# FINITE GROUPS HAVING AN INVOLUTION CENTRALIZER WITH A 2-COMPONENT OF DIHEDRAL TYPE, II

BY

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## 1. Introduction and statement of results

All groups considered in this paper are finite.

In current standard terminology, a group $L$ such that $L = L'$ and $L/O(L)$ is quasisimple is said to be 2-quasisimple. Also any subnormal 2-quasisimple subgroup of a group $G$ is called a 2-component of $G$.

Recently, a great deal of progress has been made on the fundamental problem of classifying all finite groups $G$ such that $O(G) = 1$ and such that $G$ contains an involution $t$ such that $H = C_G(t)$ has a 2-component $L$ (cf. [1, Theorem 1], [2], [3], and [13]). These results suggest the importance of investigating such groups $G$ in which $C_H(L/O(L))$ has 2-rank 1. Of particular interest is the case where $L$ is of dihedral type.

In [9], R. Solomon and the author obtained some results on groups $G$ with $O(G) = 1$ and containing an involution $t \in G - Z(G)$ such that $H = C_G(t)$ contains a 2-component $L$ such that a Sylow 2-subgroup of $L$ is dihedral, $m_2(C_H(L/O(L))) = 1$ and such that $N_H(L)/(LC_H(L/O(L)))$ is cyclic. In this paper, the methods of [9] are applied to the case in which $N_H(L)/(LC_H(L/O(L)))$ is not cyclic.

The first main result of this paper is the following.

THEOREM 1. *Let $G$ be a finite group with $O(G) = 1$. Suppose the involution $t \in G - Z(G)$ is such that $H = C_G(t)$ contains a 2-component $L$ such that a Sylow 2-subgroup of $L$ is dihedral, $m_2(C_H(L/O(L))) = 1$ and such that $N_H(L)/(LC_H(L/O(L)))$ is not cyclic. Let $S \in Syl_2(N_G(L))$ be such that $t \in S$ and let $D = S \cap L$. Then the following conditions hold:*

(i) *$L/O(L)$ is isomorphic to $PSL(2, q^2)$ for some odd prime power $q$, $N_G(L) = O(N_G(L))H$ and $S \in Syl_2(H)$.*

(ii) *$O_2(G) = F(G) = C_G(E(G)) = 1$ and $F^*(G) = E(G)$.*

(iii) *If $F^*(G)$ is not simple, then $F^*(G) = R \times R^t$ where $R$ is simple and $L = \langle rr^t \mid r \in R\rangle \cong R$.*

(iv) *If $F^*(G)$ is simple and $r_2(F^*(G)) \leq 4$, then the possibilities for $F^*(G)$ and $G$ can be obtained from [6, Main Theorem].*

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(v) *If $F^*(G)$ is simple and $r_2(F^*(G)) > 4$, then $\langle t\rangle \in Syl_2(C_G(L/O(L)))$, $H^{(\infty)} = L$,*

$$C_G(L/O(L))) = O(N_G(L))\langle t\rangle, \qquad C_H(L/O(L)) = O(H) \times \langle t\rangle,$$

*$H/(O(H) \times \langle t\rangle)$ is isomorphic to a subgroup of* Aut $(L/O(L))$ *containing* Inn $(L/O(L))$ *properly with $((O(H) \times \langle t\rangle)L)/(O(H) \times \langle t\rangle)$ corresponding to* Inn $(L/O(L))$ *and such that $H/((O(H) \times \langle t\rangle)L)$ is not cyclic. Also $S - (\langle t\rangle \times D)$ contains an involution that acts as a "field automorphism" of order* 2 *on $L/O(L)$.*

The second main result of this paper treats the open case of Theorem 1(v) in which $|D|$ is minimal:

THEOREM 2. *Let $G$, $t$, $H$, $L$, $S$, and $D$ be as in Theorem* 1. *Assume that $F^*(G)$ is simple, $r_2(F^*(G)) > 4$, and $|D| = 2^3$. Then $L \cong PSL(2, 9)$ and $G$ is isomorphic to $HS$ (the Higman-Sims sporadic simple group).*

Before presenting a corollary of our results and its proof, we give some definitions.

A subgroup $K$ of $G$ is *tightly embedded* in $G$ if $|K|$ is even and $K$ intersects its distinct conjugates in subgroups of odd order. A *standard subgroup* of $G$ is a quasisimple subgroup $A$ of $G$ such that $K = C_G(A)$ is tightly embedded in $G$, $N_G(A) = N_G(K)$, and $A$ commutes with none of its conjugates. (The importance of these concepts for the classification of simple groups is described in [1, Section 1].)

COROLLARY. *Let $G$ be a finite group with $O(G) = 1$ and assume that $A$ is a standard subgroup of $G$ such that $|Z(A)|$ is odd and a Sylow 2-subgroup of $A$ is of type $D_8$. Set $X = \langle A^G\rangle$. Then exactly one of the following holds:*

(1) *$X = A$ and $Z(A) = 1$.*

(2) *$X = F^*(G) \cong A \times A$ and $Z(A) = 1$.*

(3) *$A \cong \mathscr{A}_n$ and $X \cong \mathscr{A}_{n+4}$ for $n = 6$ or 7.*

(4) *$X = F^*(G)$ is simple, $r_2(X) \leq 4$, and the possibilities for $X$ can be determined from* [6, *Main Theorem*].

(5) *$A$ is isomorphic to the 3-fold cover of $\mathscr{A}_7$, $X = G'$, and $G \cong$* Aut $(He)$.

(6) *$A \cong \mathscr{A}_6 \cong PSL(2, 9)$, $X = G'$, and $G$ is isomorphic to* Aut $(Sp(4, 4))$, Aut $(SL(5, 2))$, *or* Aut $(PSU(5, 2))$.

(7) *$A \cong \mathscr{A}_6 \cong PSL(2, 9)$ and $X = G \cong HS$.*

*Proof.* Clearly [7, Theorem 1] implies that $A/Z(A)$ is isomorphic to $\mathscr{A}_7$ or $PSL(2, r)$ for some odd prime power $r > 3$. Assume that (1) does not hold and set $K = C_G(A)$. If $m_2(K) \geq 2$, then [3, Theorem] yields (3) since $|A|_2 = 2^3$. Suppose then that $m_2(K) = 1$ and let $t \in I(K)$. Then $H = C_G(t) \leq N_G(K) = N_G(A)$ and hence $A \lhd H$. Thus $H \neq G$, $t \notin Z(G)$, and $m_2(C_H(A/O(A))) = 1$. Applying [9, Theorem 1] and Theorem 1 we conclude that $F^*(G) = E(G)$ and

$O_2(G) = 1$. Also, if $F^*(G)$ is not simple, then $F^*(G) = R \times R^t$ where $R$ is simple and $R \cong A = \langle rr^t \mid r \in R \rangle$. Clearly $X = F^*(G)$ in this case and (2) holds in this case. Thus we may assume that $F^*(G)$ is simple. If $r_2(F^*(G)) \leq 4$, then clearly (4) holds. Suppose then that $r_2(F^*(G)) > 4$. If $A/Z(A) \cong \mathscr{A}_7$, then (5) holds by [9, Corollary]. Finally suppose that $A/Z(A) \cong PSL(2, r)$ for some odd prime power $r > 3$. Then Theorem 2 and [9, Corollary] imply that (6) or (7) hold and we are done.

The outline of this paper is as follows.

Section 2 consists of some 2-group lemmas which are utilized at various points in the later sections. In Section 3, we prove Theorem 1. In the remainder of the paper (Sections 4–11), we prove Theorem 2.

Our notation is fairly standard and tends to follow the notation of [5] and [6]. In particular, if $n$ is a positive integer, then $\mathscr{A}_n$ and $\Sigma_n$ respectively denote the alternating and symmetric groups of degree $n$. Moreover, for any finite group $J$ and any 2-power $n$, $\mathscr{E}_n(J)$ denotes the set of elementary abelian subgroups of $J$ of order $n$ and $E_n$ denotes an elementary abelian subgroup of order $n$. Also for any finite group $J$, $m_2(J)$ denotes the 2-rank of $J$, $r_2(J)$ denotes the sectional 2-rank of $J$, and $I(J)$ denotes the set of involutions of $J$.

## 2. Preliminary results

In this section, we present two auxiliary lemmas. It is straightforward to verify the first of these.

LEMMA 2.1. *Let* $S = \langle x, y, \tau \mid |y| = |\tau| = 2, [y, \tau] = 1, |x| = 2^n > 4, x^y = x^{-1}$ *and* $x^\tau = xt$ *where* $t = x^{2^{n-1}} \rangle$. *Then the following conditions hold:*

(i) $Z(S) = \langle t \rangle$, $\Omega_1(S) = S$, *and* $|S| = 2^{n+2}$.

(ii) $S' = \mho^1(S) = \Phi(S) = \langle x^2 \rangle$ *and* $\exp(S) = 2^n$.

(iii) $S - Z(S)$ *contains four S-conjugacy classes of involutions represented by* $y, yx, \tau$, *and* $y\tau$ *where* $C_S(y) = \langle y, t, \tau \rangle \cong E_8$,

$$C_S(yx) = \langle t, yx, \tau x^{2^{n-2}} \rangle \cong Z_4 \times Z_2,$$

$C_S(\tau) = \langle x^2, y \rangle \times \langle \tau \rangle$ *is a maximal subgroup of S and* $C_S(y\tau) = \langle t, y, \tau \rangle \cong E_8$.

(iv) $\langle t, \tau \rangle$ *is the unique normal subgroup of* $\mathscr{E}_4(S)$ *and* $SCN_3(S) = \emptyset$.

(v) *All elements of* $\mathscr{E}_8(S)$ *are conjugate in S to* $\langle t, y, \tau \rangle$, $\langle A \mid A \in \mathscr{E}_8(S) \rangle = \langle x^2, y \rangle \times \langle \tau \rangle$, *and* $m_2(S) = 3$.

(vi) *S has seven maximal subgroups: one of type* $Z_2 \times D_{2^n}$, *one of type* $Z_4 * D_{2^n}$, *two of type* $D_{2^{n+1}}$, *two of type* $SD_{2^{n+1}}$, *and one of type* Mod $(2^{n+1})$.

The final result of this section is:

LEMMA 2.2. *Let S be a 2-group such that* $\Omega_1(S) = S$. *Assume that* $z \in I(Z(S))$ *is such that* $\bar{S} = S/\langle z \rangle$ *is dihedral of order* $2^n$ *for some integer* $n \geq 2$. *Then either S is dihedral or* $z \notin \Phi(S)$.

*Proof.* Assume that $z \in \Phi(S)$. Then $S/\Phi(S) \cong E_4$ and $S$ is nonabelian. If $\Phi(S) = S'$, then the result follows from [5, Theorem 5.4.5]. Assume that $S' < \Phi(S)$. Since $\bar{S}' = \overline{S'} = \Phi(\bar{S})$, we conclude that $\Phi(S) = S' \times \langle z \rangle$. Setting $\tilde{S} = S/S'$, we conclude that $\tilde{S} \cong E_8$ since $\Omega_1(\tilde{S}) = \tilde{S}$. Then $z \notin \Phi(S)$ and we have a contradiction. This concludes the proof of the lemma.

## 3. The proof of Theorem 1

In this section, we present our proof of Theorem 1. Thus, throughout this section, we assume that $G$, $t$, $H = C_G(t)$, $L$, $S$ and $D = S \cap L$ are as in the hypotheses of Theorem 1 and we commence our proof of Theorem 1.

Since $N_H(L)/(LC_H(L/O(L)))$ is not cyclic, it follows from [7, Theorem 1] that

$$L/O(L) \cong PSL(2, q^2)$$

for some odd prime power $q$. Moreover, [9, Proposition 3.1] and [6, Main Theorem] imply that conditions (i)–(iv) of Theorem 1 hold. Thus we assume that $F^*(G)$ is simple and that $r_2(F^*(G)) > 4$ for the remainder of this section.

Let $Q = C_H(L/O(L)) \cap S$. Then $Q \in Syl_2(C_H(L/O(L)))$ and $Q$ is cyclic or generalized quaternion. But $C_G(L/O(L)) = O(N_G(L))C_H(L/O(L))$ and $C_G(L/O(L))$ is tightly embedded in $G$ by [9, Proposition 3.1(i)]. Thus $Q \in Syl_2(C_G(L/O(L)))$.

Suppose that $Q$ is generalized quaternion. Then [2, Theorem] implies that $F^*(G) \cong PSL(4, q)$ with $q \equiv 1 \pmod 8$ or $F^*(G) \cong PSU(4, q)$ with $q \equiv 7 \pmod 8$ and our result follows. Thus we may assume that $Q$ is cyclic and $\langle t \rangle = \Omega_1(Q)$.

Note that $8 \leq |D|$, $Q \lhd S$, $D \lhd S$, and $Q \cap D = [Q, D] = 1$. Set $\bar{S} = S/Q$ and note that $\bar{S} \hookrightarrow \mathrm{Aut}\,(L/O(L))$. Since $r_2\,(\mathrm{Aut}\,(L/O(L))) = 3$, it follows that $r_2(S) \leq 4$ and hence $S \notin Syl_2(G)$.

Since $\bar{S}/\bar{D}$ is not cyclic, there is a unique subgroup $U$ of $S$ such that $Q \times D < U \leq S$ and such that $U/(Q \times D) = \Omega_1(S/(Q \times D))$. Then $U \trianglelefteq S$ and $U/(Q \times D) \cong E_4$. Also there is a maximal subgroup $T$ of $U$ containing $Q \times D$ such that $T \lhd S$ and $\bar{T}$ is dihedral of order $2|D|$. Also there is a subgroup $W$ of $S$ containing $Q$ such that $\bar{W}$ is cyclic $\bar{W} \cap \bar{T} = 1$, $\bar{W}$ acts faithfully like a group of "field automorphisms" on $L/O(L)$), $\bar{S} = \bar{T}\bar{W}$, and $\bar{U} = \bar{T}\Omega_1(\bar{W})$. Let $w \in W$ be such that $\langle \bar{w} \rangle = \Omega_1(\bar{W})$, let $Z(D) = \langle z \rangle$, and let $V = W \cap C_S(D)$. Noting that $\overline{C_S(D)} = C_{\bar{S}}(\bar{D}) = \langle \bar{z}, \bar{w} \rangle$, we have $C_S(D) = V \times \langle z \rangle$ where $V = \langle Q, w \rangle$ and $w^2 \in Q$.

On the other hand, $C_{\bar{T}}(\bar{w}) = \bar{D}$, $Z(\bar{S}) = \langle \bar{z} \rangle$, and $\langle t, z \rangle \leq Z(T) \leq Q \times \langle z \rangle$. Since $S \notin Syl_2(G)$, $\langle t \rangle$ is not characteristic in $Z(S)$. Thus $Z(S) = \langle t, z \rangle$ and $|N_G(S)|_2 = 2|S|$. Let $S < \mathcal{S}$ where $\mathcal{S} \in Syl_2(G)$. Then $|N_{\mathcal{S}}(S)/S| = 2$ and $N_{\mathcal{S}}(S) \in Syl_2(N_G(S))$. Let $\tau \in N_{\mathcal{S}}(S) - S$.

LEMMA 3.1. *Suppose that* $\langle t\rangle$ char $C_S(D) = V \times \langle z\rangle$. *Then the following conditions hold:*

(i) $S = U$ *and* $W = V$;
(ii) $Z(N_{\mathscr{S}}(S)) = \langle tz\rangle$;
(iii) $\langle z, t\rangle$ *is the unique normal element of* $\mathscr{E}_4(N_{\mathscr{S}}(S))$;
(iv) $\mathscr{S} = N_{\mathscr{S}}(S)$;
(v) $|\mathscr{S}| = 2^8$.

*Proof.* Assuming that $\langle t\rangle$ char $C_S(D)$, it follows that $D_1 = D^\tau \neq D$, $D_1 \lhd S$, and $z^\tau = t$. Hence $Z(D_1) = \langle t\rangle$, $[D, D_1] = D \cap D_1 = 1$, $D \times D_1 \lhd N_{\mathscr{S}}(S)$ and $D_1 \times \langle z\rangle \leq C_S(D) = V \times \langle z\rangle$. Thus, as $Q$ is a cyclic maximal subgroup of $V$, we conclude that $V$ is dihedral or semidihedral and $Q \geq 4$. Also $\overline{V} = \Omega_1(\overline{W})$ and $Q \lhd W$ and hence (i) holds. Clearly (ii) holds also since $Z(S) = \langle t, z\rangle$. Next, let $Y \in \mathscr{E}_4(N_{\mathscr{S}}(S))$ be such that $Y \lhd N_{\mathscr{S}}(S)$. Hence $tz \in Y$. Suppose that $Y \not\leq S$ and let $y \in Y - S$. Choosing $d \in D$ such that $|d| = 4$ and noting that $D^y = D_1$, it follows that $|[d, y]| = 4$ which is impossible. Thus $Y \leq S$. Suppose that $t \notin Y$. Then $\overline{Y} \lhd \overline{S}$ and $\overline{Y} \cong E_4$. Then Lemma 2.1 implies that $\overline{Y} = \langle \bar{z}, \bar{w}\rangle$ and hence $Y \leq V \times \langle z\rangle$. But $Y \cap \langle z\rangle = 1$ and $Y \cap V = 1$ since $t \notin Y$ and $V$ is dihedral or semidihedral. As $Y \cong E_4$, this is impossible. Hence $t \in Y$ and (iii)–(iv) follow. Since $r_2(\mathscr{S}) > 4$, [11, Four Generator Theorem] implies that $\mathscr{S}$ contains a normal subgroup $Y \in \mathscr{E}_8(\mathscr{S})$. Then $C_Y(t) \lhd S$ and $|C_Y(t)| \geq 4$. Hence $\langle t, z\rangle \leq Y$ and $Y \leq V \times \langle z\rangle$. Since $E_4 \cong Y \cap V \lhd V$, it follows that $V \cong D_8$, $|D_1| = |D| = 8$ and (v) holds.

We can now conclude the proof of Theorem 1. If $\langle t\rangle$ char $C_S(D) = V \times \langle z\rangle$, then $|G|_2 = 2^8$ and [4, Theorem] implies that $r_2(F^*(G)) \leq 4$. Thus $\langle t\rangle$ is not characteristic in $C_S(D)$. Hence $V \cong E_4$ and $Q = \langle t\rangle$. Then we may assume that $w \in I(W)$: hence (v) of Theorem 1 holds and we are done.

## 4. Beginning the proof of Theorem 2

We now commence our proof of Theorem 2.

Let $G$, $t$, $H$, $L$, $S$, and $D$ be as in Theorem 2 and assume that $F^*(G)$ is simple, that $r_2(F^*(G)) > 4$, and that $|D| = 2^3$.

Observe that if $|F^*(G)|_2 \leq 2^{10}$, then [4] determines the structure of $F^*(G)$ and the conclusion of Theorem 2 follows. Consequently we may assume that $|F^*(G)|_2 > 2^{10}$ and we shall obtain a contradiction by showing that $|O^2(G)|_2 \leq 2^{10}$.

