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WEAKLY CLOSED ALGEBRAS OF SUBNORMAL OPERATORS[1]

BY

JOSEPH A. BALL, ROBERT F. OLIN, AND JAMES E. THOMSON

1. Introduction, preliminaries and notation

All Hilbert spaces in this paper are separable. An operator will mean a bounded linear transformation. The measures that we consider are finite positive regular Borel measures with compact support in the plane.

For two measures ν and μ, the symbolism $\nu \equiv \mu$ means that ν and μ are mutually absolutely continuous. If $\mathscr{A}$ is a linear manifold of $L^\infty(\mu)$, let $\mathscr{A}^p(\mu)$ denote the $L^p(\mu)$ closure of $\mathscr{A}$ for any p, $1 \le p < \infty$. When $p = \infty$, let $\mathscr{A}^\infty(\mu)$ be the weak-star closure of $\mathscr{A}$ in $L^\infty(\mu)$ $(= L^1(\mu)^*)$.

In Section 2 we derive the basic intersection formula

$$(1.1) \qquad \mathscr{A}^\infty(\mu) = \bigcap_{\nu \equiv \mu} [\mathscr{A}^p(\nu) \cap L^\infty(\mu)].$$

The results in this paper can be viewed as refinements, variations, and applications of this formula. In particular, an application to the case where $\mathscr{A}$ equals the set of polynomials in the variable z yields a refinement of the result of D. Sarason [16] that a normal operator is reflexive.

For an operator T, set $\mathscr{A}(T)$ equal to the ultraweakly closed algebra generated by T and the identity and $\mathscr{W}(T)$ equal to the weak closure of $\mathscr{A}(T)$. (That is, $\mathscr{W}(T)$ denotes the weakly closed algebra generated by T and 1.) If $\{T\}''$ denotes the double commutant of T then the containments

$$(1.2) \qquad \mathscr{A}(T) \subseteq \mathscr{W}(T) \subseteq \{T\}''$$

hold. For a normal operator the first inclusion is always equality, while the second can be proper (for example, the bilateral shift). In fact, the second inclusion is equality if and only if the normal operator is reductive. The problem of determining where strict inclusion occurs in (1.2), under the assumption that T is a subnormal operator with cyclic vector, is somewhat perplexing. To explain in further detail we need the following notation.

If μ is a measure, let $H^2(\mu)$ and $P^\infty(\mu)$ denote the closures of the polynomials in the norm topology of $L^2(\mu)$ and the weak-star topology of $L^\infty(\mu)$, respectively. Let M_z denote the operator multiplication by z on $H^2(\mu)$. T. Yoshino [18] has

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shown that $\{M_z\}'$, (the commutant of M_z) equals the algebra of multiplication operators of the form M_ϕ for ϕ in $H^2(\mu) \cap L^\infty(\mu)$. Observe the first commutant is commutative so it equals the second commutant. J. Conway and R. Olin [9, Theorem 2.1] have shown that $\mathscr{A}(M_z)$ equals the algebra of multiplication operators of the form M_ϕ for ϕ in $P^\infty(\mu)$. Let $W^\infty(\mu)$ denote the set of ϕ in $H^2(\mu) \cap L^\infty(\mu)$ such that M_ϕ is in $\mathscr{W}(M_z)$. With this notation the operator algebra containments in (1.2) for the case $T = M_z$ on $H^2(\mu)$ correspond to the function space containments

$$P^\infty(\mu) \subseteq W^\infty(\mu) \subseteq H^2(\mu) \cap L^\infty(\mu). \tag{1.3}$$

It is unknown whether the first containment can be proper; this question is equivalent, via [9, p. 5], to a lifting problem for subnormal operators posed by J. Bram [3]. The second containment can be proper as seen by example 2 in [10]. (The function Ψ in this example can be constructed so that the polynomials in the variable $1/\Psi$ are dense in H^2.) Our intersection theorem (for $\mathscr{A}$ equal to the set of polynomials) gives a function theoretic characterization of $W^\infty(\mu)$ that points out the "analytic" problems one has to overcome in understanding the inclusions in (1.3).

