LANDAU-TYPE INEQUALITIES FOR SOME LINEAR DIFFERENTIAL OPERATORS

BY

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1. Introduction

Recently Schoenberg [5] and later Cavaretta [1] have revived interest in Landau's problem and its generalization due to Kolmogorov. The methods of Schoenberg are quite general but require appropriate differentiation formulae which when higher derivatives are involved can be quite intricate. The approach of Cavaretta is first to prove the results for functions having an integral period and then to reduce the general case to this situation by a simple device.

In a forthcoming paper [3], Schoenberg extends Landau's problem in a different direction which is related to the study of cardinal $\mathscr{L}$-splines. For example he shows that if $\|f\|_\infty \leq A_0$, $\|f'' + f\|_\infty \leq 1$, then $\|f'\|_\infty \leq \sqrt{A_0(2 + A_0)}$, where equality holds for certain functions.

We propose to consider the linear differential operators $\mathscr{L}_{\nu,\gamma}$ given by

$$\mathscr{L}_{\nu,\gamma} \equiv (D - \gamma_1)\cdots(D - \gamma_\nu), \quad \nu = 1, 2, \ldots, n$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are real. In Section 2 we give an explicit expression for the generalized exponential Euler spline $\Phi_{n,\gamma}(x)$ introduced by Schoenberg [4] and state the first main result in Theorem 1, where the functions $\Phi_{n,\gamma}(x)$ come out as extremal functions. In particular we show that if

$$\|f\|_\infty \leq M_0, \quad \|(D^2 - \gamma^2)f\|_\infty \leq M_2$$

then $\|(D \pm \gamma)f\|_\infty \leq \sqrt{M_0(2M_2 - M_0\gamma^2)}$, provided $M_0\gamma^2 \leq M_2$. It is interesting to observe that Schoenberg [3] obtained the same bound for $\|f'\|_\infty$. In Section 6, using a different method we are able to show that the same bound holds for $\|f' - \alpha f\|_\infty$ provided

$$|\alpha| < M_2\{M_0(2M_2 - M_0\gamma^2)\}^{-1/2},$$

which includes our result also as a special case.

In Section 3 we sketch the proof of Theorem 1 which is patterned on the lines of Cavaretta's proof [1]. In Section 4 we prove our second main result as Theorem 2 using a different approach based on a Theorem of Hirschman and Widder [2]. Section 5 deals with some special cases where $M_n \leq M_0|\gamma_1\gamma_2\cdots\gamma_n|$. The results of Theorem 3 and the corollaries of Section 5 complement in a sense the results of Section 2 and Section 6.

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2. Generalized exponential spline $\Phi_{n,\gamma}(x)$

We shall denote by $\Phi_{n,\gamma}(x; t)$ the exponential $\mathscr{L}$-spline determined by the following conditions:

(i) $D\mathscr{L}_{n,\gamma}\Phi_{n,\gamma}(x; t) = 0$ in $(\nu\eta, \nu\eta + \eta)$ for every integer ν,
(ii) $\Phi_{n,\gamma}(x; t) \in C^{n-1}(\mathbf{R})$,
(iii) $\Phi_{n,\gamma}(x + \eta; t) = t\Phi_{n,\gamma}(x; t)$,

where $\eta > 0$ is a constant. Here γ refers to the vector $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are real numbers. When $t = -1$, we shall denote the $\mathscr{L}$-spline by $\Phi_{n,\gamma}(x)$.

Let $A_{n,\gamma}(x; t)$ denote the restriction of $\Phi_{n,\gamma}(x; t)$ to $[0, \eta)$. Schoenberg has shown [4] that if $0 < x < \eta$, then $A_{n,\gamma}(x; t)$ has n simple and negative t-zeros and if $x = 0$, it has $n - 1$ simple and negative t-zeros. Earlier we have shown [6] that if $\gamma_1, \gamma_2, \ldots, \gamma_n$ are distinct and nonzero, then

$$A_{n,\gamma}(x; t) = 1 + (1 - t) \sum_{\nu=1}^{n} \frac{e^{\gamma_\nu x}}{t - e^{\gamma_\nu \eta}} \frac{1}{\prod_{\substack{k=1 \\ k \neq \nu}}^{n} \left(1 - \dfrac{\gamma_\nu}{\gamma_k}\right)}.$$

Multiplying the above by a suitable constant it is easy to see that $A_{n,\gamma}(x; t)$ is the divided difference of $(t - 1)e^{yx}/(t - e^{y\eta})$ with respect to the variable y at the nodes $0, \gamma_1, \ldots, \gamma_n$. In the rest of the paper we shall set

$$A_{n,\gamma}(x; t) = [0, \gamma_1, \ldots, \gamma_n; e^{yx}(t - 1)/(t - e^{y\eta})]. \tag{2.1}$$

This formulation is valid without any restriction on γ's. From (2.1) it follows easily that

$$(D - \gamma_n)A_{n,\gamma}(x; t) = A_{n-1,\gamma}(x; t) \tag{2.2}$$

where $\gamma = (\gamma_1, \ldots, \gamma_{n-1})$ in $A_{n-1,\gamma}(x; t)$.