Since $D \cong D_8$, there is an involution $z \in D$ such that $D' = Z(D) = \langle z\rangle$. Also, by Theorem 1(v), there is an involution $u \in S - (\langle t\rangle \times D)$ that acts as a "field automorphism" on $L/O(L)$. Thus $[u, D] = 1$ and $U = \langle t, u\rangle \times D$ is a maximal subgroup of $S$ since $|S| = 2^6$. Moreover $S/(\langle t\rangle \times D) \cong E_4$ and

$\tilde{S} = S/\langle t \rangle$ is isomorphic to the group given in Lemma 2.1 with $n = 3$. Hence we have:

(4.1) $Z(S) = \langle t, z \rangle$ and there is an element $\delta \in S - U$ such that $\delta^2 \in \langle t \rangle$, $\tilde{D}\langle \tilde{\delta} \rangle \cong D_{16}$, and $u^\delta \in \{uz, uzt\}$.

(4.2) There is an involution $x \in D \times \langle u \rangle$ such that $u \sim x \sim xz$ in $L\langle u \rangle$ and there is an involution $y \in D$ such that $D = \langle y, xu \rangle$, $|yxu| = |yx| = 4$, $z \sim y \sim yz \sim xu \sim xuz$ in L, and $uz \sim yu \sim yuz$ in $L\langle u \rangle$.

(4.3) $\Phi(S) = \langle yxu \rangle$ or $\Phi(S) = \langle t \rangle \times \langle yxu \rangle$, $\mho^1(\Phi(S)) = \langle z \rangle$, and $\exp(S) = 8$.

(4.4) $r_2(\tilde{S}) = 3$, $r_2(S) = 4$, and $S \notin Syl_2(G)$.

Also by replacing $y$ by $yz$, if necessary, it follows that we may assume:

(4.5) $\delta\colon y \leftrightarrow xu$.

Let $A = \langle t, u, z, y \rangle$ and $B = \langle t, u, z, x \rangle$. Then we also have:

(4.6) $\mathscr{E}_{16}(S) = \{A, B\}$, $N_S(A) = N_S(B) = U$, $C_S(A) = A$, and $\delta\colon A \leftrightarrow B$.

Also Lemma 2.1(iv) applied to $\tilde{S} = S/\langle t \rangle$ yields:

(4.7) $X = \langle t, u, z \rangle$ is the unique normal element of $\mathscr{E}_8(S)$ and $C_S(X) = U$.

Since $L/O(L) \cong PSL(2, q^2)$ for some odd prime power $q$ and $C_{L/O(L)}(u) \cong PGL(2, q)$, we conclude:

(4.8) There is a 3-element $\rho \in C_H(u) \cap N_L(A)$ such that $x$ inverts $\rho$, $C_A(\rho) = \langle t, u \rangle$, and $[A, \rho] = \langle z, y \rangle$.

Moreover setting $\bar{H} = H/O(H)$ we have:

(4.9) $N_{\bar{H}}(\bar{A}) = \langle \bar{t}, \bar{u} \rangle \times \langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x} \rangle$ with $\langle \bar{y}, \bar{z}, \bar{\rho}, \bar{x} \rangle \cong \Sigma_4$.

Hence:

(4.10) $C_G(A) = O(C_G(A)) \times A$ and $\rho^3 \in O(C_G(A))$.

Set $\tilde{H} = \bar{H}/\langle \bar{t} \rangle$. Then $\tilde{L} \lhd \tilde{H}$, $\tilde{L} \cong L/O(L) \cong \bar{L}$, and $\tilde{H}/\tilde{L}$ is abelian. Thus $\tilde{L} \leq O^2(\tilde{H})$, $|O^2(\tilde{H})/\tilde{L}|$ is odd, and $\tilde{D} \in Syl_2(O^2(\tilde{H}))$. Then $C_{O^2(\tilde{H})}(\tilde{z})$ has a normal 2-complement by [7, Theorem 1] and we have:

(4.11) $C_G(t, z) = O(C_G(t, z))S$.

Let $\tau$ be an element in $S - U$. Thus $\tau^2 \in \langle t \rangle \times D$. If $\tau^2 \in \langle t \rangle$, then Lemma 2.1(iii) implies that $\langle \tilde{z} \rangle \in Syl_2(C_{O^2(\tilde{H})}(\tau))$. If $\tau^2 \notin \langle t \rangle$, then Lemma 2.1(ii) implies that $\Omega_1(\langle \tau \rangle) = \langle z \rangle$ or $\Omega_1(\langle \tau \rangle) = \langle tz \rangle$. Thus we have:

(4.12) If $\tau \in S - \langle t \rangle$, then $C_G(t, \tau)$ has a normal 2-complement if and only if $\tau$ is not conjugate in $H$ to $u$ or $tu$.

Also, as is well known, we have:

(4.13) $C_{\bar{H}}(\bar{u}) = \langle \bar{t}, \bar{u} \rangle \times \bar{\mathscr{A}}$ for some subgroup $\bar{\mathscr{A}}$ of $\bar{H}$ with $O^{2'}(\bar{\mathscr{A}}) = C_{\bar{L}}(\bar{u}) \cong PGL(2, q)$, $C_{\bar{H}}(\bar{u})' = \bar{\mathscr{A}}' = C_{\bar{L}}(\bar{u})'$, and with $\langle \bar{y}, \bar{z}, \bar{\rho} \rangle \leq \bar{\mathscr{A}}'$.

(4.14) $\langle y, z \rangle \in Syl_2(O^2(C_G(t, u)))$.

Let $S \leq \mathscr{S} \in Syl_2(G)$. Then $S \neq \mathscr{S}$ by (4.4) and $S < N_{\mathscr{S}}(S)$. Since $Z(S) = \langle t, z \rangle \lhd N_{\mathscr{S}}(S)$ and $\langle z \rangle \lhd N_{\mathscr{S}}(S)$ by (4.3), we have:

(4.15) $|N_{\mathscr{S}}(S)/S| = 2$, $Z(N_{\mathscr{S}}(S)) = \langle z \rangle = Z(\mathscr{S})$, $t^{N_{\mathscr{S}}(S)} = \{t, tz\}$, and $t \nsim z$ in $G$.

Also (4.3) and (4.11) imply that $O^2(N_G(S)) \leq C_G(S) = C_H(S)$. Also $\langle t, z \rangle \in Syl_2(C_G(S))$ and hence:

(4.16) $N_G(S) = O(N_G(S))N_{\mathscr{S}}(S)$.

Suppose that $Y \lhd \mathscr{S}$ and $Y \in \mathscr{E}_{32}(\mathscr{S})$. Then $|C_Y(t)| \geq 2^3$ and $C_Y(t) \lhd S$. Thus $C_Y(t) = \langle t, u, z \rangle$ by (4.7). Hence $t \in Y$, which is impossible. Thus:

(4.17) $SCN_5(\mathscr{S}) = \emptyset$.

Since $r_2(\mathscr{S}) > 4$, [11, Four Generator Theorem] implies that there is an $E \in \mathscr{E}_8(\mathscr{S})$ such that $E \lhd \mathscr{S}$. Clearly $z \in E$. Suppose that $t^G \cap E \neq \emptyset$. Then $|\mathscr{S}| \leq 2^8$, which is false. Also, if $[t, E] = 1$, then $E \lhd S$ and hence $E = \langle t, u; z \rangle$ which is impossible. Thus:

(4.18) $t^G \cap E = \emptyset$, $z \in E$, and $E \nleq S = C_{\mathscr{S}}(t)$.

Setting $E_1 = C_E(t)$, we have $z \in E_1 \cong E_4$ and $t \notin E_1 \leq S$. Thus $E_1 \times \langle t \rangle = \langle t, u, z \rangle$ and $E_1 = \langle z, u \rangle$ or $E_1 = \langle z, tu \rangle$. Replacing $u$ by $ut$ and $x$ by $xt$, if necessary, in the above, we obtain:

(4.19) $C_E(t) = \langle z, u \rangle$, $E \leq N_{\mathscr{S}}(S) \cap N_{\mathscr{S}}(A) \cap N_{\mathscr{S}}(B)$, $t^E = \{t, tz\}$, and $N_{\mathscr{S}}(S) = ES$.

Let $M = N_G(A)$ and $\bar{M} = M/O(M)$. Then $C_G(M) = O(M) \times A$, $\langle U, \rho \rangle \leq M$, and $\rho^3 \in O(M)$. Also $N_H(M) = (O(M) \times A)\langle \rho, x \rangle$ and $\overline{N_H(M)} = \overline{C_M(t)} = C_{\bar{M}}(\bar{t}) = \bar{A}\langle \bar{\rho}, \bar{x} \rangle$. Let $F = \langle y, z \rangle$. Then

$$A = F \cup tF \cup uF \cup tuF, \quad t^G \cap (F \cup uF) = \emptyset, \quad \text{and} \quad tF \subseteq t^G \cap A.$$

Since $M = N_G(A)$ controls $G$-fusion among the elements of $t^G \cap A$ by (4.6), we have:

(4.20) Either $t^M = t^G \cap A = tF$ and $|\bar{M}/\bar{A}| = 24$ or $t^M = t^G \cap A = tF \cup tuF$ and $|\bar{M}/\bar{A}| = 48$.

This concludes our investigation of the consequences of the hypotheses of Theorem 2.

## 5. The case $|\bar{M}/\bar{A}| = 24$

Throughout this section, we will assume that $t^M = t^G \cap A = tF$ and $|\bar{M}/\bar{A}| = 24$ and we shall prove that the conclusion of Theorem 2 holds in this case.

Since $C_{\bar{A}}(\bar{\rho}) = \langle \bar{t}, \bar{u} \rangle \ntriangleleft \bar{M}$, we conclude that $\bar{M}/\bar{A} \cong \Sigma_4, \bar{M} = O_2(\bar{M})\langle \bar{\rho}, \bar{x} \rangle$ and $|O_2(\bar{M})| = 2^6$. Note that $\langle t^M t^M \rangle = F \lhd M$. Let $U = N_S(A) \le \mathscr{U} \in Syl_2(M)$. Then $|\mathscr{U}| = 2^7$ and $\bar{\mathscr{U}} = O_2(\bar{M})\langle \bar{x} \rangle$. Let $W = O_{2',2}(M)$ and $V = O(M)[W, \rho]$. Clearly $O(M) \times F \trianglelefteq V$.

Lemma 5.1. *$\bar{W}/\bar{F} \cong E_{16}$ and $\bar{V} \cong E_{16}$ or $\bar{V} \cong Z_4 \times Z_4$.*

*Proof.* Clearly $|\bar{W}/\bar{F}| = 2^4$ and $C_{\bar{W}/\bar{F}}(\bar{\rho}) = \bar{A}/\bar{F} \cong E_4$. Suppose that $\bar{W}/\bar{F} \cong Z_2 \times Q_8$. Then $|C_{\bar{V}}(\bar{\rho})| = 2$ and $C_{\bar{V}}(\bar{\rho}) \le C_{\bar{W}}(\bar{\rho}) = \langle \bar{t}, \bar{u} \rangle$. Letting $\bar{q} \in V$ be such that $|\bar{q}\bar{F}| = 4$ and $\bar{\omega} \in \bar{V}$ be such that $C_{\bar{V}}(\bar{\rho}) = \langle \bar{\omega} \rangle$. Then $\bar{q}^2 \in \bar{\omega}\bar{F}$ and $Z(\bar{V}) = \langle \bar{\omega}, \bar{y}, \bar{z} \rangle$ since $\bar{F} \lhd \bar{V}$ and $\bar{\rho}$ acts on $\bar{V}$. But $\langle \bar{\omega} \rangle \le \langle \bar{t}, \bar{u} \rangle$ and $\bar{W} = \bar{V}\langle \bar{t}, \bar{u} \rangle$; thus $Z(\bar{W}) = \langle \bar{\omega}, \bar{y}, \bar{z} \rangle$ and $\bar{\omega} \in Z(\bar{\mathscr{U}})$ since $\bar{\mathscr{U}} = \bar{W}\langle \bar{x} \rangle$. Hence $\bar{\omega} \in \{\bar{u}, \bar{t}\bar{u}\}$. Since $\langle t, F \rangle \lhd M$, we conclude that $\bar{M}/\langle \bar{t}, \bar{F} \rangle \cong GL(2, 3)$. Suppose that $E_{16} \cong X \lhd \mathscr{U}$. Then

$$\bar{X}\langle \bar{t}, \bar{F} \rangle / \langle \bar{t}, \bar{F} \rangle \lhd \bar{\mathscr{U}}/\langle \bar{t}, \bar{F} \rangle \cong SD_{16}.$$

Hence $\bar{X} = \bar{A}$ and $A$ char $\mathscr{U}$. This implies that $\mathscr{U} \in Syl_2(G)$. As $|\mathscr{U}| = 2^7$, this is impossible. Applying [9, Lemma 2.1], the result follows.

Lemma 5.2. *If $\bar{V} \cong Z_4 \times Z_4$, then $|G|_2 \le 2^9$.*

*Proof.* Assume that $\bar{V} \cong Z_4 \times Z_4$ and note that $\bar{V} \lhd \bar{M}$ and $\bar{F} = \langle y, z \rangle = \Omega_1(\bar{V})$. Also $\langle \bar{t}, \bar{u} \rangle \times \langle \bar{\rho}, \bar{x} \rangle$ acts on $\bar{V}$. Thus $\bar{t}$ inverts $\bar{V}$ and there is an element $u_1 \in \langle t, u \rangle - \langle t \rangle$ such that $[\bar{u}_1, \bar{V}] = 1$. Hence

$$\bar{M} = \langle \bar{u}_1 \rangle \times \bar{V}(\langle \bar{t} \rangle \times \langle \bar{\rho}, \bar{x} \rangle) \quad \text{and} \quad \bar{\mathscr{U}} = \langle \bar{u}_1 \rangle \times (\bar{V}\langle \bar{t}, \bar{x} \rangle)$$

where $\bar{V}\langle \bar{t}, \bar{x} \rangle$ is of type $M_{12}$. Let $\mathscr{W} = \mathscr{U} \cap W$, $\mathscr{V} = \mathscr{U} \cap V$, and $E = \langle u_1 \rangle \times \mathscr{V}$. Then $\mathscr{W} \lhd \mathscr{U}$, $\mathscr{V} \lhd \mathscr{U}$, $E = C_{\mathscr{U}}(\mathscr{V}) \lhd \mathscr{U}$, and $\mathscr{W} = E\langle t \rangle$. Also let $X = \langle u_1, F \rangle = \Omega_1(E)$, so that $X \lhd \mathscr{U}$. Since

$$M = O(M)N_M(\mathscr{W}) = O(M)\mathscr{W}(N_M(\mathscr{W}) \cap C_M(t)),$$

it follows that there is a 3-element $\eta \in N_M(\mathscr{W}) \cap C_M(t)$ inverted by $x$ such that $\bar{\eta} = \bar{\rho}$ and $\bar{\eta}^3 \in C_G(\mathscr{W})$. Set $L = N_G(\mathscr{W})$ and $\bar{L} = L/O(L)$ and observe that $C_G(\mathscr{W}) = O(L) \times X$ where $X = Z(\mathscr{W}) \lhd L$. Since $J_0(\mathscr{W}) = E \lhd L$, we also have $F \lhd L$. Moreover $\bar{L}/\bar{X} \hookrightarrow \mathrm{Aut}(\mathscr{W})$ implies that $|\bar{L}|_{2'} = 3$, $\bar{L} = O_2(\bar{L})\langle \bar{\eta}, \bar{x} \rangle$, and $F \le Z(O_{2',2}(L))$. Clearly $C_{\mathscr{W}}(\eta) = \langle t, u \rangle$ and $\langle \mathscr{U}, \eta \rangle \le L$. Let $\mathscr{U} \le \mathscr{T} \in Syl_2(L)$ and observe that $X$ char $\mathscr{U}$ implies that $C_{\mathscr{U}}(X) = \mathscr{W}$ char $\mathscr{U}$. Thus $\mathscr{U} < \mathscr{T}$ and $\bar{\mathscr{T}} = O_2(\bar{L})\langle \bar{x} \rangle$.

Let $v_1 \in \mathscr{V}$ be such that $v_1^2 = y$ and set $v_2 = v_1^x$. Then $v_2^2 = yz$ and $C_{\mathscr{V}}(x) = \langle v_1 v_2 \rangle$. Also $t^G \cap E = \emptyset$, $t^E = tF$, and

$$I(tE) = t^E \cup (tu_1)^E \cup (tv_1)^E \cup (tu_1v_1)^E \cup (tv_2)^E \cup (tu_1v_2)^E \cup (tv_1v_2)^E \cup (tu_1v_1v_2)^E.$$

This implies that $\bar{L}/\bar{\mathscr{W}} \cong \Sigma_4$ and hence $|\mathscr{T}| = 2^9$.

Suppose that $O(L) \times \mathscr{W} < C_L(X) = O_{2',2}(L)$. Then $\langle u_1 \rangle \leq Z(L)$. Setting $\tilde{L} = \bar{L}/\langle \bar{u}_1 \rangle$, we have $C_{O_2(\tilde{L})}(\tilde{\eta}) = \langle \tilde{t} \rangle$ and $C_{O_2(\tilde{L})}(\tilde{t}) = (C_{O_2(\bar{L})}(\bar{t}))^\sim = \tilde{A} \cong E_8$ since $t^G \cap \langle t, u_1 \rangle = \{t\}$. Also $Z_4 \times Z_4 \cong \tilde{E} \trianglelefteq \tilde{L}$ and $O_2(\tilde{L}) = 2^7$. By [9, Lemmas 2.7 and 2.8] there is a subgroup $J$ of $L$ such that $O(L)E < J \lhd L$, $J \leq O_{2',2}(L)$, $t \notin J$, $|\bar{J}| = 2^7$, $C_J(\bar{\eta}) = \langle \bar{u}_1 \rangle$, $O_2(\bar{L}) = \bar{J}\langle \bar{t} \rangle$, and such that $\tilde{J} \cong Z_8 \times Z_8$ or $\tilde{J}$ is of type $L_3(4)$. Let $\mathscr{J} = \mathscr{T} \cap J$. Then $\mathscr{J} \lhd \mathscr{T}$ and $\mathscr{T} = \mathscr{J}\langle x, t \rangle$. Letting $x_1 = x$ if $u_1 = u$ and $x_1 = xt$ if $u_1 = ut$, we have $x_1 u_1 = xu$ and $tx_1 \nsim t \sim tx_1u_1$ in $G$. But $C_{\tilde{J}}(\tilde{t}\tilde{x}_1\tilde{u}_1) = C_{\tilde{J}}(\tilde{t}\tilde{x}_1)$ has order $2^3$ in either case by [9, Lemma 2.9]. As $(C_J(\bar{t}\bar{x}_1))^\sim = C_{\tilde{J}}(\tilde{t}\tilde{x}_1)$, we conclude that $|C_J(\bar{t}\bar{x})| = 2^4$. But $C_{\mathscr{T}}(tx_1) = C_{\mathscr{J}}(tx_1)\langle t, x \rangle$ and hence $|C_{\mathscr{T}}(tx_1)| = 2^6$. Since $\langle u_1, z \rangle \leq Z(C_{\mathscr{T}}(tx_1))$, we obtain a contradiction from (4.1). Thus $O(L) \times \mathscr{W} = C_L(X)$, $C_{\mathscr{T}}(X) = \mathscr{W}$, and $\Sigma_4 \cong \bar{L}/\bar{\mathscr{W}} \hookrightarrow GL(3, 2)$.