The third section of the paper deals with applications of the intersection theorem to the case where $\mathscr{A}$ is the algebra of rational functions with poles off of some fixed compact set. Using techniques of J. Chaumat [5], [6], we prove the following: Given a pure subnormal operator S whose minimal normal extension has a $*$ cyclic vector, there exists a pure subnormal operator with nonempty residual spectrum that has the same minimal normal extension as S and the same spectrum as S. (See [9, chapter 9] for a related result.) Combining this with an example of J. Brennan [4], one obtains a counterexample to a test question raised by M. B. Abrahamse and R. G. Douglas [1, Problem 1]. In Section 4, we give another counterexample, which is much closer to the class of operators considered by Abrahamse and Douglas.

2. Weakly closed algebras of normal and subnormal operators

The proof of the following useful fact is left to the reader.

PROPOSITION 2.1. *Let μ be a positive measure and let $L^\infty(\mu)$ have the weak-star topology induced by its duality with $L^1(\mu)$. Then*

(a) *If $\nu \equiv \mu$ then $L^\infty(\mu) = L^\infty(\nu)$ and the weak-star topologies are identical.*

(b) *For any fixed p, $1 \le p < \infty$, a net $\{f_\alpha\}$ of functions converges to f weak-star in $L^\infty(\mu)$ if and only if for every measure $\nu \equiv \mu$ the net $\{f_\alpha\}$ converges to f weakly in $L^p(\nu)$.*

PROPOSITION 2.2. *Let $\mathscr{A}$ be a linear manifold of $L^\infty(\mu)$. Then for any p, $1 \le p \le \infty$,*

$$\mathscr{A}^\infty(\mu) = \bigcap_{\nu \equiv \mu} [\mathscr{A}^p(\nu) \cap L^\infty(\mu)].$$

Proof. For a fixed $\nu \equiv \mu$, let $f \in \mathscr{A}^{\infty}(\mu)$ and let the net $\{f_\alpha\}$ converge weak-star to f where each f_α is in $\mathscr{A}$. Then by Proposition 2.1, it follows that $f_\alpha \to f$ weakly in $L^p(\nu)$. Therefore $f \in \mathscr{A}^p(\nu)$ because a subspace is closed in $L^p(\nu)$ if and only if it is weakly closed.

To prove the reverse inclusion first define $D = \{(\varepsilon, \nu) \mid \varepsilon > 0, \nu \equiv \mu\}$. Order D in the following way: $(\varepsilon, \nu) \leq (\varepsilon', \nu')$ if $\varepsilon' < \varepsilon$ and $\nu'(\Delta) \geq \nu(\Delta)$ for all Borel sets Δ. Observe that given (ε, ν) and (ε', ν') in D there exists (ε'', ν'') which dominates the two. ($\varepsilon'' = \min(\varepsilon, \varepsilon')$ and $\nu'' = \nu + \nu'$.) Hence D is a net with this ordering.

Now let f belong to the intersection. For each ν and each $\varepsilon > 0$ choose $a_{(\varepsilon,\nu)} \in \mathscr{A}$ for which the $L^p(\nu)$ norm $\|a - a_{(\varepsilon,\nu)}\|$ is less than ε. It is easy to verify for any $\nu \equiv \mu$ that $a_{(\varepsilon,\nu)} \to a$ strongly in $L^p(\nu)$. Proposition 2.1 then shows that $f \in \mathscr{A}^{\infty}(\mu)$.

For an operator T on a Hilbert space H, let Lat T be the lattice of all invariant subspaces for T; and for a collection $\mathscr{L}$ of subspaces of H, let Alg $(\mathscr{L})$ be the (weakly closed) algebra of all operators leaving each member of $\mathscr{L}$ invariant. Recall that T is said to be reflexive if $\mathscr{W}(T) =$ Alg (Lat T). Sarason [16] has shown that a normal operator is reflexive. The following theorem is a strengthening of that result for the case where the normal operator has a cyclic vector. Moreover, the theorem can be used to prove Sarason's result for a noncyclic normal operator. For a measure ν absolutely continuous with respect to a measure μ, let $d\nu/d\mu$ denote the Radon-Nikodym derivative of ν with respect to μ.

THEOREM 2.3. *If N is multiplication by z on $L^2(\mu)$ and if A is an operator on $L^2(\mu)$ such that* Lat A *contains all the reducing subspaces for N and all invariant subspaces for N of the form $(d\nu/d\mu)^{1/2}H^2(\nu)$ where $\nu \equiv \mu$, then A is in $\mathscr{W}(N)$.*

Proof. An easy computation shows that $f \to (d\nu/d\mu)^{1/2} f$ is an isometry of $H^2(\nu)$ into $L^2(\mu)$ that intertwines multiplication by z on the respective spaces, and hence $(d\nu/d\mu)^{1/2}H^2(\nu)$ is an invariant subspace for N. Since A leaves the reducing subspaces of N invariant, it follows that A commutes with the spectral projections of N, and hence $A = M_\theta$ for some θ in $L^\infty(\mu)$.