Schoenberg also showed [4] that $A_{n,\gamma}(x; t)$ for a fixed $t < 0$ has exactly one simple zero in $0 \leq x < \eta$. We shall give a direct short proof of this assertion. This is obviously true for $n = 1$. If we suppose that this is true for $A_{n-1,\gamma}(x; t)$, then it follows from (2.2) that $e^{-\gamma_n x}A_n(x; t)$ is bell-shaped and hence can have at most two zeros in $[0, \eta)$. Since $A_{n,\gamma}(\eta; t) = tA_{n,\gamma}(0; t)$ it follows that $A_{n,\gamma}(\eta; t)$ and $A_{n,\gamma}(0; t)$ are of opposite sign when $t < 0$. Hence $A_{n,\gamma}(x; t)$ can have only an odd number of zeros in $[0, \eta)$ which implies that it can have only one simple zero there. This proves the assertion.

We observe that because of (iii), $\Phi_{n,\gamma}(x) \equiv \Phi_{n,\gamma}(x; -1)$ is periodic with period 2η. Set

$$\Gamma_{n,\nu}^{(\gamma)} = \|\mathscr{L}_{\nu,\gamma}\Phi_{n,\gamma}(x)\|_\infty, \quad \nu = 0, 1, \ldots, n.$$

We now consider the class $\mathscr{A}$ of real-valued functions defined on $\mathbf{R}$ which satisfy the conditions:

(i) $f(x) = 0(1)$ as $|x| \to \infty$.
(ii) $f(x) \in C^{n-1}(\mathbf{R})$,
(iii) $f^{(n-1)}(x)$ is piecewise continuously differentiable for all real x,
(iv) $\mathscr{L}_{n,\gamma} f$ is bounded.

We now formulate:

THEOREM 1. *If $f(x) \in \mathscr{A}$ and satisfies*

$$\|f\|_\infty \leq \Gamma_{n,0}^{(\gamma)}, \quad \|\mathscr{L}_{n,\gamma} f\|_\infty \leq \Gamma_{n,n}^{(\gamma)} = 1 \tag{2.3}$$

then

$$\|\mathscr{L}_{\nu,\gamma} f\|_\infty \leq \Gamma_{n,\nu}^{(\gamma)}, \quad \nu = 1, 2, \ldots, n-1. \tag{2.4}$$

The bounds in (2.4) *are sharp, since they are attained when $f = \Phi_{n,\gamma}(x)$.*

For a more general formulation of the theorem, let $F(x) \in \mathscr{A}$. Set

$$\|\mathscr{L}_{\nu,\gamma} F\|_\infty = M_\nu, \quad \nu = 0, 1, \ldots, n \tag{2.5}$$

and $f(x) = aF(bx)$. Choose a and b such that

$$\|f\|_\infty = \Gamma_{n,0}^{(b\gamma)} \quad \text{and} \quad \|\mathscr{L}_{n,b\gamma} f\|_\infty = \Gamma_{n,n}^{(b\gamma)} = 1.$$

Then the equations which determine a and b are

$$aM_0 = \Gamma_{n,0}^{(b\gamma)}, \quad ab^n M_n = 1. \tag{2.5a}$$

We then have:

COROLLARY 1. *Suppose M_0, M_n are such that the following equation in b*

$$b^n M_n \Gamma_{n,0}^{(b\gamma)} = M_0 \tag{2.6}$$

has a solution. If $F \in \mathscr{A}$ and $\|F\|_\infty = M_0$ and $\|\mathscr{L}_{n,\gamma} F\|_\infty = M_n$, then

$$M_\nu \equiv \|\mathscr{L}_{\nu,\gamma} F\|_\infty \leq C_{n,\nu}^{(\gamma)} M_0^{1-(\nu/n)} M_n^{\nu/n} \tag{2.7}$$

where

$$C_{n,\nu}^{(\gamma)} = \Gamma_{n,\nu}^{(b\gamma)} \cdot (\Gamma_{n,0}^{(b\gamma)})^{\nu/n-1} \tag{2.8}$$

and b is given by (2.6).