Assume that $Y \in \mathscr{E}_8(\mathscr{T})$, $Y \lhd \mathscr{T}$, and $Y \neq X$. Since $Z(\mathscr{T}) = \langle z \rangle$, we have $z \in Y$. Suppose that $Y \cap (xE \cup xtE) \neq \emptyset$. Then $[Y, \mathscr{V}]$ contains an element of order 4 which is impossible. Thus $Y \cap (xE \cup xtE) = \emptyset$. Also $t^G \cap Y = \emptyset$ since $|\mathscr{T}| = 2^9$. Note also that $U = C_{\mathscr{T}}(t)$ since $X \lhd \mathscr{T}$. Thus $C_Y(t) \leq U$ and $|C_Y(t)| \geq 4$. Since $I(U) \subseteq A \cup B$, we must have $C_Y(t) \leq A = \langle t, X \rangle$. Thus $Y \leq N_{\mathscr{T}}(A) = \mathscr{U} = E\langle x, t \rangle$ and $Y \leq E\langle t \rangle$. Since $Y \nleq E$, there is an involution $\tau \in (tu_1F) \cap Y$. Thus $[\tau, E] = F \leq Y$ and $tu_1 \in Y$. Also $\mathscr{Q} = O_{2',2}(L) \cap \mathscr{T}$ is transitive on

$$\{tX, tv_1X, tv_2X, tv_1v_2X\}$$

and hence $C_{\mathscr{Q}}(tu_1) = C_{\mathscr{W}}(tu_1) = A$. Thus $|(tu_1)^{\mathscr{Q}}| = 2^4$ which is impossible. Hence $X$ char $\mathscr{T}$, $C_{\mathscr{T}}(X) = \mathscr{W}$ char $\mathscr{T}$, $\mathscr{T} \in Syl_2(G)$, and the lemma follows.

LEMMA 5.3. *If* $\bar{V} \cong E_{16}$, *then* $|G|_2 \leq 2^{10}$.

*Proof.* Assume that $\bar{V} \cong E_{16}$. Since $\bar{V} \lhd \bar{M}$, $\bar{F} = C_{\bar{V}}(\bar{t}, \bar{u}) = C_{\bar{V}}(\bar{t})$, and $\langle t, \bar{u} \rangle$ acts on $C_{\bar{V}}(\bar{x}) \cong E_4$, it follows that there is an involution $u_1 \in \langle t, u \rangle - \langle t \rangle$ such that $[\bar{u}_1, \bar{V}] = 1$. Set $\mathscr{V} = V \cap \mathscr{U}$ and $E = \langle u_1 \rangle \times \mathscr{V} \cong E_{32}$. Then $\mathscr{E}_{32}(\mathscr{U}) = \{E\}$, $\mathscr{U} = E\langle x, t \rangle$, and, as in Lemma 5.2, there is a 3-element $\eta \in N_M(E) \cap C_M(t)$ inverted by $x$ such that $\bar{\eta} = \bar{\rho}$ and $\bar{\eta}^3 \in C_G(E)$. Also $I(tE) = tF \cup tu_1F$, $C_{\mathscr{U}}(t) = \mathscr{U} = A\langle x \rangle$, and $t^G \cap tE = tF = t^E$. Let $L = N_G(E)$, $\bar{L} = L/O(L)$, and let $\mathscr{U} \leq \mathscr{T} \in Syl_2(L)$. Then $\mathscr{U} \neq \mathscr{T}$ since $E$ char $\mathscr{U}$ and $|\mathscr{U}| = 2^7$ and $\mathscr{T} \notin Syl_2(G)$ by (4.17). Since $C_{\mathscr{T}/E}(t) = UE/E$, it follows that $C_G(E) = O(L) \times E$, $\bar{L}/\bar{E} \hookrightarrow \text{Aut}(E) \cong GL(5, 2)$ and, since $UE/E \cong E_4$, that $\mathscr{T}/E$ is dihedral or semidihedral of order at most 16. Also $|\mathscr{T}/E| \geq 8$ and $|C_E(x)| = |C_E(xt)| = 8$. Let $\mathscr{T} \leq \mathscr{S} \in Syl_2(G)$. Then $\mathscr{T} \neq \mathscr{S}$ and there is an element $\tau \in N_{\mathscr{S}}(\mathscr{T}) - \mathscr{T}$ such that $\tau^2 \in \mathscr{T}$. Letting $E_1 = E^\tau$, we have

$E \neq E_1 \lhd \mathscr{T}$. Hence $Z(\mathscr{T}/E) \leq (E_1E)/E$ and $|E \cap E_1| = 8$. Thus $\mathscr{T}/E \cong E_8$.

Note that $I(xE) = x^E \cup (xu_1)^E$ and $I(xtE) = (xt)^E \cup (xtu_1)^E$. Also $x \nsim t \nsim xu$ in $G$ and $xt \nsim t \sim xtu$ in $G$. If $u_1 = u$, set $x_1 = x$ and if $u_1 = ut$, set $x_1 = xt$. Then $x_1u_1 = xu \nsim t \nsim x_1$ in $G$, $Z(\mathscr{T}/E) = \langle x_1E\rangle$ and $t \sim tx_1u_1$ in $\mathscr{T}$. Since $(E_1E)/E \cong (E\langle x_1, t\rangle)/E$, it follows that we may assume that $\tau \in C_{\mathscr{S}}(t)$ also. Note that $U = C_{\mathscr{T}}(t)$, $\tau^2 \in U$, $\tau$ normalizes $U = \langle t, u_1\rangle \times D$ and $\langle U, \tau\rangle \in Syl_2(H)$. Thus $A^\tau = B$, $\tau$ normalizes $\langle t, u, z\rangle = Z(U)$ and $Z(\langle U, \tau\rangle) = \langle t, z\rangle$.

Let $E_8 \cong Y \lhd \mathscr{S}$. Then $t^G \cap Y = \emptyset$ and $Z(\mathscr{S}) = \langle z\rangle$ since $Z(\langle U, \tau\rangle) = \langle t, z\rangle$. Hence $E_4 \cong C_Y(t) \leq A \cap B = \langle t, u, z\rangle$ and $C_Y(t) = \langle z, u\rangle$ or $\langle z, tu\rangle$. Thus $Y \leq N_{\mathscr{S}}(A) = \mathscr{U} = E\langle x, t\rangle$. If $Y \nleq E$, then $Y \cap x_1E \neq \emptyset$ and hence $[E, x_1] = C_{\mathscr{V}}(x_1) \leq Y$. Since $\langle C_Y(t), C_{\mathscr{V}}(x_1)\rangle \leq Y \cap E$ and $|\langle C_Y(t), C_{\mathscr{V}}(x_1)\rangle| = 2^3$, it follows that $Y \leq E$. Thus $Y = E \cap E_1 = C_E(x_1) = \langle u_1, z, v\rangle$ where $C_{\mathscr{V}}(x_1) = \langle z, v\rangle$ for some $v \in \mathscr{V}^\#$. Hence $Y$ is unique, $EE_1 \leq C_{\mathscr{S}}(Y) \lhd \mathscr{S}$ and $x_1 \in E_1$. Since $\mathscr{T} = (EE_1)\langle t\rangle$, $C_{\mathscr{S}}(Y) = EE_1$, $I(\mathscr{T} - (EE_1)) = I(tE \cup tx_1E)$, and $C_{\mathscr{S}}(t) = \langle U, \tau\rangle$, it follows that $N_{\mathscr{S}}(\mathscr{T}) = \langle \mathscr{T}, \tau\rangle$. Hence $C_{\mathscr{S}}(Y) = EE_1 \lhd \mathscr{S}$.

On the other hand, $\bar{L}/\bar{E}$ is of type $D_8$, $C_{\bar{L}/\bar{E}}(\bar{t}) = (\langle \bar{t}\rangle \times \langle \bar{\eta}, \bar{x}\rangle)\bar{E})/\bar{E}$, and $\bar{t}\bar{E} \nsim \bar{x}_1\bar{E}$ in $\bar{L}$. Hence $O_2(\bar{L}/\bar{E}) = 1$.

Suppose that $O(\bar{L}/\bar{E}) = 1$. Since $\bar{L}/\bar{E} \hookrightarrow GL(5, 2)$, it follows that $(\bar{L}/\bar{E})' \cong \mathscr{A}_5$ and $\bar{L}/\bar{E} \cong \Sigma_5$.

Since $\langle C_G(E), \mathscr{T}, \eta\rangle \leq C_L(u_1)$, it follows that $\langle u_1\rangle \leq Z(L)$. Let $K$ be the subgroup of index 2 in $L$ such that $C_G(E) = O(L) \times E < K$. Then $\bar{K}/\bar{E} \cong \mathscr{A}_5$, $\mathscr{T} \cap K = EE_1$, and there is a 3-element $\lambda \in K$ such that $\bar{\lambda}$ acts transitively on $((\bar{E}_1\bar{E})/\bar{E})^\#$. Noting that $Y = Z(EE_1)$, it follows that if $e_1 \in E_1 - E$, then $C_E(e_1) = Y$. Let $R \in \mathscr{E}_{32}(EE_1)$ with $R \neq E$. Then $Y = R \cap E$ and hence $x_1 \in R$ and $R \leq C_{EE_1}(x_1) = E_1$. Thus $R = E_1$ and $\mathscr{E}_{32}(EE_1) = \{E, E_1\}$. But $\mathscr{T} = N_{\mathscr{S}}(E)$ has order $2^8$ and hence $|\mathscr{S}| = 2^9$ and we are done in this case. Hence we may assume that $O(\bar{L}/\bar{E}) \neq 1$.

Since $\bar{L}/\bar{E} \hookrightarrow GL(5, 2)$, we conclude that $O(\bar{L}/\bar{E}) \cong Z_3 \times Z_3$ and $\bar{L}/\bar{E} \cong \Sigma_3 \wr Z_2$. Let $P \in Syl_3(L)$. Then, clearly $C_E(P) = \langle u_1\rangle \lhd L$ and $[P, E] = \mathscr{V} \lhd L$.

Now $\Omega_1(\mathscr{T}) = \mathscr{T}$ and $\mathscr{T}/E \cong D_8$.

Assume that $u_1\mathscr{V} \in \Phi(\mathscr{T}/\mathscr{V})$. Then $\mathscr{T}/\mathscr{V} \cong D_{16}$ by Lemma 2.2. But $E\langle x_1\rangle \lhd \mathscr{T}$ and $E_4 \cong (E\langle x_1\rangle)/\mathscr{V}$; this contradiction shows that $u_1 \notin \Phi(\mathscr{T})$. Since $\langle u_1\rangle \leq Z(L)$, [10, I, 17.4] implies that $L$ contains a normal subgroup $L_1$ such that $L = \langle u_1\rangle \times L_1$. Hence $O^2(L) = (O(L)\mathscr{V})P \leq L_1$ where $O_2(\bar{L}_1) = \bar{\mathscr{V}}$ and $\bar{L}_1/\bar{\mathscr{V}} \cong \Sigma_3 \wr Z_2$. Then [6, II, Lemma 2.2(vii)] implies that $L_1$ is of type $\mathscr{A}_{10}$. Since $\mathscr{T} = \langle u_1\rangle \times (\mathscr{T} \cap L_1)$, it follows that $|\mathscr{E}_{32}(\mathscr{T})| = 4$. Thus we may assume that

$$\mathscr{E}_{32}(\mathscr{T}) = \{E, E_1, E_2, E_3\}.$$

Also $|\langle E, E_1, E_2, E_3\rangle| = 2^7 = |EE_1|$. Hence $\mathscr{E}_{32}(\mathscr{T}) = \mathscr{E}_{32}(EE_1)$. Since

$N_{\mathscr{S}}(E) = \mathscr{T}$, $|\mathscr{T}| = 2^8$, and $EE_1 \lhd \mathscr{S}$, we conclude that $|\mathscr{S}| \leq 2^{10}$ and we have proved Lemma 5.3. Then [4, Theorem] implies:

LEMMA 5.4. *If* $|\bar{M}/\bar{A}| = 24$, *then the conclusion of Theorem* 2 *holds.*

## 6. The case $|\bar{M}/\bar{A}| = 48$

As a result of Lemma 5.4 and (4.20), it suffices to assume that $|\bar{M}/\bar{A}| = 48$ and $t^M = t^G \cap A = tF \cup tuF$ throughout the remainder of the paper.

Thus $|M|_2 = 2^8$, $\bar{M} = O_2(\bar{M})\langle\bar{\rho}, \bar{x}\rangle$, $|O_2(\bar{M})| = 2^7$, and $C_{\bar{A}}(\rho) \trianglelefteq \bar{M}$. Set $X = \langle u, y, z\rangle = \langle u, F\rangle$, so that $t^M = tX$. Clearly $C_{\bar{A}}(\bar{\rho}) \trianglelefteq \bar{M}$ implies that $O_3(\bar{M}/\bar{A}) = 1$. Since $\bar{M}/\bar{A} \hookrightarrow \mathrm{Aut}(\bar{A}) \cong GL(4, 2)$ and $GL(4, 2)$ has no subgroup isomorphic to $GL(2, 3)$, we also have $\bar{M}/\bar{A} \cong Z_2 \times \Sigma_4$. Since $t^M = t^G \cap A = tX$, we conclude from (4.1) that $u^\delta = uz$ and $X \lhd M$.

Let $U = N_S(A) \leq \mathscr{U} \in Syl_2(M)$, $W = O_{2',2}(M)$, and $V = O(M)[W, \rho]$. Also let $\mathscr{W} = \mathscr{U} \cap W$ and $\mathscr{V} = \mathscr{U} \cap V$. Clearly $M = W\langle\rho, x\rangle$, $V \lhd M$, $|\mathscr{U}| = 2^8$, $\mathscr{U} = \mathscr{W}\langle x\rangle$, $\mathscr{W}$ max $\mathscr{U}$, $C_{\mathscr{W}}(t) = A$, $t^{\mathscr{W}} = tX$. $\mathscr{W}/A \cong E_8$ and $\mathscr{W}/A$ acts regularly on $tX$. Clearly $M = O(M)N_M(\mathscr{W}) = O(M)\mathscr{W}(N_M(\mathscr{W}) \cap H)$. Hence there is a 3-element $\eta \in N_M(\mathscr{W}) \cap H$ such that $\eta^3 \in O(M)$, $\bar{\eta} = \bar{\rho}$, $\eta^x = \eta^{-1}$, $[A, \eta] = F$, $C_A(\eta) = \langle t, u\rangle$, and $\mathscr{V} = [\mathscr{W}, \eta]$.

Set $\mathscr{Y} = C_{\mathscr{W}}(\eta)$. Then clearly $C_{\mathscr{Y}}(t) = \langle t, u\rangle$, $\mathscr{Y} \cong D_8$, $t^{\mathscr{Y}} = \{t, tu\}$, $\mathscr{Y}' = Z(\mathscr{Y}) = \langle u\rangle$, $\langle\mathscr{U}, \eta\rangle \leq N_G(A\mathscr{Y})$, and $A\mathscr{Y} \leq N_{\mathscr{U}}(U)$. Also $F \leq \mathscr{V} \lhd \mathscr{W} = \mathscr{V}\mathscr{Y}$, $\eta^3 \in C_G(\mathscr{W})$, $\mathscr{V}A/A \cong E_4$, and $\langle\mathscr{U}, \eta\rangle \leq N_G(\mathscr{V})$. Moreover the arguments in [9, Section 7] yield:

LEMMA 6.1. (i) $[F, \mathscr{Y}] = 1$.

(ii) *The orbits of* $\mathscr{Y}$ *on* $t^{\mathscr{W}} = t^M = t^G \cap A = tX$ *are* $t\langle u\rangle$, $tz\langle u\rangle$, $ty\langle u\rangle$, *and* $tyz\langle u\rangle$; $\mathscr{V}A/A$ *acts regularly on these four orbits and* $O_2(Z(M)) = \langle u\rangle$.

(iii) $\mathscr{W}' = \Phi(\mathscr{W}) = \mho^1(\mathscr{W}) = X$.

(iv) $F = \langle y, z\rangle \leq \mathscr{V} \cap A \leq X$.

(v) $\langle[\mathscr{Y}, x], [\mathscr{Y}, xt]\rangle = \langle u\rangle$, *and* $x$ *or* $xt$ *centralizes* $\mathscr{Y}$.

Since $t^{\mathscr{W}} = tX$ and $C_{\mathscr{V}A}(t) = A$, it follows that no element of $tX$ is a square in $\mathscr{V}A$ and hence the proof of [9, Lemma 7.4] yields:

LEMMA 6.2. $\mathscr{V}$ *satisfies one of the following five conditions:*

(i) $\mathscr{V} \cong E_{16}$ *and* $C_{\mathscr{V}}(t) = F$.

(ii) $\mathscr{V} \cong Z_4 \times Z_4$, $F = \Omega_1(t)$, *and* $t$ *inverts* $\mathscr{V}$.

(iii) *There is a* $\langle\eta, x\rangle$ *invariant subgroup* $\mathscr{Q}$ *of* $\mathscr{V}$ *such that* $\mathscr{V} = F \times \mathscr{Q}$, $\mathscr{Q} \cong Q_8$, $\mathscr{Q}' = \langle u\rangle$, *and* $((\mathscr{Q}\langle\eta, x\rangle)/\langle\eta^3\rangle) \cong GL(2, 3)$.

(iv) $\mathscr{V}' = \langle u\rangle < X = Z(\mathscr{V}) = \Phi(\mathscr{V}) = \Omega_1(\mathscr{V})$, $\exp(\mathscr{V}) = 4$, $\mathscr{V}/\mathscr{V}' \cong Z_4 \times Z_4$, $\mathscr{V}/F \cong Q_8$, *t inverts* $\mathscr{V}/\mathscr{V}'$, *and* $((\mathscr{V}\langle\eta, x\rangle)/(\langle\eta^3\rangle \times F)) \cong GL(2, 3)$. *Also if* $\alpha \in \mathscr{V} - Z(\mathscr{V})$, *then* $|\alpha| = 4$, $C_{\mathscr{V}}(\alpha) = \langle\alpha, Z(\mathscr{V})\rangle$, *and* $\alpha^2 \notin \langle u\rangle \cup F$.

(v) $\mathscr{V}' = Z(\mathscr{V}) = \langle u \rangle$, $\mathscr{V}$ *contains subgroups* $Q_1$ *and* $Q_2$ *with* $Q_1$ *and* $Q_2$ *quaternion of order* 8 *such that* $\mathscr{V} = Q_1 * Q_2$, $\mathscr{V}$ char $\mathscr{V}A = \mathscr{V}\langle t \rangle$, $Q_1^t = Q_2$, *and* $\mathscr{V}A$ *is of type* $\mathscr{A}_8$.

Our analysis of each of these five possibilities of Lemma 7.4 is presented in one of the remaining five sections of the paper. These investigations are similar to those of [9, Sections 8–12].

