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FROBENIUS RECIPROCITY FOR SQUARE-INTEGRABLE FACTOR REPRESENTATIONS

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Suppose G is a locally compact group and H is a closed subgroup of G. When G and H are compact, the classical Frobenius reciprocity theorem provides a nice duality (or more accurately, an adjoint pairing of functors) between the operations of restricting representations from G to H and inducing representations from H to G. However this duality breaks down for more general G and H, and an important problem in the theory of group representations has been to find an effective substitute for the classical result. One approach, not relevant to the present paper, has been to abandon unitary representations on Hilbert spaces and work with group representations on more general locally convex spaces. A second approach has been to look for Frobenius reciprocity theorems applying to unitary representations under additional hypotheses. The nicest results in this direction are the theorems of Mackey [4], Mautner [6], and Anh [1], all of which use direct integral decomposition of the regular representations of G and H and require at least that G be separable and (for the Mackey and Anh theorems) that G and H have type I regular representations or (for the Mautner theorem) that H be compact and that G be unimodular. Since the conclusions of the three theorems are "almost everywhere" statements (with respect to the measure classes defined by the central decompositions of the regular representations of G and H), one suspects that stronger "local" theorems should be valid for square-integrable factor representations (which correspond to atoms in these measure classes—cf. [10, Proposition 2.3]). The purpose of this note is to show that this is indeed the case. When H is compact, G is unimodular, and one considers irreducible representations of G, our results have already been obtained by Wawrzyńczyk [13], Szmidt [12], Kunze [3], and K. and L. Maurin [5] in slightly stronger form. However, the present results apply much more generally.

Throughout this paper, G and H are as above, and Δ and δ denote the modular functions of G and H, respectively. "Square-integrable" means S.I.S.S. in the notation of [10, 2.2]. We write γ for $\delta^{-1/2}\Delta|_H^{1/2}$ (this is a real-valued character on H), $\mathscr{K}(G)$ for the continuous functions of compact support on G. Haar measures are always *left* Haar measures. Induced representations are to be formed in the sense of Mackey and Blattner—the reader may consult [8] for references. The author thanks Prof. Marc A. Rieffel for a number of valuable

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suggestions, which led in particular to a significant improvement in the formulation and proof of Lemma 1.

1. The theorems

We begin with the case where H is open in G, since the proofs are easier and the results are somewhat more precise than in the general case.

THEOREM 1. *Suppose H is open in G, and let π and σ be square-integrable factor representations of G and H, respectively. Then:*

(a) *π is quasiequivalent to a subrepresentation of* Ind σ *(the representation of G induced by σ) if and only if σ is quasiequivalent to a subrepresentation of $\pi|_H$ (the restriction of π to H).*

(b) *If π and σ are irreducible, then if either* Hom_G (Ind σ, π) *or* $\mathrm{Hom}_H(\sigma, \pi|_H)$ *is finite dimensional, so is the other, and there is a natural linear bijection between them. In particular, the intertwining numbers* I_G (Ind σ, π) *and* $I_H(\sigma, \pi|_H)$ *are always the same (if we do not distinguish between infinite cardinals).*