Example 1. Let $n = 2$. Then the restriction of $\Phi_{2,\gamma}(x)$ to $[0, \eta]$ is given by $A_{2,\gamma}(x; -1)$ where

$$A_{2,\gamma}(x; -1) = [0, \gamma_1, \gamma_2; 2e^{\gamma x}/(1 + e^{\gamma\eta})].$$

If $\gamma_2 = -\gamma_1 = \gamma > 0$, then

$$A_{2,\gamma}(x; -1) = -\frac{1}{\gamma^2}\left[1 - \frac{\cosh(x - \eta/2)\gamma}{\cosh \gamma\eta/2}\right].$$

Hence

$$\Gamma_{n,0}^{(b\gamma)} = \frac{1}{b^2\gamma^2}\left[1 - \frac{1}{\cosh b\gamma\eta/2}\right].$$

Similarly

$$\Gamma_{2,1}^{(\gamma)} = \|\mathscr{L}_{1,\gamma}A_{2,\gamma}(x; -1)\|_\infty = \left\|-\frac{1}{\gamma} + \frac{2e^{\gamma x}}{\gamma(1 + e^{\gamma\eta})}\right\|_\infty = \frac{1}{\gamma}\tanh\frac{\gamma\eta}{2},$$

so that

$$\Gamma_{2,1}^{(b\gamma)} = \frac{1}{b\gamma}\tanh\frac{b\gamma\eta}{2}.$$

Also $\Gamma_{2,2}^{(b\gamma)} = 1$. Then (2.6) becomes

$$M_2\left[1 - \frac{1}{\cosh b\gamma\eta/2}\right] = M_0\gamma^2$$

and from (2.8),

$$C_{2,1}^{(\gamma)} = \tanh\frac{b\gamma\eta}{2}\cdot\left(1 - \frac{1}{\cosh b\gamma\eta/2}\right)^{-1/2} = \left(2 - \frac{M_0\gamma^2}{M_2}\right)^{1/2}.$$

Thus we have proved that if $\|f\|_\infty \le M_0$, $\|(D^2 - \gamma^2)f\|_\infty \le M_2$ and $M_0\gamma^2 < M_2$, then

$$\|(D \pm \gamma)f\|_\infty \le \sqrt{M_0(2M_2 - M_0\gamma^2)}. \tag{2.9}$$

It is interesting to observe that this bound was obtained by Schoenberg for $\|Df\|_\infty$. We shall return to a more general problem later in Section 6.

3. Proof of Theorem 1

The proof is similar to that of Cavaretta [1].

(a) We will first prove the theorem for periodic functions with integral multiple of η as a period. Suppose $f(x + k\eta) = f(x)$ where k is an even integer. From Section 2, we know that

$$\Phi_{n,\gamma}(x + k\eta) = \Phi_{n,\gamma}(x) \tag{3.1}$$

and also that $\Phi_{n,\gamma}(x)$ has exactly one zero in $[0, \eta)$, so that there are exactly k distinct zeros of $\Phi_{n,\gamma}(x)$ in any period of length $k\eta$. Since

$$\Phi_{n,\gamma}(x + \eta) = -\Phi_{n,\gamma}(x),$$

there are at least k disjoint or abutting intervals in an interval of length $k\eta$ such that in each interval $\Phi_{n,\gamma}(x)$ takes its maximum $\Gamma_{n,0}^{(\gamma)}$ with positive and negative sign alternately. Set

$$\|\mathscr{L}_{\nu,\gamma}f\|_\infty = \alpha\Gamma_{n,0}^{(\gamma)}. \tag{3.2}$$

We will assume $\alpha > 1$ and arrive at a contradiction. Choose x_0 such that

$$|\mathscr{L}_{v,\gamma}f(x_0)| = \|\mathscr{L}_{v,\gamma}f\|_\infty \tag{3.3}$$

and we now choose x_1 such that

$$\alpha\mathscr{L}_{v,\gamma}\Phi_{n,\gamma}(x_0 - x_1) = \mathscr{L}_{v,\gamma}f(x_0). \tag{3.4}$$

We now set $h(x) = \Phi_{n,\gamma}(x - x_1) - (1/\alpha)f(x)$. Since $1/\alpha < 1$, $h(x)$ has at least one zero in each of the above intervals.

Again from $(D - \gamma_j)g = e^{\gamma_j x}D(e^{-\gamma_j x}g)$, we can use Polya's generalized Rolle's theorem and see that $\mathscr{L}_{v,\gamma}h(x)$ has at least k distinct zeros in a period for $v < n - 1$. Since x_0 is a local maximum for $\mathscr{L}_{v,\gamma}f(x)$ and also for $\mathscr{L}_{v,\gamma}\Phi_{n,\gamma}(x - x_1)$ we have

$$D\mathscr{L}_{v,\gamma}f(x_0) = D\mathscr{L}_{v,\gamma}\Phi_{n,\gamma}(x_0 - x_1) = 0$$

so that $D\mathscr{L}_{v,\gamma}h(x_0) = 0$. Hence

$$\begin{aligned}\mathscr{L}_{v+1,\gamma}h(x_0) &= (D - \gamma_{v+1})\mathscr{L}_{v,\gamma}h(x_0)\\ &= -\gamma_{n+1}[\mathscr{L}_{v,\gamma}\Phi_{n,\gamma}(x_0 - x_1) - (1/\alpha)\mathscr{L}_{v,\gamma}f(x_0)]\\ &= 0,\end{aligned} \tag{3.5}$$

so that again using Rolle's generalized theorem it follows from (3.5) that $\mathscr{L}_{v+1,\gamma}h(x)$ has at least $k + 1$ distinct zeros in a k-period. By another use of the Rolle's theorem again, we see that $\mathscr{L}_{n,\gamma}h(x)$ has at least $k + 1$ sign changes.