## 7. The case of Lemma 6.2(i)

In this section, we shall prove:

LEMMA 7.1. *If* $\mathscr{V}$ *satisfies* (i) *of Lemma* 6.2, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V} \cong E_{16}$, $C_{\mathscr{V}}(t) = F$, and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

As in [9, Section 8], we have $\mathscr{Y} \cap \mathscr{V} = 1$, $C_{\mathscr{Y}}(\mathscr{V}) = \mathscr{P}$ max $\mathscr{Y}$, $u \in \mathscr{P}$, and

$$\langle \mathscr{U}, \eta \rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P}).$$

Set $\mathscr{Q} = \mathscr{P} \times \mathscr{V}$. Then $\mathscr{W} = \mathscr{Q}\langle t \rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t \rangle$, $Z(\mathscr{U}) = \langle u, z \rangle$, $[\mathscr{P}, t] = \langle u \rangle$, and $|C_{\mathscr{V}}(x)| = |C_{\mathscr{V}}(xt)| = 4$.

LEMMA 7.2. $\mathscr{P} \cong Z_4$.

*Proof.* Assume that $\mathscr{P} = \langle u, \omega \rangle$ where $\omega^2 = 1$. Then $\mathscr{E}_{64}(\mathscr{U}) = \{\mathscr{Q}\}$ and $I(t\mathscr{Q}) = t^{\mathscr{Q}} = tX$. If $[\mathscr{P}, xt] = 1$, then $C_{\mathscr{U}}(xt) = (\mathscr{P} \times C_{\mathscr{V}}(xt))\langle x, t \rangle$ has order $2^6$ and $\mathscr{P} \times C_{\mathscr{V}}(xt) \lhd C_{\mathscr{U}}(xt)$, which is impossible. Thus $\omega^{xt} = \omega u$ and $\omega^x = \omega$. Let $N = N_G(\mathscr{Q})$ and $\bar{N} = N/O(N)$ and let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$. Then $\mathscr{U} \neq \mathscr{T}$, $t^G \cap \mathscr{Q} = \emptyset$, and $X = C_{\mathscr{Q}}(t) \lhd C_N(t)$, so that $C_N(t) \leq N_G(A) = M$ and $C_{\mathscr{T}}(t) = U$. As $I(t\mathscr{Q}) = t^{\mathscr{Q}}$ and $\langle \mathscr{U}, \eta \rangle \leq N$, it follows that $C_G(\mathscr{Q}) = O(N) \times \mathscr{Q}$, $C_{\bar{N}}(\bar{\mathscr{Q}}) = \bar{\mathscr{Q}}$, $C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\eta}, \bar{x} \rangle$, and $\bar{N}/\bar{\mathscr{Q}} \hookrightarrow \mathrm{Aut}\,(\mathscr{Q}) \cong GL(6,2)$. Also we have $C_{\bar{\mathscr{T}}/\bar{\mathscr{Q}}}(\bar{t}\bar{\mathscr{Q}}) = \langle \bar{x}\bar{\mathscr{Q}}, \bar{t}\bar{\mathscr{Q}} \rangle$ and hence $\bar{\mathscr{T}}/\bar{\mathscr{Q}}$ is dihedral or semidihedral with $8 \leq |\bar{\mathscr{T}}/\bar{\mathscr{Q}}| \leq 16$. Thus $2^9 \leq |\mathscr{T}| \leq 2^{10}$ and $\mathscr{Q}$ is not characteristic in $\mathscr{T}$. Also $t\mathscr{Q} \sim xt\mathscr{Q}$ in $\mathscr{T}/\mathscr{Q}$, $t \sim xt$ in $\mathscr{Q}$, and $Z(\mathscr{T}/\mathscr{Q}) = \langle x\mathscr{Q} \rangle$. Since $|C_{\mathscr{Q}}(x)| = 2^4$, it follows that $\mathscr{T}/\mathscr{Q} \cong D_8$ and $|\mathscr{T}| = 2^9$. Let $\mathscr{T} \leq \mathscr{S} \in Syl_2(G)$. Then $\mathscr{T} \neq \mathscr{S}$ and we may choose an element $\tau \in N_{\mathscr{S}}(\mathscr{T}) - \mathscr{T}$ such that $\tau^2 \in \mathscr{T}$. Set $\mathscr{Q}_1 = \mathscr{Q}^\tau$. Then $\mathscr{Q}_1 \lhd \mathscr{T}$, $C_{\mathscr{Q}}(x) = \langle u, \omega, z, v \rangle = \mathscr{Q}_1 \cap \mathscr{Q}$ for some $v \in \mathscr{V}^{\#}$ such that $v^t = vz$, $C_{\mathscr{V}}(x) = \langle z, v \rangle$, and $\mathscr{Q}_1 = \langle u, \omega, z, v, x, e \rangle$ for some involution $e$. Clearly $\tau\colon \mathscr{Q} \leftrightarrow \mathscr{Q}_1$ and we may assume that $\tau \in C_{\mathscr{S}}(t)$ where $U = C_{\mathscr{T}}(t)$ is a maximal subgroup of $C_{\mathscr{S}}(\tau)$. Then

$$\tau\colon A \leftrightarrow B, \quad \tau\colon \langle u, z, y \rangle \leftrightarrow \langle u, z, x \rangle,$$

$N_{\mathscr{S}}(\mathscr{T}) = \mathscr{T}C_{\mathscr{S}}(t)$, $\tau$ normalizes $\langle u, z \rangle$, $[z, \tau] = 1$, and $u^\tau = uz$. Hence $Z(\mathscr{S}) = \langle z \rangle$.

Let $Y \lhd \mathscr{S}$ with $Y \in \mathscr{E}_8(\mathscr{S})$. Thus $t^G \cap Y = \emptyset$, $z \in Y$, and $E_4 \cong C_Y(t) \leq A \cap B = \langle t, u, z \rangle$. Hence $C_Y(t) = \langle u, z \rangle$ and $Y \leq N_{\mathscr{S}}(A) = \mathscr{U} = \mathscr{Q}\,\langle x, t \rangle$.

This implies that $Y \leq \mathcal{Q}\langle x\rangle$. If $Y \not\leq \mathcal{Q}$, then $[\mathcal{Q}, x] = \langle z, v\rangle \leq Y$ and $Y = \langle u, z, v\rangle$. Thus $Y \leq \mathcal{Q}$, $Y \leq \mathcal{Q} \cap \mathcal{Q}_1 = \langle \omega, u, z, v\rangle$ and hence

$$Y \in \{\langle u, z, \omega\rangle, \langle u, z, v\rangle, \langle u, z, \omega v\rangle\} \quad \text{and} \quad \mathcal{Q}\mathcal{Q}_1 \leq C_{\mathcal{S}}(Y) \lhd \mathcal{S}.$$

But $u^\tau = uz$ implies that $C_{N_{\mathcal{S}}(\mathcal{T})}(Y) = \mathcal{Q}\mathcal{Q}_1$. However $C_{\mathcal{S}}(t) \leq N_{\mathcal{S}}(\mathcal{T})$ and $I(t\mathcal{Q}\mathcal{Q}_1) = t^{\mathcal{T}}$ since $\mathcal{T} = (\mathcal{Q}\mathcal{Q}_1)\langle t\rangle$. This yields $C_{\mathcal{S}}(Y) = \mathcal{Q}\mathcal{Q}_1$.

Suppose that all involutions $\bar{j}\bar{\mathcal{Q}}$ of $\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}/\bar{\mathcal{Q}}$ are such that $|C_{\bar{\mathcal{Q}}}(\bar{j}\bar{\mathcal{Q}})| = 2^4$ and let $E \in \mathscr{E}_{64}(\mathcal{Q}\mathcal{Q}_1)$ with $E \neq \mathcal{Q}$. Then there is an element $w \in E \cap q_1\mathcal{Q}$ for some $q_1 \in \mathcal{Q}_1 - \mathcal{Q}$. Then $C_{\mathcal{Q}}(w) = \langle u, \omega, z, v\rangle$ and hence $E \cap \mathcal{Q} = \langle u, \omega, z, v\rangle$. Since $I(q_1\mathcal{Q}) = q_1\langle u, \omega, z, v\rangle$, it follows that $q_1 \in E$. But then $E = \mathcal{Q}_1$ and $\mathscr{E}_{16}(\mathcal{Q}\mathcal{Q}_1) = \{\mathcal{Q}, \mathcal{Q}_1\}$. This implies that $|\mathcal{S}| = \mathcal{Q}^{10}$ and we have a contradiction. It follows that some involution $\bar{j}\bar{\mathcal{Q}}$ of $\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}/\bar{\mathcal{Q}}$ is such that $|C_{\bar{\mathcal{Q}}}(\bar{j}\bar{\mathcal{Q}})| \geq 2^5$.

Let $K = N_N(\mathcal{P})$. Then $\langle C_G(\mathcal{Q}), \eta, \mathcal{T}\rangle \leq K$, $N_{\bar{N}}(\bar{\mathcal{P}}) = \overline{N_N(\mathcal{P})}$ and $\bar{K}/\bar{\mathcal{Q}} \hookrightarrow \mathrm{Aut}\,(\bar{\mathcal{Q}})$. Since $Z(\bar{\mathcal{T}}/\bar{\mathcal{Q}}) = \langle \bar{x}\bar{\mathcal{Q}}\rangle$ and $\bar{x}\bar{\mathcal{Q}} \notin O_2(\bar{K}/\bar{\mathcal{Q}}) = O_2(\bar{K})/\bar{\mathcal{Q}}$, it follows that $O_2(\bar{K}/\bar{\mathcal{Q}}) = 1$. Thus $\bar{K}/\bar{\mathcal{Q}}$ has a normal 2-complement. But $\bar{P} \lhd \bar{K}$, $O_{2,2'}(\bar{K})$ acts completely reducibly on $\bar{\mathcal{Q}}$ and $[\bar{\mathcal{Q}}, \bar{\eta}] = \bar{\mathcal{V}}$, so that $\bar{\mathcal{V}} \lhd \bar{K}$. Let $\bar{M}/\bar{\mathcal{Q}}$ be a minimal normal subgroup of $\bar{K}/\bar{\mathcal{Q}}$ with $\bar{M}/\bar{\mathcal{Q}} \leq O(\bar{K}/\bar{\mathcal{Q}})$ and assume that $|\bar{M}/\bar{K}| \in \{5, 7\}$. Then $[\bar{x}, \bar{M}] \leq \bar{\mathcal{Q}}$ and hence $\bar{M}$ acts on $C_{\bar{\mathcal{V}}}(\bar{x}) = \langle \bar{z}, \bar{v}\rangle$. Since this implies that $[\bar{M}, \bar{\mathcal{Q}}] = 1$, we have $F(O(\bar{K}/\bar{\mathcal{Q}}))$ is an elementary abelian 3-group. Thus $O(\bar{K}/\bar{\mathcal{Q}})$ is an elementary abelian 3-group.

On the other hand, $\langle \bar{\mathcal{Q}}, \bar{\mathcal{Q}}_1, \bar{\eta}\rangle \leq C_{\bar{K}}(\bar{\mathcal{P}}) \trianglelefteq \bar{K}$, $O_2(C_{\bar{K}}(\bar{\mathcal{P}})) = \bar{\mathcal{Q}}$ and

$$C_{\bar{K}}(\bar{\mathcal{P}})/\bar{\mathcal{Q}} \hookrightarrow \mathrm{Aut}\,(\bar{\mathcal{V}}) \cong GL(4, 2).$$

Moreover $\bar{\mathcal{Q}}\bar{\mathcal{Q}}_1 \in Syl_2(C_{\bar{K}}(\bar{\mathcal{P}}))$ and $(\bar{\mathcal{Q}}\bar{\mathcal{Q}}_1)' = \Phi(\bar{\mathcal{Q}}\bar{\mathcal{Q}}_1) \leq \bar{\mathcal{V}}$; hence $\bar{\mathcal{P}} \cap \Phi(\bar{\mathcal{Q}}\bar{\mathcal{Q}}_1) = 1$. Then [10, III, 4.4] and [10, I, 17.4] imply that $C_{\bar{K}}(\bar{\mathcal{P}}) = \bar{\mathcal{P}} \times \bar{L}$ where $\bar{L} \lhd C_{\bar{K}}(\bar{\mathcal{P}})$. Since $O^2(C_{\bar{K}}(\bar{\mathcal{P}})) \leq \bar{L}$, we have $\bar{\mathcal{V}} = \bar{L} \cap \bar{\mathcal{Q}} = O_2(\bar{L})$. Let $L$ denote the inverse image of $\bar{L}$ in $C_K(\mathcal{P})$ and note that $C_{\bar{K}}(\bar{\mathcal{P}})$ is $\langle \bar{t}\rangle$-invariant,

$$\bar{\mathcal{V}} \lhd C_{\bar{K}}(\bar{\mathcal{P}})\langle \bar{t}\rangle, \quad \bar{P} \lhd C_{\bar{K}}(\bar{P})\langle \bar{t}\rangle, \quad \text{and} \quad C_{\bar{N}}(\bar{\mathcal{V}}) \cap (C_{\bar{K}}(\bar{\mathcal{P}})\langle \bar{t}\rangle) = \bar{\mathcal{Q}}.$$

Thus $O_2(C_{\bar{K}}(\bar{P})\langle \bar{t}\rangle) = \bar{\mathcal{Q}}$ and $(C_{\bar{K}}(\bar{P})\langle \bar{t}\rangle)/\bar{\mathcal{Q}} \hookrightarrow \mathrm{Aut}\,(\bar{\mathcal{V}}) \cong GL(4, 2)$. Then [6, II, Lemma 2.2(vii)] implies that $\bar{\mathcal{T}}/\bar{\mathcal{P}} \cong \mathcal{T}/\mathcal{P}$ is of type $\mathscr{A}_{10}$. Recall that $\mathcal{T} = (\mathcal{Q}\mathcal{Q}_1)\langle t\rangle$, $\mathcal{Q}\mathcal{Q}_1 = \mathcal{P} \times ((\mathcal{Q}\mathcal{Q}_1) \cap L)$, and let $Y \in \mathscr{E}_{64}(\mathcal{T})$. Then $Y\mathcal{P}/\mathcal{P} \leq (\mathcal{Q}\mathcal{Q}_1)/\mathcal{P}$ and hence $\mathcal{P} \leq Y$. On the other hand, $\mathcal{Q}/\mathcal{P} \cong \mathcal{Q}_1/\mathcal{P} \cong E_{16}$ and $(\mathcal{Q}\mathcal{Q}_1)/\mathcal{P}$ max $\mathcal{T}/\mathcal{P}$, so that $(\mathcal{Q}\mathcal{Q}_1)/\mathcal{P} \cong D_8 \times D_8$ and hence $(\mathcal{Q}\mathcal{Q}_1) \cap L \cong D_8 \times D_8$. This implies that $|\mathscr{E}_{64}(\mathcal{T})| = |\mathscr{E}_{64}(\mathcal{Q}\mathcal{Q}_1)| = 4$ and $|\mathcal{S}| \leq 2^{11}$. Thus $|\mathcal{S}| = 2^{11}$, $\mathcal{S}$ is transitive on $\mathscr{E}_{64}(\mathcal{Q}\mathcal{Q}_1)$, and $|\mathcal{S}/(\mathcal{Q}\mathcal{Q}_1)| = 8$. But $N_{\mathcal{S}}(\mathcal{T})$ max $\mathcal{S}$ and $\mathcal{T} = (\mathcal{Q}\mathcal{Q}_1)\langle t\rangle$, so that $\mathcal{S}/(\mathcal{Q}\mathcal{Q}_1) \cong D_8$ and $N_{\mathcal{S}}(\mathcal{T})/(\mathcal{Q}\mathcal{Q}_1) \cong E_4$. Also, as $\mathcal{Q} \not\lhd N_{\mathcal{S}}(\mathcal{T})$, it follows that $\mathcal{S}/\mathcal{Q}\mathcal{Q}_1$ acts faithfully on $\mathscr{E}_{64}(\mathcal{Q}\mathcal{Q}_1)$. Let $\mathcal{S}_1$ denote the inverse image of $Z(\mathcal{S}/\mathcal{Q}\mathcal{Q}_1))$ in $\mathcal{S}$ and observe that $t^G \cap (\mathcal{Q}\mathcal{Q}_1) = \emptyset$, $t \notin \mathcal{S}_1$, and $N_{\mathcal{S}}(\mathcal{T}) = \mathcal{T}\mathcal{S}_1$ with $\mathcal{T} \cap \mathcal{S}_1 = \mathcal{Q}\mathcal{Q}_1$. Moreover, by [8, Corollary 2.1.2], it follows that $\mathcal{S}_1$ contains an extremal $G$-conjugate $\alpha$ of $t$ in $\mathcal{S}$ since $S/(\mathcal{Q}\mathcal{Q}_1) \cong D_8$. Now

$$\mathcal{Q}\mathcal{Q}_1 \cong E_4 \times D_8 \times D_8$$

and hence $Z(\mathscr{Q}\mathscr{Q}_1) = \mathscr{Q} \cap \mathscr{Q}_1 = \mathscr{P} \times \langle z, v\rangle$. Also there exist elements $\mathscr{Q}_2$, $\mathscr{Q}_3$ of $\mathscr{E}_{64}(\mathscr{Q}\mathscr{Q}_1)$ such that $\mathscr{Q}_2\mathscr{Q}_3 = \mathscr{Q}\mathscr{Q}_1$, $\mathscr{Q}_2 \cap \mathscr{Q}_3 = Z(\mathscr{Q}\mathscr{Q}_1)$, $\mathscr{E}_{64}(\mathscr{Q}\mathscr{Q}_1) = \{\mathscr{Q}, \mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3\}$,

$$\alpha\colon \mathscr{Q} \leftrightarrow \mathscr{Q}_1 \quad \text{and} \quad \alpha\colon \mathscr{Q}_2 \leftrightarrow \mathscr{Q}_3.$$

Thus $C_{\mathscr{Q}\mathscr{Q}_1}(\alpha) \leq Z(\mathscr{Q}\mathscr{Q}_1)$. However $u^{N_{\mathscr{S}}(\mathscr{T})} = \{u, uz\}$ and hence $\alpha\colon u \leftrightarrow uz$. Since $|\mathscr{S}/(\mathscr{Q}\mathscr{Q}_1)| = 8$ and $|C_{\mathscr{S}}(\alpha)| = 2^6$, it follows that $E_8 \cong C_{\mathscr{Q}\mathscr{Q}_1}(\alpha) \lhd C_{\mathscr{S}}(\alpha)$. As $\alpha \notin C_{\mathscr{Q}\mathscr{Q}_1}(\alpha)$, we obtain a contradiction from (4.7) and the proof of Lemma 7.2 is complete.

Thus, throughout the remainder of this section, we assume that $\mathscr{P} = \langle\omega\rangle$ where $\omega^2 = u$, $\Omega_1(\mathscr{Q}) = \langle u\rangle \times \mathscr{V}$, and $\mho^1(\mathscr{Q}) = \langle u\rangle$. Set $E = \Omega_1(\mathscr{Q})$, $N = N_G(\mathscr{Q})$, $\bar{N} = N/O(N)$, $C = C_G(\mathscr{Q})$, and $D = C_N(E)$. Then $C \leq D \trianglelefteq N \leq C_G(u)$, $D/C$ is a 2-group, $O(N) = O(C) = O(D)$, $\langle \mathscr{U}, \eta\rangle \leq N$, and $C_{\mathscr{Q}}(\eta) = \mathscr{P}$.

