, i, j) = f(\beta, i, j + n) \quad \text{and} \quad p(L)g_1(b, j) = g(b, j + n)$$

for each β in A, i in E, b in B, j in Z^+.

Now (i) follows from (4), (9), and (iii) follows from (5), (8), (9) upon considering $p(L)f(\beta, 1, 0)$. If q is a polynomial of degree k, $1 \leq k \leq n - 1$, then (ii) follows from (2), (3), (8), (9) upon considering $q(L)f(\alpha, k, 0)$.

LEMMA 2. *If V is a vector space that is the direct sum of subspaces S_1 and S_2, and if M is an infinite-dimensional subspace of V with* $\dim S_1 < \dim M$, *then* $\dim (M \cap S_2) = \dim M$.

Proof. We can choose a subspace N of V such that M is a direct sum of N and $M \cap S_2$. Since $N \cap S_2 = 0$, then $\dim N \leq \dim S_1 < \dim M$. Thus $\dim (M \cap S_2) = \dim M$.

LEMMA 3. *If X is a direct sum of subspaces S_1 and S_2 with* $\dim S_2 \geq$ card A, *then there is a subspace K of X and a collection $\{S_\alpha: \alpha \in A\}$ of pairwise disjoint subsets of K such that:*

(i) *X is a direct sum of S_1 and K;*

(ii) *$\bigcup S_\alpha$ is linearly independent;*

(iii) *$S_\alpha \cap M_\alpha = \emptyset$ for each α in A;*

(iv) card $S_\alpha \geq$ card A *for each α in A.*

Proof. For each ordinal α in A we define

$$G_\alpha = \{(\beta, \delta) \in A \times A : \delta \leq \beta < \alpha\}.$$

Let $G = \bigcup G_\alpha$. We will define a function $f: G \to X$ so that:

(1) $f(G)$ is linearly independent;

(2) $S_1 \cap \vee f(G) = 0$;

(3) $f(\beta, \delta) \notin M_\delta$ if $(\beta, \delta) \in G$.

Once f has been constructed we can let K be any subspace containing $f(G)$ and satisfying (i), and, for each α in A, let

$$S_\alpha = \{f(\delta, \alpha): \alpha \leq \delta, \delta \in A\}.$$

Suppose that α is in A and that $f(\beta, \delta)$ has been defined for all (β, δ) with $\delta \leq \beta < \alpha$ so that (1)–(3) hold when G is replaced by G_α. Since

$$\dim (\vee f(G_\alpha)) < \operatorname{card} A \leq \dim X \quad \text{and} \quad \operatorname{card} (\alpha + 1) < \dim X,$$

then we can choose (using another transfinite induction) vectors $f(\alpha, \delta)$ for $\delta \leq \alpha$ so that (1)–(3) hold when G is replaced by $G_{\alpha+1}$. Proceeding by transfinite induction we can extend f to all of G so that (1)–(3) hold.

The following lemma is similar to Lemma 1, and its proof is omitted.

LEMMA 4. *If V is a vector space with a Hamel basis $\{e_k: k \in Z\}$, and if p, q are polynomials such that $p(x) + q(1/x)$ is not of the form $mx + b$ $(m \neq 0)$, then there is a Hamel basis $\{f_k: k \in Z\}$ for V such that if L is the linear transformation defined on V by $L(f_k) = f_{k+1}$, for each k in Z, then:*

(i) $f_0 = e_0, f_1 = e_1$, *and* $\{e_k: k < 0\} \subset \{f_k: k \in Z\}$;

(ii) $[p(L) + q(L^{-1})]e_0 = e_2$;

(iii) $[p(L) + q(L^{-1})]e_k = e_{k+1}$ *if* $k \geq 2$.