Since $1/\alpha < 1$, $|\mathscr{L}_{n,\gamma}f(x)| \leq 1 = \|\mathscr{L}_{n,\gamma}\Phi_{n,\gamma}(x - x_1)\|$ and since

$$\mathscr{L}_{n,\gamma}\Phi_{n,\gamma}(x - x_1)$$

is a step function with exactly k sign changes in a period, it follows that $\mathscr{L}_{n,\gamma}h(x)$ has also exactly k sign changes which is a contradiction. Hence $\alpha \leq 1$.

The case $v = n - 1$ is proved similarly with minor modifications.

(b) In order to prove the theorem in the general case we shall first show that if $f \in \mathscr{A}$ and satisfies (2.3), then $M_v \equiv M_v(f) = \|\mathscr{L}_{v,\gamma}(f)\|$ is bounded. For this it is enough to show that M_1 is bounded.

We set

$$p(x) = \beta\begin{vmatrix} 1 & e^{\gamma_1 x} & \cdots & e^{\gamma_n x}\\ 1 & e^{\eta\gamma_1} & \cdots & e^{\eta\gamma_n}\\ \cdots & \cdots & \cdots & \cdots\\ 1 & e^{n\eta\gamma_1} & \cdots & e^{n\eta\gamma_n}\end{vmatrix}.$$

Then $\eta, 2\eta, \ldots, n\eta$ are the only zeros of $p(x)$, and they are simple. We choose $\beta > 0$ such that

$$|\mathscr{L}_{n,\gamma}p| > M_n, \tag{3.6}$$

$$\min_{0\leq j\leq n}\left|p\left(\frac{\eta}{2} + j\eta\right)\right| > M_0. \tag{3.7}$$

Define $c < \eta$ and $d > n\eta$ such that

$$|p(c)| = M_0 \quad \text{and} \quad |p(d)| = M_0.$$

Let $m_1 = \max_{x \in [c,d]} |\mathscr{L}_{1,\gamma} p(x)|$. We claim that $M_1 \le m_1$. For if not there is an x_0 such that $|\mathscr{L}_{1,\gamma} f(x_0)| > m_1$. From (3.7) it follows that there is an $x_1 \in (c, d)$ such that $p(x_1) = f(x_0)$. Consider the function

$$H(x) = p(x) - f(x + x_0 - x_1).$$

Then $H(x_1) = 0$ and $|\mathscr{L}_{1,\gamma} f(x_0)| > |\mathscr{L}_{1,\gamma} p(x_1)|$. In other words,

$$e^{-\gamma_1 x_1} |(D - \gamma_1) f(x + x_0 - x_1)|_{x=x_1} > e^{-\gamma_1 x_1} |(D - \gamma_1) p(x)|_{x=x_1},$$

i.e.,

$$|D\{e^{-\gamma_1 x} f(x + x_0 - x_1)\}|_{x=x_1} > |D(e^{-\gamma_1 x} p(x))|_{x=x_1}. \tag{3.8}$$

From simple geometric considerations we see on using (3.8) that $H(x)$ has at least $n + 1$ zeros in $[c, d]$. By generalized Rolle's theorem, it follows that $\mathscr{L}_{n,\gamma} H(x)$ has at least one zero in $[c, d]$, but this is impossible because of (3.6).

This completes the proof that M_1 is bounded. We now associate a periodic function $F(x)$ with the given function $f(x)$ and show that $M_\nu(F)$ is close to $M_\nu(f)$. Following Cavaretta [1] we set

$$g(x) = \begin{cases} 1, & -1 \le x \le 1, \\ (-1)^n (x - 2)^n \sum_{k=0}^{n-1} \binom{n + k - 1}{k} (x - 1)^k, & 1 < x < 2, \\ (-1)^n (x + 2)^n \sum_{0}^{n-1} \binom{n + k - 1}{k} (x + 1)^k, & -2 < x < -1, \\ 0, & |x| \ge 2. \end{cases}$$

Set $F_k(x) = f(x) g(x/k\eta)$ for $-2k\eta \le x \le 2k\eta$ and $F_k(x + 4k\eta) = F_k(x)$ for all x. Since $\mathscr{L}_{\nu,\gamma} = \prod_{j=1}^{\nu} (D - \gamma_j)$ is a polynomial in D of degree ν, the remaining proof is exactly the same as that of Cavaretta with very minor modifications.

4. A different approach

In Section 2, we observe in Corollary 1 that (2.7) holds only when (2.6) has a solution. In Example 1 also we give the bound on $|(D \pm \gamma) f|$ under the condition $M_0 \gamma^2 < M_2$. In the case when this is not so we can use the following theorem of Hirschman and Widder in some situations.