Proof. We view $L^2(H)$ as a closed subspace of $L^2(G)$. Replacing π and σ by quasiequivalent representations if necessary, we may assume that they are contained in the respective left regular representations (both denoted by λ) of G and H, with π realized on a subspace V of $L^2(G)$ and σ realized on a subspace W of $L^2(H)$. Then Ind σ is the restriction of λ to the closed linear span U^W in $L^2(G)$ of $\{\lambda(x)f: f \in W, x \in G\}$. If $S: U^W \to V$ is a bounded G-intertwining operator, then $S|_W$ is H-intertwining and $\|S|_W\| \leq \|S\|$, so restriction defines a norm-decreasing linear map $r: \mathrm{Hom}_G(\mathrm{Ind}\, \sigma, \pi) \to \mathrm{Hom}_H(\sigma, \pi|_H)$. This map is injective since $S|_W$ clearly determines S (we have $S(\lambda(x)f) = \lambda(x)Sf$ for $f \in W$, $x \in G$). On the other hand, if $T: W \to V$ is a bounded H-intertwining operator and if P_W is the orthogonal projection of $L^2(G)$ onto W, then $TP_W: L^2(G) \to L^2(G)$ commutes with left translations by elements of H on $L^2(G)$. By the Takesaki-Nielsen-Rieffel Theorem ([7] and [9, Theorem 2.6]), $TP_W \in W^*(G, H\backslash G)$, the von Neumann algebra on $L^2(G)$ generated by the right regular representation of G and by multiplications by functions in $\mathscr{K}(H\backslash G)$ (viewed as functions on G constant on cosets of H). Therefore we can choose a net of functions $\Phi_\alpha \in \mathscr{K}(G, H\backslash G)$ (viewed as functions on $G \times G$) such that $\rho(\Phi_\alpha) \to TP_W$ strongly, where for g a continuous L^2 function on G,

$$(1) \qquad (\rho(\Phi_\alpha)g)(x) =_{\mathrm{def}} \int_G g(xy^{-1})\, \Delta(y^{-1})\Phi_\alpha(y, xy^{-1})\, dy.$$

Let $\phi_\alpha(\cdot) = \Phi_\alpha(\cdot, He)$. Then $\rho(\Phi_\alpha)$ coincides on $L^2(H)$ with right convolution $\rho(\phi_\alpha)$ by ϕ_α, since for $g \in L^2(H)$, the integrand in (1) vanishes except when $xy^{-1} \in H$. So if P_V is the orthogonal projection of $L^2(G)$ onto V, $\{P_V\rho(\phi_\alpha)P_W\}$ is a net in $\mathrm{Hom}_H(\sigma, \pi|_H)$ converging strongly to T. But if P_U is the orthogonal projection of $L^2(G)$ onto U^W, $P_V\rho(\phi_\alpha)P_U \in \mathrm{Hom}_G(\mathrm{Ind}\, \sigma, \pi)$ and $P_V\rho(\phi_\alpha)P_W =$

$r(P_V\rho(\phi_\alpha)P_U)$. Thus the image of r is strongly dense in $\mathrm{Hom}_H(\sigma, \pi|_H)$ and the theorem follows. When $\mathrm{Hom}_H(\sigma, \pi|_H)$ is finite dimensional, r must be surjective, and when $\mathrm{Hom}_H(\sigma, \pi|_H)$ is infinite dimensional, so must be the image of r.)

Now we drop the assumption that H is open in G.

LEMMA 1. *Let σ be the restriction of the left regular representation of H to an invariant subspace W of $L^2(H)$. Then for $f \in \mathcal{K}(G)$ and $g \in W$, $f * \gamma g \in L^2(G)$, and the representation* Ind σ *of G induced by σ is equivalent to the restriction of the left regular representation λ of G to the closed linear span U^W in $L^2(G)$ of the functions $f * \gamma g$, $f \in \mathcal{K}(G)$, $g \in W$. (By definition,*

$$(f * \gamma g)(x) = \int_H f(xt^{-1})\, \Delta(t)^{-1}\gamma(t) g(t)\, dt$$
$$= \int_H f(xt)\gamma(t) g(t^{-1})\, dt.)$$