Applying the proofs of [9, Lemmas 8.3–8.4], we obtain the following two lemmas:

LEMMA 7.3. (i) *$\bar{C} = C_{\bar{C}}(\bar{\eta}) \times \bar{\mathscr{V}}$ where $\bar{\mathscr{V}} = [\bar{C}, \bar{\eta}]$, $C_{\bar{C}}(\bar{\eta})$ is a cyclic 2-group and $\bar{\mathscr{P}} = \Omega_2(C_{\bar{C}}(\bar{\eta}))$.*

(ii) *$\bar{U}$ normalizes $C_{\bar{C}}(\bar{\eta})$ and $C_{\bar{C}}(\bar{\eta})\langle \bar{t}\rangle$ is dihedral.*

(iii) *$\bar{\mathscr{Q}} \leq \bar{C} \leq \bar{D} \leq O_2(\bar{N})$.*

LEMMA 7.4. (i) *$\bar{D} = C_{\bar{D}}(\eta) \times \bar{\mathscr{V}}$ where $= [\bar{D}, \bar{\eta}]$.*

(ii) *$\bar{U}$ normalizes $C_{\bar{D}}(\bar{\eta})$ and $C_{\bar{D}}(\bar{\eta})\langle \bar{t}\rangle$ is dihedral or semidihedral.*

(iii) *Either $C_{\bar{D}}(\bar{\eta}) = C_{\bar{C}}(\bar{\eta})$ (and $C = D$) or $C_{\bar{D}}(\bar{\eta})$ is dihedral or generalized quaternion and $C_{\bar{C}}(\bar{\eta})$ is the unique cyclic maximal subgroup of $C_{\bar{D}}(\bar{\eta})$ when $C_{\bar{D}}(\bar{\eta})$ is not isomorphic to $Q_8$.*

(iv) *$t^G \cap D = \emptyset$.*

(v) *$\bar{\mathscr{Q}} = \bar{\mathscr{P}} \times \bar{\mathscr{V}}$ char $\bar{D}$ if $C_{\bar{D}}(\bar{\eta})$ is not isomorphic to $Q_8$.*

(vi) *$C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\eta}, \bar{x}\rangle$.*

From the nature of the remainder of the proof of Lemma 7.1 and in order to simplify the notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$.

Set $\mathscr{R} = C_D(\eta)$. Then $D = \mathscr{R} \times \mathscr{V}$, $\mathscr{R}\langle t\rangle$ is dihedral or semidihedral, $Z(\mathscr{R}\langle t\rangle) = \langle u\rangle$, $\mathscr{R} = C_{\mathscr{R}\langle t\rangle}(E)$ is cyclic, dihedral, or generalized quaternion, $E = \langle u\rangle \times \mathscr{V} \leq Z(D)$ and $t^G \cap D = \emptyset$. Let $\gamma$ be a generator of the cyclic maximal subgroup of $\mathscr{R}\langle t\rangle$. Then $\mathscr{P} \leq \langle\gamma\rangle$, $\gamma \in C$ if $C = D$ and $\langle\gamma^2\rangle = C_C(\eta)$ if $C \neq D$. Also $I(tD) = I(t\mathscr{R}) \times F$ and hence $I(tD) = t^D$ if $\mathscr{R}\langle t\rangle$ is semidihedral and $I(tD) = t^D \cup (t\gamma)^D$ if $\mathscr{R}\langle t\rangle$ is dihedral. However, if $\mathscr{R}\langle t\rangle$ is dihedral and $C \neq D$, then $\mathscr{R}$ is dihedral and $t^G \cap tD = t^D$ since $t^G \cap D \neq \emptyset$.

Now $\mathscr{U} = \mathscr{Q}\langle x, t\rangle = (\mathscr{P} \times \mathscr{V})\langle x, t\rangle$, $|C_{\mathscr{V}}(xt)| = |C_{\mathscr{V}}(x)| = 4$, $|C_E(xt)| = 2^3$, and $C_E(xt) \lhd C_{\mathscr{U}}(xt)$. Then (4.7) implies that $[\omega, xt] = \mathscr{U}$ and hence $[\omega, x] = 1$. Similarly, since $\mathscr{U} \leq D\langle x, t\rangle$, we conclude that $C_{\mathscr{R}}(xt) = \langle u\rangle$

and hence $\mathscr{R}\langle xt \rangle$ is dihedral or semidihedral. Also, it is clear that $\mathscr{P} \lhd \mathscr{R}\langle xt \rangle$ and if $\tau \in I(xtD)$, then

$$C_{D\langle xt \rangle}(\tau) = \langle \tau, u \rangle \times C_{\mathscr{V}}(xt) \cong E_{16}.$$

LEMMA 7.5. $t^G \cap (D\langle x \rangle) = \emptyset$.

*Proof.* Suppose that $t^G \cap (D\langle x \rangle) \neq \emptyset$.

Since $D\langle x \rangle = \langle \mathscr{R} \times \mathscr{V})\langle x \rangle$, $t^G \cap D = \emptyset$, and $|C_{\mathscr{V}}(x)| = 4$, it follows that there is an involution $\tau \in t^G \cap (\mathscr{R}\langle x \rangle) - \mathscr{R}$. Now $\mathscr{P} \times \langle x \rangle \leq C_{\mathscr{R}\langle x \rangle}(x)$ and hence $\mathscr{R}\langle x \rangle$ is neither dihedral nor semidihedral. Thus $|C_{\mathscr{R}\langle x \rangle}(\tau)| > 4$. But

$$C_{D\langle x \rangle}(\tau) = C_{\mathscr{R}\langle x \rangle}(\tau) \times C_{\mathscr{V}}(x) \quad \text{and} \quad E_{16} \cong \langle \tau, u \rangle \times C_{\mathscr{V}}(x) \leq Z(C_{D\langle x \rangle}(\tau)),$$

so that we obtain a contradiction from (4.10). Thus Lemma 7.5 follows.

Let $\hat{E} = E/\langle u \rangle$. Then $\hat{E} \cong E_{16}$, $N$ acts on $\hat{E}$ and $D \leq C_N(\hat{E}) \trianglelefteq N = N_G(\mathscr{Q})$.

The proof of [9, Lemma 8.5] yields:

LEMMA 7.6. *$C_N(\hat{E}) = D$ and $N/D \hookrightarrow$ Aut $(\hat{E}) \cong GL(4, 2)$.*

Choose $v_1 \in \mathscr{V}^{\#}$ such that $C_{\mathscr{V}}(x) = \langle z, v_1 \rangle$. Then $v_1^t = v_1 z$ and $C_{\mathscr{V}}(xt) = \langle z, v_1 y \rangle$. Also let $D\langle x, t \rangle \leq \mathscr{T} \in Syl_2(N)$.

The proofs of [9, Lemmas 8.6–8.7] yield:

LEMMA 7.7. (i) $\mathscr{T} \neq D\langle x, t \rangle$.

(ii) $C_{N/D}(tD) = \langle tD \rangle \times \langle \eta D, xD \rangle = C_N(t)D/D$.

(iii) $O_2(N) = D$.

(iv) $\mathscr{T}/D \cong D_8$, $Z(\mathscr{T}/D) = \langle xD \rangle$, *and* $tD \sim xtD$ *in* $\mathscr{T}$.

Next we prove:

LEMMA 7.8. $\mathscr{T} \in Syl_2(G)$.

*Proof.* Assume that $\mathscr{T}$ is a maximal subgroup of the 2-group $\mathscr{S}$ and let $\tau \in \mathscr{S} - \mathscr{T}$. Then $\tau^2 \in \mathscr{T}$ and $\mathscr{Q} \neq \mathscr{Q}_1 = \mathscr{Q}^\tau \lhd \mathscr{T}$. Let $E_1 = \Omega_1(\mathscr{Q}_1) = E^\tau$. Note that $U = C_{\mathscr{T}}(t)$, $t^G \cap \mathscr{Q}_1 = \emptyset$, and $C_{E_1}(t) = \langle u, y, z \rangle$ or $C_{E_1}(t) = \langle u, x, z \rangle$. As in the proof of [9, Lemma 8.8], it follows that

$$C_{E_1}(t) = \langle u, x, z \rangle \quad \text{and} \quad D \cap \mathscr{Q}_1 = \mathscr{Q} \cap \mathscr{Q}_1 = \mathscr{P} \times \langle z, v_1 \rangle.$$

Since $\mathscr{S}$ normalizes $\mathscr{Q} \cap \mathscr{Q}_1$, it follows that $\langle u \rangle \leq Z(\mathscr{S})$. If $t^\tau \sim t$ in $\mathscr{T}$, then we may assume that $\tau \in C_G(t, u)$. Then $\tau$ normalizes $UC_{\mathscr{T}}(t)$ and hence $\tau \in C_G(t, u, z)$. Thus $|C_{\mathscr{S}}(t)| = 2^6$ and $Z(C_{\mathscr{S}}(t)) = \langle t, u, z \rangle$ which contradicts (4.1). Hence $t^\tau \nsim t$ in $\mathscr{T}$ and, utilizing (4.14), the proof of [9, Lemma 8.8] yields a contradiction. Thus Lemma 7.8 is established.

Finally the argument at the end of [9, Section 8] can now be applied to establish Lemma 7.1.

## 8. The case of Lemma 6.2(ii)

In this section, we shall prove:

LEMMA 8.1. *If* $\mathscr{V}$ *satisfies* (ii) *of Lemma* 6.2, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V} \cong Z_4 \times Z_4$, $F = \Omega_1(\mathscr{V})$, $t$ inverts $\mathscr{V}$, and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

As in [9, Section 9], let $v_1 \in \mathscr{V}$ be such that $v_1^2 = y$ and set $v_2 = v_1^x$ and $v = v_1v_2$. Then $v_2^2 = yz$, $v^2 = z$, $\mathscr{V} = \langle v_1, v_2\rangle$, and $C_{\mathscr{V}}(x) = \langle v\rangle$. Also $\mathscr{P} = C_{\mathscr{Y}}(\mathscr{V})$ max $\mathscr{Y}$, $u \in \mathscr{P}$, $[\mathscr{P}, \mathscr{V}] = 1$, and $\langle \mathscr{U}, \eta\rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P})$. Set $\mathscr{Q} = \mathscr{P} \times \mathscr{V}$. Then $\mathscr{W} = \mathscr{Q}\langle t\rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t\rangle$, and $Z(\mathscr{U}) = \langle u, z\rangle$. Note also that

$$C_{\mathscr{V}}(xt) = \langle vy\rangle, \quad ((\langle u\rangle \times \langle vy\rangle)\langle x, t\rangle) \leq C_{\mathscr{U}}(xt) \quad \text{and} \quad \langle u, z, vy\rangle \leq Z(C_{\mathscr{U}}(xt)).$$

Thus $C_{\mathscr{U}}(xt) = (\langle u\rangle \times \langle vy\rangle)\langle x, t\rangle$, $[\mathscr{P}, x] = 1$, and $[\mathscr{P}, xt] = \langle u\rangle$. Clearly $t^G \cap \mathscr{Q} = \emptyset$ and if $\tau \in I(x\mathscr{Q})$, then $C_{\mathscr{Q}}(\tau) = \mathscr{P} \times \langle v\rangle$, $|C_{\mathscr{Q}\langle x\rangle}(\tau)| = 2^5$, and $C_{\mathscr{Q}\langle x\rangle}(\tau)$ is abelian. Thus $t^G \cap (\mathscr{Q}\langle x\rangle) = \emptyset$ by Lemma 2.1(vi). We also clearly have $\mathscr{U}' = \langle u\rangle \times \langle v\rangle \times \langle y\rangle$, $C_{\mathscr{U}}(\mathscr{U}') = \mathscr{Q}$, $\Omega_1(\mathscr{U}') = X$, $C_{\mathscr{U}}(X) = \mathscr{W}$, $\mho^1(\mathscr{U}') = \langle z\rangle$, and $|\mathscr{U}| = 2^8$.

LEMMA 8.2. $\mathscr{P} \cong Z_4$.

*Proof.* Assume that $\mathscr{P} = \langle u, \omega\rangle$ where $\omega^2 = 1$. Let $\mathscr{U}$ be a maximal subgroup of the 2-subgroup $\mathscr{T}$ of $G$ and let $L = N_G(\mathscr{U})$. Clearly $\mathscr{T} \leq L$, $\mathscr{Q} \lhd L$, $\mathscr{W} = \mathscr{Q}\langle t\rangle \lhd L$, $Z(\mathscr{W}) = \langle u, y, z\rangle \lhd L$, $\mathscr{Q}\langle x\rangle \lhd L$, and $\mathscr{Q}\langle xt\rangle \lhd L$ as $t^G \cap (\mathscr{Q}\langle x\rangle) = \emptyset$. Thus $C_{\mathscr{Q}}(x) = \langle \omega, u\rangle \times \langle v\rangle \lhd L$, $\langle \omega, u, z\rangle \lhd L$, and $\langle z\rangle \lhd L$. Since $C_L(\mathscr{W}) \leq C_L(A) = O(C_L(A)) \times A$, it follows that $C_L(\mathscr{U}) = O(L) \times \langle u, z\rangle$ where $Z(U) = \langle u, z\rangle$. Hence $L$ has a normal 2-complement. As $I(xt\mathscr{Q}) = (xt)^{\mathscr{Q}}$, $|C_{\mathscr{U}}(xt)| = 2^5$, and $Z(C_{\mathscr{U}}(xt)) = \langle u, z, xt\rangle$, it follows that $\mathscr{T} \in Syl_2(L)$, $L = O(L)\mathscr{T}$, $\mathscr{T} = \mathscr{U}C_{\mathscr{T}}(xt)$, and $u \notin Z(C_{\mathscr{T}}(xt))$ since $\langle z\rangle \leq Z(C_{\mathscr{T}}(xt))$ and $|C_{\mathscr{T}}(xt)| = 2^6$.

Note that $\mathscr{Q} = J_0(\mathscr{W})$ char $\mathscr{W}$ and set $N = N_G(\mathscr{W})$, $\bar{N} = N/O(N)$, and $E = \Omega_1(\mathscr{Q}) = \mathscr{P} \times F$. Clearly $X = Z(\mathscr{W}) \lhd N$, $C_E(t) = X = C_{\mathscr{Q}}(t)$,

$$I(t\mathscr{Q}) = t^{\mathscr{Q}} \cup (tv_1)^{\mathscr{Q}} \cup (tv_2)^{\mathscr{Q}} \cup (tv)^{\mathscr{Q}}$$

and $\langle \mathscr{U}, \eta\rangle \leq N$. Also $C_N(t) = O(C_N(t))A\langle \eta, x\rangle$ where $O(C_N(t)) \leq C_G(\mathscr{W})$ and $C_G(\mathscr{W}) = O(N) \times Z(\mathscr{W})$. Then $C_{\bar{N}}(\bar{\mathscr{W}}) = \bar{X}$, $C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\eta}, \bar{x}\rangle$ where $\bar{\eta}^3 = 1$ and $\bar{N} = O_2(\bar{N})\langle \bar{\eta}, \bar{x}\rangle$ as $\bar{N}/\bar{X} \hookrightarrow \mathrm{Aut}(\mathscr{W})$ and $|\mathrm{Aut}(\mathscr{W})|_{2'} = 3$. Now $\mathscr{T} \leq N$, $C_{O_2(\bar{N})}(\bar{\eta}) = \bar{\mathscr{Y}}$ since $C_{O_2(\bar{N})}(\bar{t}) = \bar{A}$ and $C_{\bar{E}}(\bar{\eta}) = \bar{\mathscr{P}} \lhd C_{O_2(\bar{N})}(\bar{\eta})$. Thus $\bar{N}/\bar{\mathscr{W}} \cong \Sigma_4$ and $O_2(\bar{N})/\bar{\mathscr{W}} \cong E_4$. As $F \lhd N$, we conclude that $[O_2(\bar{N}), \bar{F}] = 1$. Since $\bar{\mathscr{T}} = (O_2(\bar{N}) \cap \bar{\mathscr{T}})\langle \bar{x}\rangle$ and $\bar{u} \notin Z(\bar{\mathscr{T}})$, it follows that $C_{\bar{N}}(\bar{X}) = \bar{\mathscr{W}}$ and hence $C_{\bar{N}}(\bar{E}) = \bar{\mathscr{Q}}$. Then $\bar{N}/\bar{\mathscr{Q}} \hookrightarrow \mathrm{Aut}(\bar{E}) \cong GL(4, 2)$. But then $\bar{N}/\bar{\mathscr{Q}} \cong Z_2 \times \Sigma_4$ where $\langle \bar{t}\bar{\mathscr{Q}}\rangle \lhd \langle \bar{N}/\bar{\mathscr{Q}}\rangle$. Hence $[\bar{E}, \bar{t}] = \langle \bar{u}\rangle \lhd \bar{N}$ and we have a contradiction since $\bar{u} \notin Z(\bar{\mathscr{T}})$. This completes the proof of Lemma 9.2.

Let $\mathscr{P} = \langle \omega \rangle$ where $\omega^2 = u$. Thus $t$ inverts $\mathscr{Q} = \mathscr{P} \times \mathscr{V} \cong Z_4 \times Z_4 \times Z_4$ and $\Omega_1(\mathscr{Q}) = X$. Also $I(t\mathscr{Q}) = t\mathscr{Q}$, $I(x\mathscr{Q}) = x(\langle u \rangle \times \langle vy \rangle)$ and $I(xt\mathscr{Q}) = xt(\mathscr{P} \times \langle v \rangle)$. Note also that if $\tau_1 \in I(t\mathscr{Q})$, $\tau_2 \in I(x\mathscr{Q})$, $\tau_3 \in I(xt\mathscr{Q})$, then $\langle \tau_1, X \rangle \leq \langle \tau_1^{\mathscr{Q}} \rangle$, $\langle \tau_2, vy \rangle \leq \langle \tau_2^{\mathscr{Q}} \rangle$, and $\langle \tau_3, v \rangle \leq \langle \tau_3^{\mathscr{Q}} \rangle$.

LEMMA 8.3. *Let $\mathscr{U} \leq \mathscr{R}$ where $\mathscr{R}$ is a 2-subgroup of $G$. Then $X \lhd \mathscr{R}$ and $X$ is the unique normal element of $\mathscr{E}_8(\mathscr{R})$.*

*Proof.* Since $X = \Omega_1(\mathscr{U}')$ char $\mathscr{U}$, it suffices, by induction on $|\mathscr{R}|$, to assume that $X \lhd \mathscr{R}$ and to show that $X$ is unique. Thus let $X \neq Y \lhd \mathscr{R}$ where $Y \in \mathscr{E}_8(\mathscr{R})$. Assume that $t^G \cap Y \neq \emptyset$. Then $|\mathscr{R}| \leq 2^8$ and hence $\mathscr{R} = \mathscr{U}$. This implies that $Y \leq \mathscr{Q}$ and hence $Y = X$, which is impossible. Thus $t^G \cap Y \neq \emptyset$. Also, since $X \lhd C_{\mathscr{R}}(t)$, it follows that $U = C_{\mathscr{R}}(t)$. Now the proof of [9, Lemma 9.3] applies to yield Lemma 8.3.