THEOREM (Hirschman and Widder [2]). *Suppose $\phi(x)$ is continuous and bounded on* $\mathbf{R}$. *Then the only function satisfying*

$$\mathscr{L}_{n,\gamma} f = \phi(x), \quad x \in \mathbf{R}, \tag{4.1}$$

$$f(x) = \begin{cases} 0(e^{\alpha_2 x}), & x \to +\infty, \\ 0(e^{\alpha_1 x}), & x \to -\infty, \end{cases} \tag{4.2}$$

where α_1 is the largest negative γ_k (or $-\infty$ if all the γ_k in Section 1 are positive) and α_2 is the smallest γ_k (or $+\infty$ if all the γ_k are negative) is given by

$$f(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)\, dt \tag{4.3}$$

where $G(t) = g_1 * g_2 * \cdots * g_n$, $g_k(t) = (\operatorname{sgn} \gamma_k) g(\gamma_k t)$ and

$$g(t) = \begin{cases} e^t, & -\infty < t < 0, \\ 1/2, & t = 0, \\ 0, & 0 < t < \infty. \end{cases} \tag{4.4}$$

Consider the operator $P_m(D) = \sum_0^m a_k D^k$, $1 \le m \le n - 1$, $a_m \ne 0$. Then by the above theorem

$$P_m(D)f(0) = \int_{-\infty}^{\infty} K_{m,\gamma}(t)\phi(t)\, dt \tag{4.5}$$

where

$$K_{m,\gamma}(t) = [P_m(D)G(x - t)]_{x=0}. \tag{4.6}$$

By a reasoning similar to that of Hirschman and Widder ([2], p. 91, Theorem 5.1) we can show that $K_{m,\gamma}(t)$ does not have more than m changes of sign for any choice of $\gamma_1, \gamma_2, \ldots, \gamma_n$.

Set $\phi_0(t) = \operatorname{sgn} K_{m,\gamma}(t)$ and

$$f_{0,\gamma}(t) = \int_{-\infty}^{\infty} G(x - t)\phi_0(t)\, dt. \tag{4.7}$$

Then $f_{0,\gamma}$ is bounded and $\mathscr{L}_{n,\gamma} f_{0,\gamma} = \phi_0(x)$. Also

$$P_m(D) f_{0,\gamma}(0) = \int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt. \tag{4.8}$$

On the other hand if f is any function which satisfies (4.2) and $\|\mathscr{L}_{n,\gamma} f\|_\infty \le 1$, then by (4.5),

$$|P_m(D)f(0)| \le \int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt.$$

Since the point $x = 0$ is as good as any other, we have proved:

THEOREM 2. *If $f(x)$ satisfies* (4.2) *and if $\|\mathscr{L}_{n,\gamma} f\|_\infty \le 1$, then*

$$\|P_m(D)f\| \le \int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt \tag{4.9}$$

where $K_{m,\gamma}(t)$ is given by (4.5). *The bound in* (4.9) *is sharp.*

In particular, we have:

COROLLARY 2. *If* $\|f\|_\infty \leq M_0$ *where* M_0 *is a constant* $\geq \|f_{0,\gamma}\|$ *and if*

$$\|\mathscr{L}_{n,\gamma}f\| \leq 1,$$

then (4.9) *holds.*

A slightly more general result is given by

COROLLARY 3. *Suppose* $\|f\|_\infty \leq M_0$, $\|\mathscr{L}_{n,\gamma}f\|_\infty \leq M_n$ *are such that there exists a real positive b which satisfies the condition*

$$\|f_{0,b\gamma}\| b^n M_n \leq M_0. \tag{4.10}$$

Then

$$\|P_m(D)f\|_\infty \leq M_n \int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt. \tag{4.11}$$

Proof. In order to prove (4.11), we set $F(x) = af(bx)$. Then $\|F\|_\infty = aM_0$ and $\|\mathscr{L}_{n,b\gamma}F\|_\infty = ab^n M_n$. Choose a and b such that F satisfies the hypothesis of Corollary 2. Then $aM_0 \geq \|f_{0,b\gamma}\|$, $ab^n M_n = 1$. Choose b to be a solution of (4.10) and $a \geq \|f_{0,b\gamma}\|/M_0$. By (4.9), we have

$$\begin{aligned} b^m \|P_m(D/b)F\|_\infty &= ab^m \|P_m(D)f\|_\infty \\ &\leq b^m \int_{-\infty}^{\infty} |K_{m,b\gamma}(t)|\, dt \\ &= b^{m-n} \int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt. \end{aligned}$$

Hence

$$\|P_m(D)f\|_\infty \leq M_n \int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt.$$