Proof. We use the realization of Ind σ given in [8] on p. 228. If $f \in \mathcal{K}(G)$ and $g \in W$, $f * \gamma g$ exists as a continuous function and $f \otimes g$ exists as an element of $\mathcal{K}(G) \otimes_{\mathcal{K}(H)} W$, the preinner product space on which Ind σ is constructed. It is easily checked (cf. [8, Proposition 4.6]) that the map $(f, g) \mapsto f * \gamma g$ is $\mathcal{K}(H)$-balanced, hence defines a linear map $T\colon \mathcal{K}(G) \otimes_{\mathcal{K}(H)} W \to \{$continuous functions on $G\}$ taking $f \otimes g$ to $f * \gamma g$. Since this map clearly commutes with left translations by elements of G, T will provide the desired intertwining operator from the Hilbert space of Ind σ to U^W once we show that T is isometric with respect to the preinner product norm on $\mathcal{K}(G) \otimes_{\mathcal{K}(H)} W$ and the $L^2(G)$-norm. Let $f_1, f_2 \in \mathcal{K}(G)$ and let $g_1, g_2 \in W$. Then

$$\begin{aligned}
\langle f_1 \otimes g_1, f_2 \otimes g_2\rangle &= \langle P(f_2^* * f_1) g_1, g_2\rangle_{L^2(H)} \\
&= \int_H \int_H P(f_2^* * f_1)(s) g_1(s^{-1}t)\, ds\, \bar{g}_2(t)\, dt \\
&= \int_H \int_H \gamma(s) \int_G \bar{f}_2(x) f_1(xs)\, dx\, g_1(s^{-1}t)\, ds\, \bar{g}_2(t)\, dt \\
&= \int_G \int_H \int_H \gamma(ts)\bar{f}_2(x) f_1(xts) g_1(s^{-1})\, ds\, \bar{g}_2(t)\, dt\, dx \\
&= \int_G \int_H (f_1 * \gamma g_1)(xt)\bar{f}_2(x)\gamma(t)\bar{g}_2(t)\, dt\, dx \\
&= \int_G (f_1 * \gamma g_1)(x) \int_H \bar{f}_2(xt^{-1})\gamma(t)\bar{g}_2(t)\, \Delta(t)^{-1}\, dt\, dx \\
&= \int_G (f_1 * \gamma g_1)(x)(f_2 * \gamma g_2)^{-}(x)\, dx \\
&= \langle f_1 * \gamma g_1, f_2 * \gamma g_2\rangle_{L^2(G)}.
\end{aligned}$$

Hence T is isometric and the lemma is proved.

LEMMA 2. *For $f \in L^2(H)$ and $\phi \in \mathscr{K}(G)$, the function $f * \phi$, defined by*

$$(f * \phi)(x) = \int_H f(t)\phi(t^{-1}x)\, dt,$$

is in $L^2(G)$.

Proof. Let $f_1(t) = f(t^{-1})\, \delta(t)^{-1/2}$, $\phi_1(x) = \phi(x^{-1})\, \Delta(x)^{-1/2}$, for $t \in H$ and $x \in G$. Then $f_1 \in L^2(H)$ and $\phi_1 \in \mathscr{K}(G)$, so by Lemma 1, $\phi_1 * \gamma f_1 \in L^2(G)$. But an easy calculation shows that

$$(f * \phi)(x) = (\phi_1 * \gamma f_1)(x^{-1})\, \Delta(x)^{-1/2},$$

and thus $f * \phi \in L^2(G)$.

THEOREM 2. *Let π and σ be square-integrable factor representations of G and H, respectively, with σ irreducible. Then π is quasiequivalent to a subrepresentation of* Ind σ *if and only if σ is contained in $\pi|_H$. Furthermore, if π is also irreducible, then the intertwining numbers I_G* (Ind σ, π) *and $I_H(\sigma, \pi|_H)$ are the same (again if we do not distinguish between infinite cardinals).*

Proof. Replacing π by a quasiequivalent representation if necessary, we may assume π is the restriction of the left regular representation λ of G to an invariant subspace V of $L^2(G)$. Using Lemma 1, we view Ind σ as realized on the space U^W defined above, where W is an irreducible subspace of $L^2(H)$ containing a coefficient function for σ.