THEOREM C′. *Suppose X is the direct sum of K and the infinite-dimensional subspaces N_t, $t \in T$, and that* $\dim K$, $\dim (\sum N_t) \geq \operatorname{card} A$. *If $\{p_t: t \in T\}$ and $\{q_t: t \in T\}$ are collections of polynomials such that, for each t in T, $p_t(x) + q_t(1/x)$ is not of the form $mx + b$ $(m \neq 0)$, then there is an $\mathcal{M}$-transitive bijective linear transformation L on X such that, for each t in T, $p_t(L) + q_t(L^{-1})$ leaves N_t invariant.*

Proof. By Lemma 3 we may suppose that $\dim (\vee (K \cap M_\alpha')) \geq \operatorname{card} A$ for each α in A. To simplify our notation we will suppose $s \notin T$ and define $T_1 = T \cup \{s\}$ and $N_s = K$. We are going to define a function $f: A \times T_1 \times Z \to X$ so that:

(1) $f(A \times T_1 \times Z) \cap \{0\}'$ is linearly independent;

(2) for each β in A, $f(\beta, t, 0) \neq 0$ for at least one but at most finitely many values of t in T_1;

(3) if $f(\beta, t, 0) \neq 0$, then $f(\beta, t, n) \neq 0$ for every n in Z;

(4) if $f(\beta, t, 0) = 0$, then $f(\beta, t, n) = 0$ for every n in Z;

(5) $f(\beta, t, n) \in N_t$ if $\beta \in A$, $t \in T_1$, $n \geq 0$, $n \neq 1$;

(6) $f(\beta, t, n) \in N_s = K$ if $\beta \in A$, $t \in T_1$, $n = 1$ or $n < 0$;

(7) $\sum_{t \in T_1} f(\beta, t, 0) \in M_\beta$ for each β in A;

(8) $\sum_{t \in T_1} f(\beta, t, 1) \notin M_\beta$ for each β in A;

Suppose that $\alpha \in A$ and f has been defined on $\alpha \times T_1 \times Z$ so that (1)–(8) hold when A is replaced by α. Let $V = \vee f(\alpha \times T_1 \times Z)$. Since V has a Hamel basis contained in $\bigcup N_t$, then V is the direct sum $\sum_{t \in T_1} V \cap N_t$.

For each t in T_1 we choose a subspace V_t of N_t so that N_t is a direct sum of $V \cap N_t$ and V_t. Then X is the direct sum $V + \sum V_t$.

We now consider the possibility that $0 < \dim V_t < \aleph_0$ for some t in T_1. In this case we choose the smallest ordinal β_t for which $f(\beta_t, t, 0) \neq 0$. We then redefine $f(\beta_t, t, n)$ for each n in Z so that $f(\{\beta_t\} \times \{t\} \times Z)$ contains a Hamel basis for N_t. Once this definition has been made for each appropriate t in T_1, we redefine V and the V_t's accordingly. We may therefore assume that, for each t in T_1, $V_t = 0$ or V_t is infinite-dimensional.

This redefinition does not affect our induction process since the redefinition occurs at most once for each t in T_1.

Since $\dim V < \operatorname{card} A \leq \dim M_\alpha$, it follows from Lemma 2 that there is a nonzero vector x in $M_\alpha \cap \sum_{t \in T_1} V_t$. For each t in T_1 we define $f(\alpha, t, 0)$ to be the component of x in V_t (relative to the direct sum of the V_t's).

Since $\dim N_s \geq \operatorname{card} A$, $\dim (\vee (N_s \cap M_\alpha')) \geq \operatorname{card} A$, and V_t is infinite-dimensional whenever $f(\alpha, t, 0) \neq 0$, then it follows that vectors $f(\alpha, t, n)$ can be chosen for each t in T_1 and n in $Z \cap \{0\}'$ so that (1)–(8) hold when A is replaced by $\{\beta: \beta \leq \alpha\}$.

We can therefore proceed by transfinite induction to extend f to all of $A \times T_1 \times Z$ so that (1)–(8) hold.