5. Some special cases

(a) $n = 2, \gamma_2 > 0, \gamma_1 < 0$. In this case

$$G(t) = \begin{cases} -\dfrac{e^{\gamma_1 t}}{\gamma_2 - \gamma_1}, & t \geq 0, \\[2ex] -\dfrac{e^{\gamma_2 t}}{\gamma_2 - \gamma_1}, & t \leq 0. \end{cases}$$

Here $P_1(D) = D - \alpha$ so that from (4.5),

$$\begin{aligned} f'(0) - \alpha f(0) &= \int_{-\infty}^{\infty} \{G'(-t) - \alpha G(-t)\}\phi(t)\, dt \\ &= \int_{-\infty}^{\infty} K_{1,\gamma}(t)\phi(t)\, dt \end{aligned}$$

where

$$K_{1,\gamma}(t) = \begin{cases} \dfrac{\alpha - \gamma_2}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}, & t > 0, \\ \dfrac{\alpha - \gamma_1}{\gamma_2 - \gamma_1} e^{-\gamma_1 t}, & t < 0. \end{cases} \tag{5.1}$$

Hence

$$\phi_0(t) = \operatorname{sgn} K_{1,\gamma}(t) = \begin{cases} \operatorname{sgn}(\alpha - \gamma_1), & t < 0, \\ \operatorname{sgn}(\alpha - \gamma_2), & t > 0, \end{cases}$$

so that (4.7) gives on simplification

$$f_{0,\gamma}(x) = \begin{cases} \dfrac{\operatorname{sgn}(\alpha - \gamma_1) - \operatorname{sgn}(\alpha - \gamma_2)}{\gamma_1(\gamma_2 - \gamma_1)} e^{\gamma_1 x} + \dfrac{\operatorname{sgn}(\alpha - \gamma_2)}{\gamma_2\gamma_1}, & x \geq 0, \\ \dfrac{\operatorname{sgn}(\alpha - \gamma_2) - \operatorname{sgn}(\alpha - \gamma_1)}{\gamma_2(\gamma_1 - \gamma_2)} e^{\gamma_2 x} + \dfrac{\operatorname{sgn}(\alpha - \gamma_1)}{\gamma_1\gamma_2}, & x \leq 0. \end{cases}$$

Then

$$\|f_{0,\gamma}\| = \frac{1}{|\gamma_1\gamma_2|}, \quad \|f_{0,b\gamma}\| = \frac{1}{b^2|\gamma_1\gamma_2|}.$$

Condition (4.10) thus becomes

$$M_2 \leq M_0|\gamma_1\gamma_2|. \tag{5.2}$$

From (5.1), we have

$$\int_{-\infty}^{\infty} |K_{1,\gamma}(t)|\, dt = \frac{1}{\gamma_2 - \gamma_1}\left[\frac{|\alpha - \gamma_2|}{\gamma_2} - \frac{|\alpha - \gamma_1|}{\gamma_1}\right]. \tag{5.3}$$

Finally from Corollary 3, we get:

COROLLARY 4. *Suppose* $\gamma_1 < 0, \gamma_2 > 0$. *If*

$$\|f\|_\infty \leq M_0, \quad \|(D - \gamma_1)(D - \gamma_2)f\|_\infty \leq M_2$$

and if (5.2) *is satisfied, then*

$$|f'(x) - \alpha f(x)| \leq \frac{M_2}{\gamma_2 - \gamma_1}\left\{\frac{|\alpha - \gamma_2|}{\gamma_2} - \frac{|\alpha - \gamma_1|}{\gamma_1}\right\} \tag{5.4}$$

where α *is any real number.*

(b) $n = 2, 0 < \gamma_1 < \gamma_2$. In this case

$$G(t) = \begin{cases} 0, & t \geq 0, \\ \dfrac{e^{\gamma_1 t} - e^{\gamma_2 t}}{\gamma_2 - \gamma_1}, & t \leq 0. \end{cases}$$

Also

$$K_{1,\gamma}(t) = \begin{cases} 0, & t < 0, \\ \dfrac{(\gamma_1 - \alpha)e^{-\gamma_1 t} - (\gamma_2 - \alpha)e^{-\gamma_2 t}}{\gamma_2 - \gamma_1}, & t > 0. \end{cases} \tag{5.5}$$

We consider the cases (i) $\alpha < \gamma_1$ and (ii) $\alpha \geq \gamma_1$. In case (i), $K_{1,\gamma}(t)$ has only one change of sign at τ_α where

$$\tau_\alpha = \frac{1}{\gamma_2 - \gamma_1} \ln \frac{\gamma_2 - \alpha}{\gamma_1 - \alpha}. \tag{5.6}$$

In this case we choose

$$\phi_0(t) = \begin{cases} -1 & \text{for } t < \tau_\alpha, \\ +1 & \text{for } t > \tau_\alpha. \end{cases}$$