Clearly $\mathscr{Q} = J_0(\mathscr{W})$ char $\mathscr{W} = \mathscr{Q}\langle t \rangle$,

$$\langle \mathscr{U}, \eta \rangle \leq N_G(\mathscr{W}) \leq N_G(\mathscr{Q}),\ C_G(\mathscr{W}) = O(C_G(\mathscr{W})) \times X, \text{ and } \eta^3 \in O(C_G(\mathscr{W})).$$

LEMMA 8.4. (i) *$\mathscr{Q} \leq N_G(\mathscr{W}) \cap C_G(\mathscr{Q}) \trianglelefteq N_G(\mathscr{W})$ and $O(N_G(\mathscr{W}))$ is a normal 2-complement of $N_G(\mathscr{W}) \cap C_G(\mathscr{Q})$.*

(ii) *Either $\mathscr{Q} \in Syl_2(N_G(\mathscr{W}) \cap C_G(\mathscr{Q}))$ and*

$$t^{(N_G(\mathscr{W}) \cap C(G(\mathscr{Q}))} = tX = t^{\mathscr{Q}}$$

*or $\mathscr{Q}$ is a maximal subgroup of a Sylow 2-subgroup of $N_G(\mathscr{W}) \cap C_G(\mathscr{Q})$ and*

$$t^{(N_G(\mathscr{W}) \cap C_G(\mathscr{Q}))} = t(\mathscr{P} \times F).$$

*Proof.* Let $N = N_G(\mathscr{W})$, $\bar{N} = N/O(N)$, and $J = C_N(\mathscr{Q})$. Clearly $\mathscr{Q} \leq Z(J)$, $J \lhd N \leq N_G(\mathscr{Q})$, $O(N) = O(J) = O(C_G(\mathscr{W}))$, $O^2(J) = O(J)$, and (i) holds. Thus $\bar{J} = C_{\bar{N}}(\bar{\mathscr{Q}})$ is a 2-group and $\langle \bar{x}, \bar{t} \rangle$ normalizes $\bar{J}$. Also

$$\bar{\mathscr{U}} \leq \bar{J}\langle \bar{x}, \bar{t} \rangle \quad \text{and} \quad \langle \bar{u}, \bar{z}, \bar{x}\bar{t} \rangle \leq Z(C_{\bar{J}\langle \bar{x}, \bar{t} \rangle}(\bar{x}\bar{t})).$$

Hence $C_{\bar{\mathscr{U}}}(\bar{x}\bar{t}) = (\langle \bar{u} \rangle \times \langle \bar{v}\bar{y} \rangle)\langle \bar{x}, \bar{t} \rangle \in Syl_2(C_{\bar{J}\langle \bar{x}, \bar{t} \rangle}(\bar{x}\bar{t}))$ and the proof of [9, Lemma 9.4] yields Lemma 8.4.

For the remainder of this section, let $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar{N} = N/O(N)$. Clearly $\langle \mathscr{U}, \eta \rangle \leq N$. Let $Y = C\langle t \rangle$ and let $\mathscr{U} = \mathscr{Q}\langle x, t \rangle \leq \mathscr{T} \in Syl_2(N)$. Clearly $\eta^3 \in C$ and $Y \lhd N$. Also let $O(N) \leq \mathscr{R} \leq C$ be such that $\bar{\mathscr{R}} = C_{\bar{C}}(\eta)$.

Applying the proof of [9, Lemma 9.5], we obtain:

LEMMA 8.5. (i) *$\bar{C} = \bar{\mathscr{R}} \times \bar{\mathscr{V}}$, $\bar{\mathscr{V}} = [\bar{C}, \bar{\eta}]$, $\bar{\mathscr{R}}$ is a cyclic 2-group and $\bar{\mathscr{P}} = \Omega_2(\bar{\mathscr{R}})$.*

(ii) *$\bar{U}$ normalizes $\bar{\mathscr{R}}$ and $\bar{\mathscr{R}}\langle \bar{t} \rangle$ is dihedral or semidihedral.*

(iii) *$C_{\bar{N}}(\bar{t}) = \bar{A}\langle \bar{\eta}, \bar{X} \rangle$.*

As in the previous section, without loss of generality, we assume that $O(N) = 1$. Thus $C = \mathscr{R} \times \mathscr{V}$ and $C\langle x, t\rangle \leq \mathscr{T} \in Syl_2(N)$. Since $C_C(x) = \mathscr{P} \times \langle v\rangle$, it follows that $t^G \cap (C\langle x\rangle) = \emptyset$. Then, as in [9, Lemmas 9.6–9.7], we obtain:

LEMMA 8.6. (i) $\mathscr{Q} = \Omega_2(C)$ char $C\langle x, t\rangle$.
(ii) $\mathscr{T} \neq C\langle x, t\rangle$.
(iii) $N = O_2(N)\langle \eta, x\rangle$ *and* $\mathscr{T} = O_2(N)\langle x\rangle$.

Next we prove:

LEMMA 8.7. *$\mathscr{R}\langle t\rangle$ is dihedral and $t$ inverts $C$.*

*Proof.* Assume that $\mathscr{R}\langle t\rangle$ is semidihedral. Then, as in [9, Lemma 9.8], we conclude that $N/C \cong Z_2 \times \Sigma_4$ where $Z(N/C) = \langle tC\rangle$, $X$ char $\mathscr{T}$, $\mathscr{Q}$ char $\mathscr{T}$, $\mathscr{T} \in Syl_2(G)$, and $N_G(\mathscr{T}) = \mathscr{T}$. Since $C \lhd \mathscr{T}$, we also have $\langle u\rangle \leq Z(\mathscr{T})$ and hence $Z(\mathscr{T}) = \langle u\rangle$ by (4.15), $\langle u\rangle = Z(N)$, $C_{O_2(N)}(X) = Y$ char $\mathscr{T}$, and $C$ char $\mathscr{T}$. Since $[y, x] = z$, it follows that $\langle u, z\rangle$ is the unique element of $\mathscr{E}_4(X)$ that is normal in $\mathscr{T}$.

Since $u^\delta = uz$, it follows from the proof of [9, Lemma 9.8] that there is an element $g \in N$ such that $(uz)^g = u$. Since $\langle u\rangle = Z(N)$, we have a contradiction and we are done.

Setting $\mathscr{X} = O_2(N)$, we prove:

LEMMA 8.8. (i) $\mathscr{R}\langle t\rangle = C_{\mathscr{X}}(\eta)$.
(ii) $\mathscr{X}/C \cong E_8$.
(iii) $\mathscr{Q}$ char $\mathscr{T}$ *and* $\mathscr{T} \in Syl_2(G)$.

*Proof.* Assume that $\mathscr{R}\langle t\rangle \neq C_{\mathscr{X}}(\eta)$. Then, as $C_{\mathscr{X}}(\eta, t) = \langle t, u\rangle$, it follows that $C_{\mathscr{X}}(\eta)$ is dihedral or semidihedral. Since $C_Y(\eta) = \mathscr{R}\langle t\rangle \lhd C_{\mathscr{X}}(\eta)$, we conclude that $\mathscr{R}\langle t\rangle$ max $C_{\mathscr{X}}(\eta)$. Let $\alpha$ generate the cyclic maximal subgroup of $C_{\mathscr{X}}(\eta)$. Then $C_{\mathscr{X}}(\eta) = \langle \alpha, t\rangle$ and $\mathscr{R} = \Phi(C_{\mathscr{X}}(\eta)) = \mho^1(\langle\alpha\rangle)$. Set $\mathscr{S} = \langle C, t, \alpha, x\rangle$. Then, as in [9, Lemma 9.9], it follows that $\mathscr{V} \lhd \mathscr{S}$, $\alpha$ inverts $\mathscr{V}$, $t \sim t\alpha$ in $G$, $\mathscr{S}/C \cong E_8$, $C\langle x, t\rangle$ max $\mathscr{S}$, $\langle u, z\rangle \leq Z(\mathscr{S})$, $C\langle x\rangle \lhd \mathscr{S}$, and $C\langle xt\rangle \lhd \mathscr{S}$. Moreover, if $\mathscr{R}\langle xt\rangle$ is not dihedral, then it follows that $|C_{\mathscr{S}}(xt)| = 2^6$ and $\langle u, z, xt\rangle \leq Z(C_{\mathscr{S}}(xt))$ which is impossible. Thus $xt$ inverts $\mathscr{R}$ and $[\mathscr{R}, x] = 1$. Suppose that $\mathscr{S} \neq \mathscr{T}$ and let $\gamma \in N_{\mathscr{X}}(\mathscr{S}) - \mathscr{S}$ be such that $\gamma^2 \in \mathscr{S} \cap \mathscr{X}$. Then as in [9, Lemma 9.9], it follows that $\gamma$ acts trivially on $\mathscr{S}/C$ and that $[\gamma, \langle u, z\rangle] = 1$. But $I(xtC) = (xt)^{\langle C, \alpha\rangle}$ and hence $|C_{\langle \mathscr{S}, \gamma\rangle}(xt)| = 2^6$. Since $\langle u, z, xt\rangle \leq Z(C_{\langle \mathscr{S}, \gamma\rangle}(xt))$, this is impossible and we must have $\mathscr{S} = \mathscr{T}$. Then, as in [9, Lemma 9.9], we obtain a contradiction. Thus (i) holds. Moreover, as in [9, Lemma 9.9], we conclude that (ii) holds and $X$ char $\mathscr{T}$. Suppose that (iii) is false. Then $C_{\mathscr{T}}(X) = \mathscr{Z}$ char $\mathscr{T}$, $Z(\mathscr{T}) = \langle u, z\rangle$, and $\langle u\rangle = Z(N)$. Moreover, setting $\mathscr{J} = [\mathscr{X}, \eta]$, as in [9, Lemma 9.9], we conclude that $\mathscr{Z}' = \langle u\rangle \times \mathscr{V}$, $\mathscr{J}' = \langle u\rangle$, $\mathscr{Z}' \leq Z(\mathscr{J})$, and $N_{\mathscr{T}}(A) = \mathscr{Z}'$. Then it follows that $\langle u, z, xt\rangle \leq C_{\mathscr{S}}(xt)$ and $|C_{\mathscr{T}}(xt)| = 2^6$, which is impossible. Thus (iii) also holds.

Hence $N_G(\mathscr{T}) = N_N(\mathscr{T}) = \mathscr{T}$, $|\mathscr{T}| \geq 2^{11}$, and $|\mathscr{R}| \geq 2^3$. Thus $\langle u\rangle \leq Z(N)$. Then the argument at the end of Lemma 8.7 yields a contradiction. Hence the proof of Lemma 8.1 is complete.

## 9. The case of Lemma 6.2(iii)

In this section, we shall prove:

LEMMA 9.1. *If $\mathscr{V}$ satisfies* (iii) *of Lemma* 6.2, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus throughout this section, we assume that $\mathscr{V}$ contains a $\langle \eta, x\rangle$-invariant subgroup $\mathscr{Q}$ such that $\mathscr{V} = F \times \mathscr{Q}$, $\mathscr{Q} \cong Q_8$, $\mathscr{Q}' = \langle u\rangle$, and

$$((\mathscr{Q}\langle \eta, x\rangle)/\langle \eta^3\rangle \cong GL(2, 3).$$

We shall also assume that $|O^2(G)|_2 \geq 2^{11}$ and we shall proceed to obtain a contradiction.

Clearly $\mathscr{Y} \cap \mathscr{V} = \langle u\rangle$ and $\mathscr{Y}$ acts on

$$C^*_{\mathscr{V}}(x) = \{v \in \mathscr{V} \mid v^x = v \text{ or } v^x = v^{-1}\} = \langle z\rangle \times \langle q\rangle$$

where $q \in \mathscr{Q}$ is such that $q^x = q^{-1} = qu$. Also $t^{\mathscr{Q}} = tF$ or $t^{\mathscr{Q}} = tuF$ and hence $q^t \in \{qz, quz\}$. Since $\mathscr{Q} = \langle q, q^\eta, q^{\eta^2}\rangle$, it follows that no element of $\mathscr{Y}$ can invert $q$. Then, as in [9, Section 10], $C_{\mathscr{Y}}(\mathscr{V}) = \mathscr{P}$ is a maximal subgroup of $\mathscr{Y}$ and $\langle \mathscr{U}, \eta\rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P})$. Also $\langle \mathscr{V}, \mathscr{P}, \eta, x\rangle \leq N_G(\mathscr{Q})$ and $I(t\mathscr{V}) = t^{\mathscr{V}} \cup (tu)^{\mathscr{V}}$. Set $E = \mathscr{P}\mathscr{V} = \mathscr{P} * \mathscr{V}$. Then $\mathscr{W} = E\langle t\rangle$, $E \lhd \mathscr{U} = E\langle x, t\rangle$, $Z(\mathscr{U}) = \langle x, t\rangle$, $Z(E) = \mathscr{P} \times F$, and $[\mathscr{P}, t] = \langle u\rangle$.

Suppose that $\mathscr{P} = \langle u, \omega\rangle$ where $\omega^2 = 1$. Then $E = \langle \omega, y, z\rangle \times \mathscr{Q}$ and $t^G \cap E = \emptyset$ by (4.6). But then the proof of [9, Lemma 10.2] implies that $\mathscr{U} \in Syl_2(G)$. Since $|\mathscr{U}| = 2^8$, we have:

LEMMA 9.2. $\mathscr{P} \cong Z_4$.

Let $\mathscr{P} = \langle \omega\rangle$ where $\omega^2 = u$ and $\omega^t = \omega^{-1}$. Thus $I(tE) = tX \cup (t\omega)X$, $E = F \times (\mathscr{P} * \mathscr{Q})$, $|E| = 2^6$, and if $j \in I(E) - Z(E)$, then $C_E(j)$ is abelian of order $2^5$. Then (4.1) and the fact that $\tilde{S} = S/\langle t\rangle$ is isomorphic to the group given in Lemma 2.1 with $n = 3$, imply that $t^G \cap E = \emptyset$. Also $C_E(t) = C_E(t\omega)$ $X = \Omega_1(Z(E))$ and $Z(E) = F \times \mathscr{P}$. Moreover $\mathscr{U}' = F \times \langle q\rangle$, $C_{\mathscr{U}}(\Omega_1(\mathscr{U}')) = C_{\mathscr{U}}(X) = E\langle t\rangle$ char $\mathscr{U}$, and hence $\Omega_1(E) = E$ char $\mathscr{U}$.

Set $N = N_G(E)$, $\bar{N} = N/O(N)$, and $C = C_G(E)$. Thus $\langle \mathscr{U}, \eta\rangle \leq N$, $\eta^3 \in C$, and $Z(E) = F \times \mathscr{P} \leq Z(C)$. Also let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$ and set $Y = C\langle t\rangle$. Note that $E' = \langle u\rangle \leq Z(N)$ and $X = \Omega_1(Z(E)) \lhd N$. Let $O(N) \leq \mathscr{R} \leq C$ be such that $\bar{\mathscr{R}} = C_{\bar{C}}(\bar{\eta})$. Since $C_E(t) = X \lhd C_N(t)$, we have $U \in Syl_2(C_N(t))$. The proof of [9, Lemma 10.3] yields:

LEMMA 9.3. (i) $\bar{C} = \bar{\mathscr{R}} \times \bar{F}$ *where* $[\bar{C}, \bar{\eta}] = \bar{F}$, $\bar{\mathscr{R}} = C_{\bar{C}}(\bar{\eta})$ *is a cyclic 2-group*, $\bar{\mathscr{P}} = \Omega_2(\bar{\mathscr{R}})$, *and* $\bar{X} = \Omega_1(\bar{C})$.

(ii) $\bar{U} = \bar{A}\langle\bar{x}\rangle$ *normalizes* $\bar{\mathscr{R}}$, $C_{\bar{\mathscr{R}}}(\bar{t}) = \langle\bar{u}\rangle$, *and* $\bar{\mathscr{R}}\langle\bar{t}\rangle$ *is dihedral or semidihedral.*

(iii) $C_{\bar{N}}(\bar{t}) = \bar{A}\langle\bar{\eta}, \bar{x}\rangle$.

From the nature of the remainder of the proof of Lemma 9.1 and in order to simplify notation, it is clear that, without loss of generality, we may (and shall) assume that $O(N) = 1$. Then $C = \mathscr{R} \times F$,

$$EC = F \times (\mathscr{R} * \mathscr{Q}) \lhd N, \quad EC \leq C_N(X) \lhd N,$$

and $t^G \cap (EC) = \emptyset$ since $Z(EC) = F \times \mathscr{R}$ has order at least $2^4$. Since $X \lhd N$, we also have:

(9.1) $EC\langle x\rangle \nsim EC\langle t\rangle \nsim EC\langle xt\rangle$ in $N$.

Since $\exp(S) = 2^3$, the proof of [9, Lemma 10.4] yields:

LEMMA 9.4. $\mathscr{R}\langle t\rangle$ *is dihedral.*

Let $\mathscr{R} = \langle\gamma\rangle$. Then $I(tEC) = t^{EC} \cup (t\gamma)^{EC}$. Since $U \leq EC\langle x, t\rangle$ and $t^G \cap (EC) = \emptyset$, we conclude that $|N_N(EC\langle t\rangle)\colon EC(\langle t\rangle \times \langle\eta, x\rangle)| \leq 2$. Moreover, the proof of [9, Lemma 10.5] yields:

LEMMA 9.5. $N_N(EC\langle t\rangle) \neq EC(\langle t\rangle \times \langle\eta, x\rangle)$.

Set $J = N_N(EC\langle t\rangle)$. Then $J = O_2(J)\langle\eta, x\rangle$,

$$[O_2(J), \eta] = [EC, \eta] = F \times \mathscr{Q} = \mathscr{V} \lhd J.$$

Also $O_2(J)$ acts on $X$ and $|O_2(J)/C_{O_2(J)}(X)| \leq 2$. It follows that $[O_2(J), X] = 1$, $F \lhd J$, and $O_2(J) = \mathscr{V}C_{O_2(J)}(\eta)$. Then $\mathscr{R}\langle t\rangle = C_{EC\langle t\rangle}(\eta)$ is a maximal subgroup of $C_{O_2(J)}(\eta)$, $C_{O_2(J)}(\eta, t) = \langle t, u\rangle$, and $C_{O_2(J)}(\eta)$ is $\langle x\rangle$-invariant and dihedral or semidihedral. Also, as in [9, Section 10], it follows that $C_{O_2(J)}(\eta, \mathscr{V}) = \mathscr{R}_1$ is a maximal subgroup of $C_{O_2(J)}(\eta) = \mathscr{R}_1\langle t\rangle$, $\mathscr{R}_1$ is dihedral or generalized quaternion, $\mathscr{R}$ is the cyclic maximal subgroup of $\mathscr{R}_1$,

$$\mathscr{R}_1 EC = F \times (\mathscr{R}_1 * \mathscr{Q}) \quad \text{and} \quad \mathscr{S} = (F \times (\mathscr{R}_1 * \mathscr{Q}))\langle x, t\rangle \in Syl_2(J).$$

Moreover, it also follows that $X$ char $\mathscr{S}$,

$$C_{\mathscr{S}}(X) = (F \times (\mathscr{R}_1 * \mathscr{Q}))\langle t\rangle \text{ char } \mathscr{S}$$

and $I(t(F \times (\mathscr{R}_1 * \mathscr{Q})) = t^{(F \times \mathscr{R}_1 * \mathscr{Q}))}$. It is easy to see that $t^G \cap (F \times (\mathscr{R}_1 * \mathscr{Q})) = \emptyset$. Since $X$ char $\mathscr{S}$ and $\mathscr{U} = N_{\mathscr{S}}(A) \in Syl_2(N_G(A))$, it follows that $\mathscr{S} \in Syl_2(G)$. Then the last portion of [9, Section 10] applies and the proof of Lemma 9.1 is complete.