Fix $\nu \equiv \mu$. Since A leaves $(d\nu/d\mu)^{1/2}H^2(\nu)$ invariant, it follows that

$$\theta\left(\frac{d\nu}{d\mu}\right)^{1/2} \cdot 1 \in \left(\frac{d\nu}{d\mu}\right)^{1/2} H^2(\nu).$$

Since $d\nu/d\mu$ is nonzero almost everywhere with respect to μ, it follows that $\theta \in H^2(\nu)$.

Proposition 2.2 now implies that $\theta \in P^\infty(\mu)$. Since the weak-star topology on $L^\infty(\mu)$ corresponds to the weak operator topology on the corresponding operator algebra, it follows that $A = M_\theta \in \mathscr{W}(N)$.

Recall that if m is normalized Lebesgue measure on the unit circle, then the bilateral shift is defined as M_z on $L^2(m)$. By Beurling's theorem [13], [14], the simply invariant subspaces are of the form $\theta H^2(m)$ where θ is a unimodular function in $L^\infty(m)$. In view of the last theorem it is natural to ask which of these subspaces is of the form $(dv/dm)^{1/2}H^2(v)$ for some measure v with $v \equiv m$.

PROPOSITION 2.4. *With notation as above,*

$$\left\{\left(\frac{dv}{dm}\right)^{1/2} H^2(v) \,|\, v \equiv m \text{ and } \int \log\left[\left(\frac{dv}{dm}\right)^{1/2}\right] dm > -\infty\right\}$$
$$= \left\{\frac{\bar{\Psi}}{|\Psi|} H^2(m) \,|\, \Psi \in H^2(m),\ \Psi \neq 0\right\}.$$

Proof. Note that there is no loss of generality in assuming the integrability condition. For if the integral equals $-\infty$ then the invariant subspace is $L^2(m)$. (See [14, p. 50].)

Fix the measure $dv = (dv/dm)\,dm$ and set $g = dv/dm$. By Szego's Theorem [14] there exists an outer function $\Psi \in H^2(m)$ so that $g = |\Psi|^2$. Observe that $H^2(v)$ and $H^2(m)$ are isometrically isomorphic under the map which sends $f \to f\Psi$. (Recall that an outer function is a cyclic vector for the unilateral shift.) If we set

$$\theta = \frac{\sqrt{g}}{\Psi}\left(= \frac{\bar{\Psi}}{|\Psi|}\right)$$

then

$$\theta H^2(m) = \left(\frac{H^2(m)}{\Psi}\right)\sqrt{g} = H^2(v)\sqrt{g}.$$

This proves one inclusion relationship.

To prove the other inclusion relationship the following fact will be useful.

Fact. Set $\theta = \bar{\Psi}/|\Psi|$ for some $\Psi \in H^2(m)$, $\Psi \neq 0$. Then there exists an outer function $U \in H^2(m)$ such that $\theta U > 0$ almost everywhere m.

If we write $\Psi = IU_1$ into its inner-outer factorization and let

$$U = U_1(1 + 2I + I^2),$$

an easy computation shows that $\theta U > 0$. It now suffices to show that $(1 + 2I + I^2)$ is outer. This last statement follows from the facts that the polynomial $(1 + z)^2$ is an outer function [14, p 76] and that the composition of an outer function with an inner function is outer [17, p. 120].

We now establish the other inclusion. Let $\Psi \in H^2(m)$; and find an outer function U with the property stated in the fact above. We let $g = (\theta U)^2 = |\theta U|^2 = |U|^2$ and then define $dv = g\,dm$. (Szego's theorem shows that $\int \log g\,dm > -\infty$.) We then have (applying the same ideas used in the other inclusion) that

$$H^2(v)\sqrt{g} = \left(\frac{H^2(m)}{U}\right)\sqrt{g} = H^2(m)\left(\frac{\theta U}{U}\right) = H^2(m)\frac{\bar{\Psi}}{|\Psi|}.$$

Remark. If θ is a nonconstant inner function then θ is an example of a unimodular function which is not of the form $\bar{\Psi}/|\Psi|$ for any Ψ in $H^2(m)$.