Then

$$f_{0,\gamma}(x) = \begin{cases} \dfrac{1}{\gamma_1\gamma_2}, & x > \tau_\alpha \\ -\dfrac{1}{\gamma_1\gamma_2} + \dfrac{2}{\gamma_2 - \gamma_1}\left[\dfrac{e^{\gamma_1(x-\tau_\alpha)}}{\gamma_1} - \dfrac{e^{\gamma_2(x-\tau_\alpha)}}{\gamma_2}\right], & x < \tau_\alpha. \end{cases} \tag{5.7}$$

Since $f'_{0,\gamma}(x) > 0$ for $x < \tau_\alpha$, it follows that $\|f_{0,\gamma}\|_\infty = 1/\gamma_1\gamma_2$. Further

$$\int_{-\infty}^{\infty} |K_{1,\gamma}(t)|\, dt = \frac{\alpha}{\gamma_1\gamma_2} + \frac{2(\gamma_1 - \alpha)e^{-\gamma_1\tau_\alpha}}{\gamma_1\gamma_2}. \tag{5.8}$$

However in case (ii), $K_{1,\gamma}(t)$ is negative for $t > 0$ and then we choose $\phi_0(t) = -1$ for all t. Here $f_{0,\gamma}(x) = -1/\gamma_1\gamma_2$ and $\|f_{0,\gamma}\| = 1/\gamma_1\gamma_2$. Further

$$\int_{-\infty}^{\infty} |K_{1,\gamma}(t)|\, dt = \frac{\alpha}{\gamma_1\gamma_2}.$$

Combining cases (i) and (ii), we have (as in Corollary 4):

COROLLARY 5. *Suppose* $0 < \gamma_1 < \gamma_2$ *and suppose*

$$\|f\|_\infty \leq M_0, \quad \|(D - \gamma_1)(D - \gamma_2)f\| \leq M_2.$$

If (5.2) *is satisfied, then*

$$\|f' - \alpha f\|_\infty \leq \begin{cases} \dfrac{\alpha M_2}{\gamma_1\gamma_2}, & \alpha \geq \gamma_1 \\ \dfrac{\alpha M_2}{\gamma_1\gamma_2} + \dfrac{2M_2(\gamma_1 - \alpha)}{\gamma_1\gamma_2} e^{-\gamma_1\tau_\alpha}, & \alpha < \gamma_1. \end{cases} \tag{5.9}$$

(c) $n > 2$, $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$. In this case we can see by induction that

$$G_n(t) = \begin{cases} 0, & t > 0, \\ (-1)^{n+1}[\gamma_1, \ldots, \gamma_n; e^{\gamma t}], & t < 0, \end{cases} \tag{5.10}$$

where $[\gamma_1, \ldots, \gamma_n; e^{\gamma t}]$ is the divided difference with respect to γ of $e^{\gamma t}$ on the nodes $\gamma_1, \ldots, \gamma_n$. In other words for $t < 0$,

$$G_n(t) = (-1)^{n+1} \sum_1^n \frac{e^{\gamma_k t}}{\omega'(\gamma_k)} \quad \text{where } \omega(\gamma) = \prod_1^n (\gamma - \gamma_k). \tag{5.11}$$

For any positive integer $m \le n - 1$, let $P_m(D) = \sum_0^m a_k D^k$, $a_m \ne 0$. Suppose $P_m(x)$ satisfies the condition

(5.12) $\varepsilon(-1)^k P_m^{(k)}(\gamma) \le 0$ for $\gamma_1 < \gamma < \gamma_n$, $\varepsilon = \pm 1$ $(k = 0, 1, \ldots, m)$.

Then from (4.6), we have

$$K_{m,\gamma}(t) = [P_m(D)G_n(x - t)]_{x=0}$$

(5.13)
$$= \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k} P_m^{(k)}(\bar\gamma)(-1)^k t^{n-k-1} e^{-\bar\gamma t}.$$

where $\gamma_1 < \bar\gamma < \gamma_n$ and the last equality follows from (5.11) and the mean-value theorem.

We set $\phi_0(t) = -\varepsilon$ $(\varepsilon = \pm 1)$. Then

$$f_{0,\gamma}(x) = -\int_{-\infty}^{\infty} G_n(x - t)\, dt = -\int_{-\infty}^{\infty} G_n(t)\, dt = -\frac{\varepsilon}{\gamma_1\gamma_2\cdots\gamma_n}.$$

Also

$$\int_{-\infty}^{\infty} |K_{m,\gamma}(t)|\, dt = (-1)^n \sum_1^n \frac{1}{\gamma_k}\frac{P_m(\gamma_k)}{\omega'(\gamma_k)} = (-1)^n \left[\gamma_1, \ldots, \gamma_n; \frac{P_m(\gamma)}{\gamma}\right]\varepsilon.$$

We have thus proved:

THEOREM 3. *Let* $0 < \gamma_1 < \cdots < \gamma_n$. *Suppose* $\|f\|_\infty \le M_0$, $\|\mathscr{L}_{n,\gamma}f\|_\infty \le M_n$ *and*

$$M_n \le M_0\gamma_1\gamma_2\cdots\gamma_n.$$

If $P_m(x)$ *is a polynomial of degree* m $(1 \le m \le n - 1)$ *satisfying* (5.12), *then*