## 10. The case of Lemma 6.2(iv)

In this section, we shall prove:

LEMMA 10.1. *If* $\mathscr{V}$ *satisfies* (iv) *of Lemma* 6.2, *then* $|O^2(G)|_2 \leq 2^{10}$.

Thus, throughout this section, we assume that $\mathscr{V}$ satisfies (iv) of Lemma 6.2 and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

As in [9, Section 11], we conclude that if $q \in \mathscr{V} - X$, then $q^t = q^{-1}u = q^3u$ and that $\mathscr{P} = C_{\mathscr{Y}}(\mathscr{V})$ max $\mathscr{Y}$. Clearly $u \in \mathscr{P}$, $\langle \mathscr{U}, \eta \rangle \leq N_G(\mathscr{V}) \cap N_G(\mathscr{P})$, and $I(t\mathscr{V}) = t^{\mathscr{V}} \cup (tu)^{\mathscr{V}}$. Set $\mathscr{Q} = \mathscr{P}\mathscr{V} = \mathscr{P} * \mathscr{V}$. Then $\mathscr{W} = \mathscr{Q}\langle t \rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t \rangle$, $Z(\mathscr{U}) = \langle u, z \rangle$, and $[\mathscr{P}, t] = \langle u \rangle$. Also, as in [9, Section 11], there is an element $v \in \mathscr{V} - X$ such that $v^2 = uz$ and $v^{xt} \in v\langle u \rangle$.

Suppose that $[v, xt] = 1$. Then since $v^t = vz$, $v$ normalizes $B = \langle u, z, x, t \rangle$. Then $B \lhd B\langle v \rangle \leq C_G(xt)$ and hence $B\langle v \rangle \cong U$. Hence $\langle z \rangle = (B\langle v \rangle)' = \mho^1(B\langle v \rangle) = \langle uz \rangle$, a contradiction. Thus $v^{xt} = vu$.

We shall now describe how the proof of [9, Lemma 11.2] can be adapted to prove:

LEMMA 10.2. $\mathscr{P} \cong E_4$.

*Proof.* Assume that $\mathscr{P} = \langle \omega \rangle$ where $\omega^2 = u$. Clearly $\omega^t = \omega^{-1}$ and $\Omega_1(\mathscr{Q}) = X$. Suppose that $xt$ inverts $\mathscr{P}$. Then $B = \langle u, z, t, xt \rangle$ is $\langle \omega v \rangle$-invariant, $B \lhd B\langle \omega v \rangle \leq C_G(xt)$, $(\omega v)^2 = z$, and $[\omega v, t] = \langle uz \rangle$ and we have a contradiction. Thus $[\mathscr{P}, xt] = 1$ and $\omega^x = \omega^{-1}$.

Set $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar{N} = N/O(N))$. Thus $Z(\mathscr{Q}) = \mathscr{P} \times F \leq Z(C)$, $\langle \mathscr{U}, \eta \rangle \leq N$, $\eta^3 \in C$, $Z(\mathscr{Q}) = \mathscr{P} \times F = \mathscr{Q} \cap C \lhd N$, and $X \leq Z(C\langle t \rangle)$. Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$. Then, as in the proof of [9, Lemma 11.2], we conclude that $\bar{C} = C_{\bar{C}}(\bar{\eta}) \times \bar{F}$ where $C_{\bar{C}}(\bar{\eta})$ is cyclic, $\bar{\mathscr{P}} = \Omega_2(C_{\bar{C}}(\bar{\eta}))$, and $\bar{F} = [\bar{C}, \bar{\eta}]$. Moreover, without loss of generality and in order to simplify notation, it follows that we may assume that $O(N) = 1$.

Set $\mathscr{R} = C_C(\eta)$. Then $C = \mathscr{R} \times F$, $\mathscr{R}\langle xt \rangle$ is abelian or modular, $C\langle x, t \rangle \leq \mathscr{T}$, $[xt, \mho^1(\mathscr{R})] = 1$, and $|\mathscr{R}| \leq 2^4$ by (4.3).

Suppose that $\mathscr{P} < \mathscr{R}$ and let $\mathscr{R} = \langle \gamma \rangle$. If $\mathscr{R}\langle xt \rangle$ is abelian, then $|\mathscr{R}| = 2^3$ since $xt \sim t$ in $G$ and $((\mathscr{R} \times \langle z \rangle \times \langle xt \rangle)\langle t \rangle) \leq C_{\mathscr{T}}(xt)$. Now (4.7) yields a contradiction. So suppose that $\mathscr{R}\langle xt \rangle$ is modular. Then $v\gamma \in C_{\mathscr{T}}(xt)$, $(v\gamma)^2 = v^2\gamma^2 = uz\gamma^2$ and hence $|v\gamma| = |\mathscr{R}|$. Since $((\langle v\gamma \rangle \times \langle z, xt \rangle)\langle t \rangle) \leq C_{\mathscr{T}}(xt)$, we again obtain a contradiction by (4.7). Thus $\mathscr{R} = \mathscr{P}$ and $C = \mathscr{P} \times F$. Also, as in the proof of [9, Lemma 11.2], we have $N = O_2(N)\langle \eta, x \rangle$, $\mathscr{T} = O_2(N)\langle x \rangle$ and $\mathscr{Q}\langle t \rangle = (C\mathscr{Q})\langle t \rangle = (\mathscr{P} * \mathscr{V})\langle t \rangle \leq O_2(N)$. Setting $\mathscr{X} = O_2(N)$, we also have $C_{\mathscr{X}}(t) = A$ and $t^G \cap \mathscr{Q} = \emptyset$ since $\Omega_1(\mathscr{Q}) = X \leq Z(\mathscr{Q})$.

Suppose that $\mathscr{X} = \mathscr{Q}\langle t \rangle$. Then $\mathscr{T} = \mathscr{U} \in Syl_2(G)$ as in [9, Lemma 11.2] and we have a contradiction. Thus $\mathscr{W} = \mathscr{Q}\langle t \rangle < \mathscr{X}$. Let $\mathscr{X}_1 = N_{\mathscr{X}}(\mathscr{W})$. Then as in [9, Lemma 11.2], it follows that $|\mathscr{X}_1/(\mathscr{Q}\langle t \rangle)| = 2$, $[\mathscr{X}_1, \eta] = \mathscr{V} \lhd \mathscr{X}_1$,

$\mathscr{Y} = \mathscr{P}\langle t \rangle$ is of index 2 in $\mathscr{Y}_1 = C_{\mathscr{X}_1}(\eta)$, $\mathscr{Y}_1$ is dihedral or semidihedral, $\mathscr{Y}_1' = \mathscr{P}$, $\mathscr{Y}_1 \cap \mathscr{V} = \langle u \rangle$, $\mathscr{R}_1 = C_{\mathscr{Y}_1}(\mathscr{V})$ is a maximal subgroup of $\mathscr{Y}_1$, $R_1$ is dihedral or quaternion of order 8, and $\mathscr{X}_1 = (\mathscr{R}_1 * \mathscr{V})\langle t \rangle$. Suppose that $\tau \in t^G \cap (\mathscr{R}_1 * \mathscr{V})$. Then, since $|\mathscr{R}_1 * \mathscr{V}| = 2^7$ and $(\mathscr{R}_1 * \mathscr{V})' = \langle u \rangle$, it follows that $C_{\mathscr{R}_1 * \mathscr{V}}(\tau) = \langle \tau \rangle \times \mathscr{V}$ which contradicts (4.3). Thus $t^G \cap (\mathscr{R}_1 * \mathscr{V}) = \emptyset$. Also, it is easy to see that $I(t(\mathscr{R}_1 * \mathscr{V})) = t^{\mathscr{X}_1}$. Since $\Omega_1(\mathscr{Q}) = X \lhd N$, we have $A \trianglelefteq C_{\mathscr{T}}(t)$ and hence $\mathscr{X} = \mathscr{X}_1$ and $\mathscr{T} = (\mathscr{R}_1 * \mathscr{V})\langle t, x \rangle$. Moreover $X = \Omega_1(\mathscr{T}')$, as in [9, Lemma 11.2] and hence $X$ char $\mathscr{T}$ and $C_{\mathscr{T}}(X) = \mathscr{X}_1$ char $\mathscr{T}$. Assume that $\mathscr{T}$ is a maximal subgroup of the 2-subgroup $\mathscr{S}$ of $G$. Then $X \lhd \mathscr{S}$, $\mathscr{X}_1 \lhd \mathscr{S}$, and $X \trianglelefteq C_{\mathscr{S}}(t)$. Thus $A \trianglelefteq C_{\mathscr{S}}(t)$, we have a contradiction and the proof of Lemma 10.2 is complete.

Hence $\mathscr{P} = \langle u, \omega \rangle$ for some involution $\omega$, $\mathscr{Q} = \langle \omega \rangle \times \mathscr{V}$, $\mathscr{Q}' = \langle u \rangle$, $\mho^1(\mathscr{Q}) = X$, $\mathscr{W} = \mathscr{Q}\langle t \rangle$, and $\mathscr{U} = \mathscr{Q}\langle x, t \rangle$. Let $E = \Omega_1(\mathscr{Q}) = \langle \omega \rangle \times X \cong E_{16}$. Then $E = Z(\mathscr{Q})$ and $t^G \cap \mathscr{Q} = \emptyset$. Since $X = \Omega_1(\mathscr{U}')$ char $\mathscr{U}$, we conclude that $\mathscr{W} = C_{\mathscr{U}}(X)$ char $\mathscr{U}$. Also, as in [9, Section 11], we have $E$ char $\mathscr{W}$ and $\mathscr{Q}$ char $\mathscr{W}$.

Set $N = N_G(\mathscr{W})$, $C = C_G(\mathscr{W})$, and $\bar{N} = N/O(N)$. Clearly

$$\langle \mathscr{U}, \eta \rangle \leq N \leq N_G(\mathscr{Q}) \leq N_G(X) \quad \text{and} \quad \mathscr{U} \cap C = X = \mathscr{W} \cap C.$$

Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$; thus $\mathscr{U} \neq \mathscr{T}$ since $\mathscr{W}$ char $\mathscr{U}$. Also $N_{\mathscr{T}}(A) = \mathscr{U}$. Then $X \in Syl_2(C)$, $C = O(N) \times X$ and $\bar{C} = \bar{X}$. Moreover, as in [9, Section 11], we have $\bar{N} = O_2(\bar{N})\langle \bar{\eta}, \bar{x} \rangle$, $\bar{\eta}^3 = 1$ and $\bar{\mathscr{U}} < \bar{\mathscr{T}} = O_2(\bar{N})\langle \bar{x} \rangle$. Again, for convenience, we assume that $O(N) = 1$.

Set $\mathscr{X} = O_2(N)$. Note that $N \leq N_G(X)$ and hence $C_N(t) \leq N_G(A) = M$, so that $C_{\mathscr{X}}(t) = A$, $C_{\mathscr{X}}(\eta, t) = \langle t, u \rangle$ and $C_{\mathscr{X}}(\eta) = \langle t, \omega \rangle = \mathscr{Y}$. Then we conclude that $\mathscr{X}$ is transitive on $I(t\mathscr{Q})$, $|\mathscr{X}| = 2^9$, $\Omega_1(\mathscr{Q}) = E$ is strongly closed in $\mathscr{W}$ with respect to $G$, $E \lhd N$ and $N/\mathscr{W} \cong \Sigma_4$. Also we always have $|[E, \mathscr{Q}x]| \neq |[E, \mathscr{Q}xt]|$ and, as in [9, Section 11], we have $N/\mathscr{Q} \cong Z_2 \times \Sigma_4$, $\langle t\mathscr{Q} \rangle = C_{\mathscr{X}/\mathscr{Q}}(\eta)$, $\mathscr{V} < [\mathscr{X}, \eta]$, $t \notin [\mathscr{X}, \eta]$, $\mathscr{T} = \mathscr{X}\langle x \rangle$, $|\mathscr{T}| = 2^{10}$, and $\mathscr{T} \notin Syl_2(G)$.

LEMMA 10.3. *$E$ char $\mathscr{T}$ and $E$ char $\mathscr{X}$.*

*Proof.* Clearly $E \lhd \mathscr{T}$. Let $E_{16} \cong Y \lhd \mathscr{T}$. Suppose that $Y \nleq \mathscr{X}$. Then

$$Y \cap (x\mathscr{Q} \cup xt\mathscr{Q}) \neq \emptyset$$

and $Y$ contains an element of order 4, which is impossible. Thus $Y \leq \mathscr{X}$. Noting that if $r \in \mathscr{X} - \mathscr{W}$, then $|[t, r]| = 4$, it follows that $Y \leq \mathscr{Q}$ and hence $Y = E$ and the lemma follows.

Clearly $\mathscr{Q} \leq C_{\mathscr{T}}(E) = C_{\mathscr{Q}}(E)$ char $\mathscr{T}$.

Suppose that $\mathscr{Q} = C_{\mathscr{T}}(E)$ and set $J = N_G(E)$ and $\bar{J} = J/O(J)$. Let $\mathscr{T} \leq \mathscr{S} \in Syl_2(J)$ and suppose that there is an element $\tau \in \mathscr{S} - \mathscr{T}$ such that $\tau$ normalizes $\mathscr{Q}$ and $\mathscr{Q}\langle t \rangle$. Then, we may assume that $\tau \in C_{\mathscr{S}}(t) - \mathscr{T}$. But $X = C_E(t) \lhd C_{\mathscr{S}}(t)$ and hence $U = C_{\mathscr{S}}(t)$, so that we have a contradiction. Thus

$C_{\mathscr{S}}(E) = \mathscr{Q}$. Now $\mathscr{Q}/E \cong E_4$ and any element of odd order in $N_J(\mathscr{Q}) \cap C_G(E)$ must centralize $\mathscr{Q}$. Thus $C_G(E) = O(J)\mathscr{Q}$, $\bar{J} = N_{\bar{J}}(\bar{\mathscr{Q}}) = \overline{N_J(\mathscr{Q})}$, $C_{\bar{J}}(\bar{\mathscr{Q}}) = E$, and $\bar{J}/\bar{E} \hookrightarrow \operatorname{Aut}(\bar{\mathscr{Q}})$. Then $\bar{J} = O_2(\bar{J})\langle\bar{\eta}, \bar{x}\rangle$, $O_2(\bar{J}) = C_{\bar{J}}(\bar{\mathscr{Q}}/\bar{E})$, and hence $O_2(\bar{J})$ acts trivially on $\bar{X} = \mho^1(\bar{\mathscr{Q}})$. But then $|O_2(\bar{J}):\bar{\mathscr{Q}}|$ divides $2^3$ and hence $|\mathscr{S}| = |\mathscr{T}|$ which contradicts Lemma 10.3.

Thus $\mathscr{Q} < C_{\mathscr{T}}(E) = C_{\mathscr{X}}(E)$. Setting $\mathscr{X}_1 = C_{\mathscr{X}}(C)$, we have $\mathscr{X} = \mathscr{X}_1\langle t\rangle$ and $t \notin \mathscr{X}_1 = C_{\mathscr{T}}(E)$ char $\mathscr{T} = \mathscr{X}_1\langle x, t\rangle$.

Also $\mathscr{X}_1 = \mathscr{Q}[\mathscr{X}, \eta] \lhd N$, $|\mathscr{X}_1| = 2^8$, and $\mathscr{P} = C_{\mathscr{X}_1}(\eta) \lhd N$. Set $\tilde{N} = N/\mathscr{P}$. As in [9, Section 11], we conclude that $\tilde{\mathscr{X}}_1$ is of type $L_3(4)$ and $\langle u\rangle < \mathscr{X}_1' \leq \Phi(\mathscr{X}_1) \leq E = Z(\mathscr{X}_1)$. If $\Phi(\mathscr{X}_1) = X$, then $\mathscr{V} \leq [\mathscr{X}, \eta] < \mathscr{X}_1$, $[\mathscr{X}, \eta] \lhd N$, and $[\mathscr{X}, \eta] \cap \mathscr{Q} = \mathscr{V}$ and we obtain a contradiction as above. Thus $E = \Phi(\mathscr{X}_1) = Z(\mathscr{X}_1) = E$, $\exp(\mathscr{X}_1) = 4$, $C_{\mathscr{X}_{1/E}}(tE) = \mathscr{Q}/E$, $I(t\mathscr{Q}_1) = t^{\mathscr{X}_1}$, $t^G \cap \mathscr{X}_1 = \emptyset$, and $(\mathscr{X}_1\langle x\rangle)/\mathscr{Q} \cong D_8$. Set $J = N_G(E)$, $\bar{J} = J/O(J)$, and let $\mathscr{T} \leq \mathscr{S} \in Syl_2(J)$. Then $\mathscr{T} \neq \mathscr{S}$ and $U = C_{\mathscr{S}}(t)$ since $C_E(t) = X < C_{\mathscr{S}}(t)$.

LEMMA 10.4. *$E$ char $\mathscr{S}$.*

*Proof.* Let $E_{16} \cong Y \lhd \mathscr{S}$. Since $2^{11} \leq |\mathscr{S}|$, we have $t^G \cap Y = \emptyset$. Note that $4 \leq |C_Y(t)|$ and $C_{\mathscr{S}}(t) = U = A\langle x\rangle$. Also, if $\tau \in x\mathscr{Q} \cup xt\mathscr{Q}$, then there is an element $v \in \mathscr{V} - X$ such that $|[v, \tau]| = 4$. Thus $C_Y(t) \leq X = \langle u, y, z\rangle$ and hence $[Y, t] \leq C_Y(t) \leq X$. But $\mathscr{Q}$ is transitive on $tX$. Hence $Y \leq \mathscr{Q}C_{\mathscr{S}}(t) = \mathscr{U}$ and then $Y \leq \mathscr{Q}\langle t\rangle = \mathscr{W}$. As usual, this implies that $Y \leq \mathscr{Q}$. Thus $Y = E$ and Lemma 10.4 follows.