First suppose $S\colon U^W \to V$ is a nonzero bounded G-intertwining operator. Then SP_U (where P_U is the orthogonal projection of $L^2(G)$ onto U^W) may be viewed as an operator on $L^2(G)$ commuting with λ, and so SP_U is in the von Neumann algebra on $L^2(G)$ generated by the right regular representation of G. Hence we may choose a net of functions $\psi_\alpha \in \mathscr{K}(G)$ such that $\rho(\psi_\alpha) \to SP_U$ strongly, where $\rho(\psi_\alpha)$ denotes right convolution by ψ_α. Letting $\gamma = \Delta|_H^{1/2}\, \delta^{-1/2}$ as usual, we see from [2, Proposition 6] that $W \cap \gamma^{-1}L^2(H)$ is dense in W. Define an operator

$$T_\alpha\colon W \cap \gamma^{-1}L^2(H) \to L^2(G)$$

by $T_\alpha(f) = \gamma f * \psi_\alpha$, where the convolution product is defined as in Lemma 2. This makes sense since, by Lemma 2, $\gamma f * \psi_\alpha \in L^2(G)$ for $f \in \gamma^{-1}L^2(H)$. We show that the densely defined operator T_α is closable by showing that its adjoint is also densely defined. For $\phi \in \mathscr{K}(G)$ and $f \in \gamma^{-1}L^2(H) \cap W$, we have

$$\begin{aligned}
\langle \phi, T_\alpha(f)\rangle_{L^2(G)} &= \int_G (\gamma f * \psi_\alpha)^-(x)\phi(x)\, dx \\
&= \int_G \int_H \gamma(t)\bar{f}(t)\bar{\psi}_\alpha(t^{-1}x)\, dt\, \phi(x)\, dx \\
&= \int_H \bar{f}(t)\gamma(t) \int_G \phi(x)\psi_\alpha^\vee(x^{-1}t)\, dx\, dt \\
&= \langle \gamma \cdot (\phi * \psi_\alpha^\vee)|_H, f\rangle_{L^2(H)}
\end{aligned}$$

if we let $\psi_\alpha^\vee(x) = \bar{\psi}_\alpha(x^{-1})$, so that $\phi \in \operatorname{dom}(T_\alpha^*)$ and $T_\alpha^*\phi = P_W(\gamma \cdot (\phi * \psi_\alpha^\vee)|_H)$, where P_W is the orthogonal projection of $L^2(H)$ onto W. In particular, T_α is closable. Furthermore, T_α is semi-invariant with weight γ^{-1} for the left action of H, since for f as above and $t \in H$,

$$T_\alpha(\sigma(t)f) = (\gamma \cdot \sigma(t)f) * \psi_\alpha = \gamma(t)\lambda(t)(\gamma f * \psi_\alpha) = \gamma(t)\lambda(t)(T_\alpha f).$$

So by [2, Theorem 1], polar decomposition of the closure of T_α produces an H-intertwining isometry $U_\alpha\colon W \to L^2(G)$ with $U_\alpha W = (T_\alpha W)^-$. Since S is nonzero, we claim $P_V U_\alpha$ must be nonzero for some α (here P_V is the orthogonal projection of $L^2(G)$ onto V), so that σ is contained in $\pi|_H$. To prove this, suppose on the contrary that $P_V U_\alpha = 0$ for all α. Then for all α and for all $f \in W \cap \gamma^{-1}L^2(H)$, $T_\alpha f \in V^\perp$, the orthogonal complement of V. But $V^\perp$ is invariant under left convolutions, so that for all $\phi \in \mathscr{K}(G)$,

$$\phi * T_\alpha f = (\phi * \gamma f) * \psi_\alpha \in V^\perp$$

and

$$S(\phi * \gamma f) = \lim (\phi * \gamma f) * \psi_\alpha \in V \cap V^\perp = (0).$$

However, functions of the form $\phi * \gamma f$ are total in U^W by Lemma 1, so $S = 0$, a contradiction.