(5.13)
$$\|P_m(D)f\|_\infty \le M_n|[\gamma_1, \ldots, \gamma_n; P_m(\gamma)/\gamma]|.$$

6. Case $n = 2$, $\gamma_2 = -\gamma_1 = \gamma > 0$

Suppose $\|f\|_\infty = M_0$ and $\|(D^2 - \gamma^2)f\|_\infty = 1$ where $M_0\gamma^2 < 1$. We are interested in finding the best bound for $\|(D - \alpha)f\|_\infty$. If $\alpha = \pm\gamma$, we have this result in (2.9). If $\alpha = 0$, Schoenberg [3] gave the best bound which turns out to be the same number as in (2.9). We shall follow the idea of Schoenberg and find an expression for $f'(0) - \alpha f(0)$.

Indeed we have

(6.1)
$$f'(0) - \alpha f(0) = Af(-\eta) + Bf(\eta) + Rf$$

where $Rf = 0$ when $f = e^{\gamma x}$ and $e^{-\gamma x}$.

Then we have

(6.2)
$$A = -\tfrac{1}{2}(\alpha \operatorname{sech} \gamma\eta + \gamma \operatorname{cosech} \gamma\eta) = -\frac{1}{2}\frac{\alpha \sinh \gamma\eta + \gamma \cosh \gamma\eta}{\cosh \gamma\eta \sinh \gamma\eta}$$
$$B = -\tfrac{1}{2}(\alpha \operatorname{sech} \gamma\eta - \gamma \operatorname{cosech} \gamma\eta) = -\frac{1}{2}\frac{\alpha \sinh \gamma\eta - \gamma \cosh \gamma\eta}{\cosh \gamma\eta \sinh \gamma\eta}.$$

It follows from Peano's theorem that

$$Rf = \int_{-\eta}^{\eta} K(t)(D^2 - \gamma^2)f(t)\,dt \tag{6.3}$$

where

$$K(t) = \begin{cases} \dfrac{\alpha \sinh \gamma\eta - \gamma \cosh \gamma\eta}{\gamma \sinh 2\gamma\eta} \sinh \gamma(\eta - t), & t > 0 \\ \dfrac{\alpha \sinh \gamma\eta + \gamma \cosh \gamma\eta}{\gamma \sinh 2\gamma\eta} \sinh \gamma(\eta + t), & t < 0. \end{cases} \tag{6.4}$$

Choose η such that

$$(1/\gamma^2 - M_0) \cosh \gamma\eta = +1/\gamma^2. \tag{6.5}$$

Set

$$f_0(x) = -(1/\gamma^2 - M_0) \cosh \gamma(x - \eta) + 1/\gamma^2, \quad 0 \le x \le 2\eta \tag{6.6}$$

and $f_0(x + 2\eta) = -f_0(x)$ for all x. Hence $\|f_0(x)\|_\infty = M_0$. Also

$$(D^2 - \gamma^2)f_0 = \begin{cases} -1, & 0 \le x \le 2\eta \\ +1, & -2\eta \le x \le 0. \end{cases}$$

If $|\alpha| < \gamma \coth \gamma\eta$, then $K(t) < 0$ for $t > 0$ and is >0 for $t < 0$. Hence

$$\begin{aligned} f_0'(0) - \alpha f_0(0) &= -M_0 A + BM_0 + \int_{-\eta}^{\eta} |K(t)|\,dt \\ &= M_0 \frac{\gamma}{\sinh \gamma\eta} + \int_{-\eta}^{0} K(t)\,dt - \int_0^{\eta} K(t)\,dt. \end{aligned}$$

Using (6.5), we get

$$|f_0'(0) - \alpha f_0(0)| = \sqrt{M_0(2 - M_0\gamma^2)}. \tag{6.7}$$

From (6.1) and (6.3), we get

$$\begin{aligned} |f'(0) - \alpha f(0)| &\le |A|M_0 + |B|M_0 + \int_{-\eta}^{\eta} |K(t)|\,dt \\ &= |f_0'(0) - \alpha f_0(0)|. \end{aligned}$$

By an appropriate change of scale we get on using (6.5):

THEOREM 4. *Suppose* $\|f\|_\infty \le M_0$, $\|f'' - \gamma^2 f\|_\infty \le M_2$ *where* $M_0\gamma^2 < M_2$. *Let* α *be a real number with* $|\alpha| < M_2\{M_0(2M_2 - M_0\gamma^2)\}^{-1/2}$, *then*

$$\|f'(x) - \alpha f(x)\|_\infty \le \sqrt{M_0(2M_2 - M_0\gamma^2)}. \tag{6.8}$$

Remark. The case $\alpha = 0$ was studied by Schoenberg [3] and when $\alpha = \pm\gamma$, this was done in Example 1 above.

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