We can now conclude the proof of Lemma 10.1. Clearly Lemma 10.4 implies that $\mathscr{S} \in Syl_2(G)$. Since $\mathscr{X}_1 \leq C_{\mathscr{S}}(E)$ and $I(t\mathscr{X}_1) = t^{\mathscr{X}_1}$, we conclude that $\mathscr{X}_1 = C_{\mathscr{S}}(E)$ and hence $\mathscr{S}/\mathscr{X}_1 \hookrightarrow \operatorname{Aut}(E)$. But $C_{\mathscr{S}/\mathscr{X}_1}(t\mathscr{X}_1) = \mathscr{T}/\mathscr{X}_1 \cong E_4$, so that $\mathscr{S}/\mathscr{X}_1 \cong D_8$. On the other hand, $I(t\mathscr{X}_1) = t^{\mathscr{X}_1}$ and hence $t\mathscr{X}_1 \sim xt\mathscr{X}_1$ in $\mathscr{S}$. Thus $\langle x\mathscr{X}_1\rangle = (\mathscr{S}/\mathscr{X}_1)' = Z(\mathscr{S}/\mathscr{X}_1)$, $|\mathscr{S}| = 2^{11}$, $|C_E(xt)| = |C_E(t)| = 8$, $\omega^x = \omega u$, and $\omega^{xt} = \omega$. But then $I(x\mathscr{X}_1) = x^{\mathscr{X}_1}$ and $t^G \cap (\mathscr{X}_1\langle x\rangle) = \emptyset$. Now [12, Lemma 5.38] implies that $|O^2(G)|_2 \leq 2^{10}$. This contradiction completes the proof of Lemma 10.1.

## 11. The case of Lemma 6.2(v)

In this section, we conclude the proof of Theorem 2 by proving:

LEMMA 11.1. *If $\mathscr{V}$ satisfies* (v) *of Lemma* 6.2, *then* $|O^2(G)|_2 \leq 2^{10}$.

As usual, throughout this section, we assume that $\mathscr{V}$ satisfies (v) of Lemma 11.1 and that $2^{10} < |O^2(G)|_2$ and we shall proceed to a contradiction.

Thus $\mathscr{V}' = Z(\mathscr{V}) = \langle u\rangle$, $\mathscr{V}$ contains subgroups $Q_1, Q_2$ quaternion of order 8 such that $\mathscr{V} = Q_1 * Q_2$, $\mathscr{V}$ char $\mathscr{V}A = \mathscr{V}\langle t\rangle$, $Q_1^t = Q_2$, and $\mathscr{V}A$ is of type $\mathscr{A}_8$. Note also that $O_2(\bar{M}) = \bar{\mathscr{V}}\bar{A}$, $\bar{M}/O_2(\bar{M}) \cong \Sigma_3$, and $N_{\bar{M}}(\langle\bar{\eta}\rangle) = \langle\bar{t}, \bar{u}, \bar{x}\rangle$ and hence the proof of [6, VI, Lemma 2.7(iii)] implies that $\mathscr{U} =$

$\mathscr{V}A\langle x\rangle$ is of type $\mathscr{A}_{10}$. Thus $\mathscr{V}A\langle x\rangle \cong D_8 \wr Z_2$. Also we clearly have $\mathscr{W} = \mathscr{V}A = \mathscr{V}\langle t\rangle$, $[\mathscr{W}, \eta] = [\mathscr{V}, \eta] = \mathscr{V}$, $\mathscr{E}_{16}(\mathscr{W}) = \{A\}$, and every element of $\mathscr{V}A - \mathscr{V}$ interchanges $Q_1$ and $Q_2$. Moreover, as in [9, Section 12], $\mathscr{Y}$ contains a maximal subgroup $\mathscr{P}$ such that $[\mathscr{P}, \mathscr{V}] = 1$, $\langle \mathscr{U}, \eta\rangle \leq N_G(\mathscr{P}) \cap N_G(\mathscr{V})$, and $\mathscr{P} \cap \mathscr{V} = \langle u\rangle$. Set $\mathscr{Q} = \mathscr{P}\mathscr{V} = \mathscr{P} * \mathscr{V}$. Then $\mathscr{W} = \mathscr{Q}\langle t\rangle$, $\mathscr{Q} \lhd \mathscr{U} = \mathscr{Q}\langle x, t\rangle$, $Z(\mathscr{U}) = \langle u\rangle$, and $Z(\mathscr{Q}) = U$.

LEMMA 11.2. $\mathscr{P} \cong Z_4$.

*Proof.* Assume that $\mathscr{P} = \langle u, \omega\rangle$ where $\omega^2 = 1$. Then $\mathscr{Q} = \langle \omega\rangle \times \mathscr{V}$, $I(t\mathscr{Q}) = tX = t^{\mathscr{Q}}$ and $\mathscr{U} = \mathscr{Q}\langle x, t\rangle \in Syl_2(G)$. As in [9, Lemma 12.2], we conclude that $\mathscr{Q}$ char $\mathscr{U}$ and $\mathscr{P} = Z(\mathscr{Q})$ char $\mathscr{U}$. Set $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar N = N/O(N)$ and let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$. Then $\mathscr{U} \neq \mathscr{T}$, $C_{\mathscr{S}}(t) = U$ since $C_{\mathscr{Q}}(t) = X \lhd C_{\mathscr{T}}(t)$, $I(t\mathscr{P}) = t\langle u\rangle = t^{\mathscr{P}}$ and hence $\mathscr{U} \cap C = \mathscr{T} \cap C = \mathscr{P}$. Thus $C = O(N) \times \mathscr{P}$, $\bar C = \bar{\mathscr{P}}$, and $\bar N/\bar{\mathscr{P}} \hookrightarrow \operatorname{Aut}(\mathscr{Q})$. As in [9, Lemma 12.2], it follows that $\mathscr{T}/\mathscr{Q} \cong D_8$, $|\mathscr{T}| = 2^9$, $Z(\mathscr{T}/\mathscr{Q}) = \langle x\mathscr{Q}\rangle$, and $t\mathscr{Q} \sim xt\mathscr{Q}$ in $\mathscr{T}/\mathscr{Q}$. Let $\mathscr{T}$ be a maximal subgroup of the 2-subgroup $\mathscr{S}$ of $G$. Since $I(t\mathscr{Q}) = t^{\mathscr{Q}}$ and $\mathscr{Q} \not\trianglelefteq \mathscr{S}$, we also have $|C_{\mathscr{S}}(t)| = 2^6$. But $Z(\mathscr{S}) = \langle u\rangle$, $U = A\langle x\rangle \lhd C_{\mathscr{S}}(t)$, $U' = \langle z\rangle \lhd C_{\mathscr{S}}(t)$, and hence $\langle t, u, z\rangle \leq Z(C_{\mathscr{S}}(t))$. Thus (4.1) yields a contradiction and the proof of Lemma 12.2 is complete.

Let $\mathscr{P} = \langle \omega\rangle$ where $\omega^2 = u$. Then $I(t\mathscr{Q}) = t^{\mathscr{Q}} \cup (t\omega)^{\mathscr{Q}}$, $\mathscr{Q}$ char $\mathscr{U}$, and $\mathscr{P}$ char $\mathscr{U}$ as in [9, Section 12]. Set $N = N_G(\mathscr{Q})$, $C = C_G(\mathscr{Q})$, and $\bar N = N/O(N)$. Then $X = C_{\mathscr{Q}}(t) \lhd C_N(t)$, $\langle \mathscr{U}, \eta\rangle \leq N$, and $\eta^3 \in C$. Let $\mathscr{U} \leq \mathscr{T} \in Syl_2(N)$; thus $\mathscr{U} \neq \mathscr{T}$.

Applying the proof of [9, Lemma 12.3], we obtain:

LEMMA 11.3. (i) $C = O(N)(C \cap \mathscr{T})$ *where* $C \cap \mathscr{T}$ *is cyclic,* $C \cap \mathscr{T} \lhd \mathscr{T}$, $(C \cap \mathscr{T})\langle t\rangle$ *is dihedral or semidihedral, and* $\mathscr{P} \leq (C \cap \mathscr{T}) \cap Z(C)$.

(ii) $(C \cap \mathscr{T})\mathscr{Q} = (C \cap \mathscr{T}) * \mathscr{V}$.

(iii) $\bar N/(\bar C\bar{\mathscr{V}}) \hookrightarrow Z_2 \times \Sigma_6$.

(iv) $C_N(t) = (O(C_N(t)) \cap C)A\langle \eta, x\rangle$, $O(C_N(t) \cap C) \leq O(N)$, $\eta^3 \in O(N)$, *and* $C_{\bar N}(\bar t) = \bar A\langle \bar\eta, \bar x\rangle$.

As in [9, Section 12], without loss of generality, we assume that $O(N) = 1$. Then $C = C \cap \mathscr{T}$, $(C * \mathscr{V})\langle x, t\rangle = C\mathscr{U} \leq \mathscr{T}$, and $C\mathscr{Q} = C * \mathscr{V} \leq O_2(N)$. Let $C = \langle \gamma\rangle$ and $|C| = 2^a$ for some integer $a \geq 2$. Clearly $\mathscr{P} = \langle \omega\rangle = \Omega_2(C)$ and the proof of [9, Lemma 12.4] yields:

LEMMA 11.4. $C * \mathscr{V}$ char $C\mathscr{U} = (C * \mathscr{V})\langle x, t\rangle$, $\mathscr{Q}$ char $C\mathscr{U}$, *and* $\mathscr{T} \neq C\mathscr{U}$.

Assume that $N_N((C * \mathscr{V})\langle t\rangle) = (C * \mathscr{V})(\langle t\rangle \times \langle \eta, x\rangle)$. Then, as in [9, Lemma 12.5], we conclude that $O_2(N) = C * \mathscr{V}$, $\mathscr{T}/O_2(N) \cong D_8$, and $\mathscr{T} \notin Syl_2(G)$. Let $\mathscr{T}$ be a maximal subgroup of the 2-subgroup $\mathscr{S}$ of $G$ and suppose that $C_{\mathscr{S}}(t) \neq C_{\mathscr{T}}(t)$. Then $C_{\mathscr{T}}(t) = U$ is a maximal subgroup of $C_{\mathscr{S}}(t)$. Hence $U' = \langle z\rangle \lhd C_{\mathscr{T}}(t)$. Since $Z(\mathscr{S}) = \langle u\rangle$, we have $\langle t, u, z\rangle \leq$

$Z(C_{\mathscr{S}}(t))$ which contradicts (4.1). Thus $C_{\mathscr{T}}(t) = C_{\mathscr{S}}(t) = U$. But then the proof of [9, Lemma 12.5] yields:

LEMMA 11.5. (i) $|N_N((C * \mathscr{V})\langle t\rangle): ((C * \mathscr{V})(\langle t\rangle \times \langle \eta, x\rangle))| = 2$.
(ii) *$C\langle t\rangle$ is dihedral.*

Let $Y = N_N((C * \mathscr{V})\langle t\rangle)$. Then

$$|O_2(Y)/((C * \mathscr{V})\langle t\rangle)| = 2 = |C_{O_2(Y)}(\eta)/C\langle t\rangle)|,$$

$C_{O_2(Y)}(\eta)$ is dihedral or semidihedral, $[O_2(Y), \eta] = \mathscr{V} \lhd Y$, and there is a maximal subgroup $\mathscr{R}$ of $C_{O_2(Y)}(\eta)$ such that $[\mathscr{R}, \mathscr{V}] = 1$, $C$ max $\mathscr{R}$, $C_{O_2(Y)}(\eta) = \mathscr{R}\langle t\rangle$, $\mathscr{R}$ is dihedral or generalized quaternion, $\mathscr{R}\mathscr{V} = \mathscr{R}\mathscr{Q} = \mathscr{R} * \mathscr{V}$, $\mathscr{R} \cap \mathscr{V} = \langle u\rangle$, and

$$Y = (\mathscr{R} * \mathscr{V})(\langle t\rangle \times \langle \eta, x\rangle).$$

LEMMA 11.6. (i) $t^G \cap (\mathscr{R} * \mathscr{V}) = \emptyset$.
(ii) $I(t(\mathscr{R} * \mathscr{V})) = t^{(\mathscr{R} * \mathscr{V})}$.
(iii) $\mathscr{R} * \mathscr{V} \leq O_2(N)$.
(iv) $O_2(N) = \mathscr{R} * \mathscr{V}$ *or* $O_2(N) = (\mathscr{R} * \mathscr{V})\langle t\rangle$.

*Proof.* Let $T = \mathscr{R} * \mathscr{V}$ and $\tau \in t^G \cap T$. Then [9, Lemma 2.12] implies that $|C_T(\tau)| \geq 2^6$ and if $|C_T(\tau)| = 2^6$, then $\exp(C_T(\tau)) = 4$. Thus (i) follows from (4.3). Noting that (ii) is clear and that (iii)–(iv) also hold as in [9, Section 12], the lemma is proved.

LEMMA 11.7. $O_2(N) = \mathscr{R} * \mathscr{V}$ *and* $\mathscr{T} \neq (\mathscr{R} * \mathscr{V})\langle x, t\rangle$.

*Proof.* If $O_2(N) \neq \mathscr{R} * \mathscr{V}$, then $\mathscr{T} = (\mathscr{R} * \mathscr{V})\langle x, t\rangle$. Thus assume that

$$\mathscr{T} = (\mathscr{R} * \mathscr{V})\langle x, t\rangle.$$

Clearly, $Z(\mathscr{T}) = \langle u\rangle \leq \mho^1(\mathscr{T}') = \mho^1(C)$ since $\mathscr{T}' = C * (\mathscr{V} \cap \mathscr{T}')$. Also, as in [9, Lemma 12.6], we have $\mathscr{R} * \mathscr{V}$ char $\mathscr{T}$.

Assume that $|\mathscr{R}| \geq 2^4$. Then, as in [9, Lemma 12.6], it follows that $\mathscr{T} \in Syl_2(G)$. Hence $|\mathscr{T}| \geq 2^{11}$, $|C| \geq 2^4$, and we obtain a contradiction as in [9, Lemma 12.6]. Thus $|\mathscr{R}| = 2^3$, $C = \mathscr{P}$, $|\mathscr{T}| = 2^9$, and $\mathscr{R} * \mathscr{V}$ is extra-special of order $2^7$.

Let $J = N_G(\mathscr{R} * \mathscr{V})$ and let $\mathscr{T} \leq \mathscr{S} \in Syl_2(J)$. Thus $\mathscr{T} \neq \mathscr{S}$ and $Z(\mathscr{S}) = \langle u\rangle$. Then, it follows that $C_{\mathscr{S}}(t) = C_{\mathscr{T}}(t) = U$. Moreover, the argument at the end of [9, Lemma 12.6] yields a contradiction and we are done.

As in [9, Section 12], we have $\mathscr{T}/O_2(N) \cong D_8$, $tO_2(N) \sim xtO_2(N)$, and $t \sim xt$ in $\mathscr{T}$, $Z(\mathscr{T}/O_2(N)) = \langle xO_2(N)\rangle$, and $x \in \mathscr{T}'$. Also, when $|\mathscr{R}| = 2^3$. we obtain a contradiction as in [9, Section 12]. So, let $|\mathscr{R}| = 2^a$ with $a \geq 4$, Then $\mathscr{P} \leq O_2(N)' = \mathscr{R}'$, $\mathscr{P} = \Omega_2(O^2(N)')$, $C_{O_2(N)}(\mathscr{P}) = C * \mathscr{V}$, $C = Z(C * \mathscr{V})$, and $\mathscr{Q} = \Omega_2(C * \mathscr{V})$. Thus $C \lhd N$, $C * \mathscr{V} \lhd N$, $[C, x] = 1$, $O_2(N)\langle x\rangle \lhd \mathscr{T}$ and every involution of $xO_2(N)$ is conjugate via $O_2(N)$ to an involution of $\mathscr{R}x$.

Also $t^G \cap O_2(N) = \emptyset$, $|Z(\mathscr{R}\langle x\rangle)| = 4$ by [9, Lemmas 2.2–2.3] and $C_{\mathscr{R}\langle x\rangle}(t) = \langle u, x\rangle$. Note that

$$Z(\mathscr{R}\langle x\rangle) \leq N_{\mathscr{T}}(A) \cap (\mathscr{R}\langle x\rangle) = \langle x\rangle \times \langle \omega\rangle.$$

Suppose that $\delta \in t^G \cap \mathscr{R}\langle x\rangle$. Thus $\delta \in \mathscr{R}x$. Assume that $Z(\mathscr{R}\langle x\rangle) = \langle x, u\rangle$. Then $\mathscr{R}\langle x\rangle = \mathscr{R} \times \langle x\rangle$ and $\mathscr{R}$ is dihedral. Let $u, r_1, r_2$ be representatives for the conjugacy classes of involutions of $\mathscr{R}$. Then $\delta \in \{xr_1, xr_2\}$ and $E_{16} \cong \langle x, u, \delta, z\rangle \leq C_{O_2(N)\langle x\rangle}(\delta)$. Since $\{x, u, z\} \cap t^G = \emptyset$, we have $\delta \sim x\delta$ in $G$ by Lemma 6.1(ii). This is impossible since $x\delta \in \mathscr{R}^{\#}$. Suppose that $Z(\mathscr{R}\langle x\rangle) = \langle x\omega\rangle$. Then $\mathscr{R}\langle x\rangle = \mathscr{R} * \langle \omega, x\rangle$ and $\mathscr{R}$ is generalized quaternion and $\mathscr{R}\langle t\rangle$ is semidihedral. Let $\omega, q_1, q_2$ be representatives for the conjugacy classes of elements of order 4 or $\mathscr{R}$. Then $\delta \in \{x\omega q_1 x\omega q_2\}$ and $\langle x\omega, \delta, z\rangle \leq C_{O_2(N)\langle x\rangle}(\delta)$. If $x\colon Q_1 \to Q_2$, then $C_{\mathscr{V}}(\delta) = C_{\mathscr{V}}(x) \cong E_8$ and hence $C_{O_2(N)\langle x\rangle}(\delta)$ is abelian of order $2^5$. Since this is impossible, $x$ normalizes $Q_1$ and $Q_2$ and there is an element $\beta_1 \in Q_1$ such that $\beta_1^2 = u$, $\beta_1^x = \beta_1^{-1}$ and $x \sim x\beta_1$ in $Q_1\langle x\rangle$. Clearly $\langle u, z\rangle = \langle u, \beta_1\beta_1^t\rangle$. Also, it is easy to see that $\delta^{x\beta_1} = \delta$. Hence $E_{16} \cong \langle \delta, u, z, x\beta_1\rangle \leq C_{O_2(N)\langle x\rangle}(\delta)$. Since $t^G \cap \{u, x, z\beta_1\} = \emptyset$, as above, we have $\delta \sim \delta x\beta_1$. However $\delta x\beta_1 \in O_2(N)$ and we again have a contradiction. Thus $t^G \cap (O_2(N))\langle x\rangle) = \emptyset$.

If $\mathscr{T} \in Syl_2(G)$, we obtain a contradiction as in [9, Section 12]. Suppose that $\mathscr{T}$ is a maximal subgroup of the 2-subgroup $\mathscr{S}$ of $G$. Since $Z(\mathscr{T}) = \langle u\rangle = Z(\mathscr{S})$, we have $C_{\mathscr{S}}(t) = C_{\mathscr{T}}(t) = U$. But now the argument at the end of [9, Section 12] yields a contradiction. Thus the proofs of Lemma 11.1 and Theorem 2 are complete.

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