We conclude this section with an application of the intersection theorem to algebras of operators generated by a subnormal operator with a cyclic vector. Recall that if S is a subnormal operator with a cyclic vector, then S is unitarily equivalent to M_z on $H^2(\mu)$ for some measure μ [3]. For a measure μ, let $\mathscr{F}H^2(\mu)$ denote the set of measures $\nu \equiv \mu$ such that $d\nu = (\sum_1^n |\Psi_j|^2)\, d\mu$ for some finite sequence $\{\Psi_j\}_1^n \subset H^2(\mu)$ and let $\mathscr{I}H^2(\mu)$ denote the set of measures $\nu \equiv \mu$ such that $d\nu = (\sum_1^\infty |\Psi_j|^2)\, d\mu$ for some sequence $\{\Psi_j\}_1^\infty \subset H^2(\mu)$ with $\sum_1^\infty \|\Psi_j\|^2_{H^2(\mu)} < \infty$. The following theorem reposes the problems concerning the containments

$$P^\infty(\mu) \subset W^\infty(\mu) \subset H^2(\mu) \cap L^\infty(\mu)$$

discussed in the introduction.

THEOREM 2.5. *Let μ be a measure. Then*

(i) $P^\infty(\mu) = \bigcap_{\nu \in \mathscr{I}H^2(\mu)} [H^2(\nu) \cap L^\infty(\mu)]$

and

(ii) $W^\infty(\mu) = \bigcap_{\nu \in \mathscr{F}H^2(\mu)} [H^2(\nu) \cap L^\infty(\mu)]$.

Proof. First we recall some general operator theory facts. These ideas are originally due to W. Arveson [2]. For an operator T on a Hilbert space $\mathscr{H}$ and each integer $n = 1, 2, \ldots, \infty$, let $T^{(n)}$ be the direct sum of n copies of the operator T and set

$$B_n(T) = \{A\colon \text{Lat } T^{(n)} \subset \text{Lat } A^{(n)}\}.$$

Using Theorem 7.1 of [15] and the ultraweak continuity of the map $T \to T^{(\infty)}$, one sees that

$$\bigcap_{n<\infty} B_n(T) = \mathscr{W}(T) \tag{2.6}$$

and

$$B_\infty(T) = \mathscr{A}(T). \tag{2.7}$$

It is trivial consequence of the intersection theorem (2.2) that $P^\infty(\mu)$ is contained in the right-hand side of (i). The proof of the reverse inclusion is similar to the proof of the reverse inclusion of (ii). We shall thus proceed to the proof of the equality in (ii).

Let $\theta \in W^\infty(\mu)$. Then $M_\theta \in \mathscr{W}(S)$ where S is M_z on $H^2(\mu)$. Fix a measure

$$\nu = \left(\sum_1^n |\Psi_j|^2\right)\mu \quad \text{in } \mathscr{F}H^2(\mu).$$

Define a map $R\colon H^2(\nu) \to \bigoplus_1^n H^2(\mu)$ as follows: for a polynomial p, $R(p) = \bigoplus_1^n \Psi_j p$. Observe $\|R(p)\| = \|p\|$ so we can extend R to an isometry from $H^2(\nu)$ into $\bigoplus_1^n H^2(\mu)$. It is easy to verify that R intertwines M_z on $H^2(\nu)$ and $S^{(n)}$ on

$\bigoplus_1^n H^2(\mu)$. Consequently, $R(H^2(\mu))$, the range of R, is an invariant subspace of $S^{(n)}$. From (2.6) and the fact that $M_\theta \in \mathscr{W}(S)$, it now follows that

$$\bigoplus_1^n \theta\Psi_j \cdot 1 \in R(H^2(\nu)).$$

Therefore, $\theta\Psi_j = \Psi_j f$ for some f in $H^2(\nu)$ and all $j = 1, 2, \ldots, n$. Since $(\sum_1^n |\Psi_j|^2)\mu$ is mutually absolutely continuous with respect to μ, it follows that $\theta = f$ and hence $\theta \in H^2(\nu)$. Thus $W^\infty(\mu)$ is contained in the right-hand side of (ii).