Suppose on the other hand that σ is contained by $\pi|_H$. By [2, Proposition 6], there exists a positive self-adjoint operator in W semi-invariant with weight γ^{-1}; composition with an H-intertwining operator $W \to V$ then yields a nonzero closed, densely defined operator T from W to V which is semi-invariant for the action of H, with weight γ^{-1}. Given $\phi \in \mathscr{K}(G)$, define an operator $R_\phi\colon W \cap \gamma^{-1}L^2(H) \to L^2(G)$ as before by $R_\phi(f) = \gamma f * \phi$. We have seen that R_ϕ is closable and semi-invariant with weight γ^{-1}. We can clearly choose ϕ so that R_ϕ is nonzero; having done this, we let R be the closure of R_ϕ. Since R must be injective (by irreducibility of σ), we can factor T as SR, where $S\colon L^2(G) \to L^2(G)$ commutes with left translations by elements of H. By the Takesaki-Nielsen-Rieffel Theorem ([7] and [9, Theorem 2.6]), $S \in W^*(G, H\backslash G)$, and we can choose a net of functions $\Phi_\alpha \in \mathscr{K}(G, H\backslash G)$ such that $\rho(\Phi_\alpha) \to S$ strongly. ($\rho(\Phi_\alpha)$ is defined as in (1).) Then for $f \in W \cap \gamma^{-1}L^2(H)$,

$$Tf = \lim_\alpha \rho(\Phi_\alpha)Rf = \lim_\alpha \rho(\Phi_\alpha)(\gamma f * \phi) = \lim_\alpha \gamma f * \phi_\alpha,$$

$$\text{where } \phi_\alpha = \rho(\Phi_\alpha)\phi \in \mathscr{K}(G).$$

We claim that since T is nonzero, some $P_V\rho(\phi_\alpha)P_U$ is nonzero and hence π is quasiequivalent to a subrepresentation of Ind σ. Indeed, if this were false, we would have for all $f \in W \cap \gamma^{-1}L^2(H)$, for all $\psi \in \mathscr{K}(G)$, and for all α, $(\psi * \gamma f) * \phi_\alpha \in V^\perp$, hence $\psi * Tf = \lim_\alpha (\psi * \gamma f) * \phi_\alpha \in V \cap V^\perp = (0)$. This being true for all ψ, we have $Tf = 0$, and f being arbitrary, this forces $T = 0$, a contradiction.

Finally, suppose π is irreducible. If $I_G(\text{Ind } \sigma, \pi) < \infty$, then in the first part of the proof we can actually choose a ψ_α with $P_V\rho(\psi_\alpha)P_U = SP_U$ (since the space of operators $P_V\rho(\psi)P_U$ is dense in the finite dimensional space of possible SP_U's), and the argument shows that $I_H(\sigma, \pi|_H) \geq I_G$ (Ind σ, π) (since $S \neq 0$ implies that $P_V T_\alpha \neq 0$). Similar reasoning in the second half of the proof shows the reverse inequality (again under the assumption I_G (Ind σ, π) $< \infty$). And it is clear that if I_G (Ind σ, π) $= \infty$, then the space of closed γ^{-1}-semi-invariants from W to V is infinite dimensional, so that $I_H(\sigma, \pi|_H) = \infty$.

Remark. The requirement in Theorem 2 that σ be irreducible (rather than, say, primary of type II or III) is probably unnecessary, but to eliminate it while retaining the framework of the proof above would require some results on semi-invariant operators for non-type I square-integrable representations, to replace Proposition 6 of [2]. Of course, if the regular representation of H is primary (of arbitrary type), the theorem is clear, since any subrepresentation of the regular representation λ_G of G must restrict to something quasicontained in the regular representation λ_H of H, while conversely, if σ is quasiequivalent to λ_H, then Ind σ is quasiequivalent to Ind $\lambda_H \cong \lambda_G$.