If we let $W = \vee f(A \times T_1 \times Z)$, then (as in the case with V) we can write W as the direct sum of the $W \cap N_t$'s ($t \in T_1$), and we can choose subspaces W_t of N_t, for each t in T_1, so that N_t is the direct sum of W_t and $W \cap N_t$.

As in the case of the V_t's we can redefine f, if necessary, to insure that, for each t in T_1, either $W_t = 0$ or W_t is infinite-dimensional.

We define a linear transformation L on X by defining L separately on W and on each W_t.

Using Lemma 4 we can define L bijectively on W so that $p_t(L) + q_t(L^{-1})$ sends $f(\alpha, t, 0)$ onto $f(\alpha, t, 2)$ and sends $f(\alpha, t, n)$ onto $f(\alpha, t, n + 1)$ for all α in A, t in T_1, $n \geq 2$.

On each nonzero W_t we define L so that L maps W_t bijectively onto W_t and leaves invariant no proper finite-dimensional subspaces of W_t.

From (7), (8) it follows that L is $\mathscr{M}$-transitive. Furthermore, for each t in T, it follows from (5), (6) that $p_t(L) + q_t(L^{-1})$ leaves both $W \cap N_t$ and W_t invariant. Since $N_t = (W \cap N_t) + W_t$ for each t in T, it follows that $p_t(L) + q_t(L^{-1})$ leaves N_t invariant for each t in T. This completes the proof.

Our remaining task is to show how Theorems A, B, C are derived from their algebraic counterparts. This is done in the following lemma. The proof of this lemma uses a theorem of Bessaga and Pelczynski [1, Theorem 1] which states that any separable infinite-dimensional Banach space contains a closed infinite-dimensional subspace that has a Schauder basis. In the case of a Hilbert space any orthonormal basis will do.

LEMMA 5. *Let Y be a separable infinite-dimensional Banach space. Then:*

(i) *Y has exactly $2^{\aleph_0}$ infinite-dimensional closed subspaces, and each of these subspaces has Hamel dimension $2^{\aleph_0}$;*

(ii) *there is a collection $\{N_t: t \in [0, 1]\}$ of infinite-dimensional closed subspaces of Y such that $\sum N_t$ is a linear direct sum.*

Proof. The fact that a separable infinite-dimensional Banach space has Hamel dimension at least $2^{\aleph_0}$ is well known and can be found in [2]. The rest of (i) can be deduced from (ii) using the fact that any separable metric space contains at most $2^{\aleph_0}$ closed subsets.

To prove (ii) let M be an infinite-dimensional closed subspace of Y which has a Schauder basis. We write the vectors in this basis as $e(m, n)$ for $(m, n) \in Z \times Z$. For each n in Z define M_n to be the closed subspace of Y spanned by $\{e(m, n): m \in Z\}$. Then each M_n is a closed infinite-dimensional subspace of Y and thus has Hamel dimension $2^{\aleph_0}$. Hence there is a function f: $[0, 1] \times Z \to M$ such that, for each n in Z, $f([0, 1] \times \{n\})$ is a Hamel basis for M_n. For each t in $[0, 1]$ we define N_t to be the closed subspace of Y spanned by $f(\{t\} \times Z)$. It is now easy to verify (ii).

We can see from the above theorems that on any separable infinite-dimensional Banach space there is a rich supply of (not necessarily bounded) transitive linear transformations.

However, there seems to be very little chance of using these techniques to construct a bounded transitive operator. There does not seem to be any nice way of describing continuity in terms of a Hamel basis.

It should be noted that the assumption that card $A > \aleph_0$ is not necessary. All of the proofs given here can easily be changed into standard induction proofs in the case when card $A = \aleph_0$.

It should also be noted that the conclusions of Theorems A, B, C hold when Y is replaced by the separable locally convex Frechet space (s) of all complex sequences with the coordinate seminorms. However, Johnson and Shields [3] proved that every continuous linear transformation on (s) which is not a scalar multiple of the identity has a hyperinvariant subspace.

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