2. An example

The following example does not reflect the full power of Theorem 2 (since Anh's theorem would also apply), but it does illustrate the sort of situation in which the theorem might be useful. Let $G = SL(2, \mathbf{R})$, and let

$$H = \left\{\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} : a \in \mathbf{R}^*, b \in \mathbf{R}\right\}$$

be the subgroup of lower triangular matrices in G. Then H is the semidirect product of the subgroups

$$N = \left\{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} : b \in \mathbf{R}\right\} \quad \text{and} \quad D = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R}^*\right\},$$

and it is easy to see by the Mackey "little group" method that H has (up to equivalence) exactly four infinite-dimensional irreducible unitary representations, all of which are square-integrable. These are parameterized by the two-element dual of the center of H and by the two-element set of open orbits of D on $N^\wedge$ (which, when $\mathbf{R}$ is identified with $N^\wedge$, are just the positive and negative half-lines). We shall apply Theorem 2 to compute the multiplicities of the discrete series representations of G in Ind σ, where σ is one of the square-integrable irreducible representations of H. To this end we must compute the restrictions to H of the discrete series representations π_h of G. (In keeping with tradition, we let the index parameter h range over $\pm 1/2$, ± 1, $\pm 3/2, \ldots$.) According to [11, pp. 20–21], π_h is realized on a Hilbert space $\mathscr{H}_h$ of functions on the upper half-plane (holomorphic functions for $h > 0$, antiholomorphic functions for $h < 0$), and $\pi_h(x)$, for $x \in H$, is given by the formula

$$\left(\pi_h\left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}\right)f\right)(z) = a^{2|h|}f(a^2 z + ab). \tag{2}$$

In order to identify the representation $\pi_h|_H$, it is convenient to replace π_h by the equivalent representation π'_h of G on the Fourier transforms of the "boundary values" of the functions in $\mathscr{H}_h$ on the real line. As also indicated in [11, pp. 20–21], it follows from the Paley-Wiener Theorem that the Hilbert space $\mathscr{H}'_h$ of π'_h is just

$$L^2((0, \infty), t^{-2h+1}\, dt) \quad \text{for } h > 0$$

and

$$L^2((-\infty, 0), |t|^{2h+1}\, dt) \quad \text{for } h < 0.$$

Taking Fourier transforms in (2) shows that $\pi'_h|_H$ is given by the formula

$$\left(\pi'_h\left(\begin{pmatrix} a & 0 \\ b & a^{-1}\end{pmatrix}\right) f\right)(t) = a^{2(|h|-1)} f(a^{-2}t) e^{itb/a}.$$

By using the isometry of $\mathscr{H}'_h$ onto $L^2((0, \infty), dt/t)$ given by $f \mapsto f_1$, where $f_1(t) = t^{1-h} f(t)$ for $h > 0$ and $f_1(t) = t^{1+h} f(-t)$ for $h < 0$, we may replace π_h by still another equivalent representation π^h, this time on $L^2((0, \infty), dt/t)$. The formula for $\pi^h|_H$ is

$$\left(\pi^h\left(\begin{pmatrix} a & 0 \\ b & a^{-1}\end{pmatrix}\right) f\right)(t) = \varepsilon_h(a) \exp\,(i\, \mathrm{sgn}(h) tb/a) f(a^{-2}t), \tag{3}$$

where $\varepsilon_h(a) = +1$ if h is an integer, sgn (a) if h is a half-integer. But (3) is easily recognized as the standard form of one of the four infinite-dimensional irreducible representations of H—which one we get depends on ε_h and sgn (h). We conclude that the restriction to H of any discrete series representation of G is irreducible, and hence that if σ is a square-integrable irreducible representation of H, then Ind σ contains each discrete series representation of G with multiplicity 0 or 1. (Of course, Ind σ also contains a direct integral of principal series representations.) The discrete series representations that *do* appear in Ind σ are exactly those restricting to the same character of the center as σ and having index h of the sign corresponding to the orbit in $N^\wedge$ associated with $\sigma|_N$.

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