To conclude the proof of (ii), it suffices to show that M_θ is in $\mathscr{W}(S)$ whenever $\theta \in \cap \{H^2(\nu) \cap L^\infty(\mu) \mid \nu \in \mathscr{F}H^2(\mu)\}$. For any such θ and for each $\varepsilon \geq 0$ and ν in $\mathscr{F}H^2(\mu)$ let $p_{(\varepsilon,\nu)}$ be a polynomial for which

$$\int |p_{(\varepsilon,\nu)} - \theta|^2 \, d\nu < \varepsilon.$$

Order the set of pairs (ε, ν) as in the proof of Proposition 2.2. It now follows from the definition of the weak topology on $\mathscr{B}(H^2(\mu))$ and of $\mathscr{F}H^2(\mu)$ that the net of operators $\{M_{p_{(\varepsilon,\nu)}}\}$ converges weakly to M_θ, and hence $M_\theta \in \mathscr{W}(S)$.

Remark. For T an operator on H, $n = 1, 2, 3, \ldots, \infty$, define

$$C_n(T) = \{A\colon \mathrm{Lat}_G\, T^{(n+1)} \subset \mathrm{Lat}\, A^{(n+1)}\}$$

where $\mathrm{Lat}_G\,(T^{(n+1)})$ is the set of graph invariant subspaces for $T^{(n+1)}$ with dense domain [15, p. 142]. It is thus apparent that $B_{n+1}(T) \subset C_n(T)$, and hence (2.6) and (2.7) imply that for any operator T,

$$\bigcap_{n<\infty} C_n(T) \supset W(T), \tag{2.8}$$

$$C_\infty(T) \supset \mathscr{A}(T). \tag{2.9}$$

An examination of the proof of Theorem 2.5 shows that (2.8) and (2.9) hold with equality for $T = M_z$ on $H^2(\mu)$. Whether this holds in general does not seem to be known.

3. An answer to a question by Abrahamse and Douglas

Let μ be a measure and K a compact set (in the plane) with $\operatorname{spt}\mu \subseteq K$. (Here $\operatorname{spt}\mu$ denotes the support of the measure μ.) Let $R(K)$ denote the uniform closure of the rational functions with poles off K. Following the notation of Chaumat [6], $R^\infty(\mu, K)$ denotes the weak-star closure of $R(K)$ in $L^\infty(\mu)$ and $R^2(\mu, K)$ denotes the $L^2(\mu)$ closure of $R(K)$. Observe that if K is the polynomial convex hull of $\operatorname{spt}\mu$, then $R^\infty(\mu, K) = P^\infty(\mu)$ and $R^2(\mu, K) = H^2(\mu)$.

The following proposition appears in [6]. It and its proof are essential to the ideas in this section. For μ a measure and w a complex number, write $\mu(w)$ instead of $\mu(\{w\})$.

PROPOSITION 3.1. *Let μ be a measure and K a compact set with* spt $\mu \subseteq K$. *Then $R^\infty(\mu, K) \neq L^\infty(\mu)$ if and only if there exist a point w in K and a complex* *representing measure λ_w for evaluation at w for $R(K)$ such that $\lambda_w(w) = 0$ and $\lambda_w \ll \mu$.*

Proof. Suppose there exists λ_w as claimed, and write $\lambda_w = g\, d\mu$ where $g \in L^1(\mu)$. Set $h = (z - w)g$. Then for each function r in $R(K)$, we see that

$$\int rh\, d\mu = \int r(z - w)\, d\lambda_w = 0.$$

Since $\lambda_w(w) = 0$ it follows that $h\, d\mu$ is not the zero measure and therefore, $R^\infty(\mu, K) \neq L^\infty(\mu)$.

Now assume that $R^\infty(\mu, K) \neq L^\infty(\mu)$, and let h in $L^1(\mu)$ be such that h is nonzero and $h \perp R^\infty(\mu, K)$. That is, $\int gh\, d\mu = 0$ for all g in $R^\infty(\mu, K)$. If r is a rational function with poles off K, then, for every w in K,

$$\int \left[\frac{r - r(w)}{z - w}\right] h\, d\mu = 0. \tag{3.2}$$

By basic results about the Cauchy transform of a measure [12, pp. 46–47], there exists a point z_0 in K such that $\mu(z_0) = 0$,

$$c = \int \frac{h}{z - z_0}\, d\mu \neq 0, \tag{3.3}$$

and the integral in (3.3) converges absolutely. Letting

$$dv = \frac{h}{c(z - z_0)}\, d\mu \tag{3.4}$$

and using (3.2), we see that v is a complex representing measure for z_0 and $v \ll \mu$. Since $\mu(z_0) = 0$ it follows that $v(z_0) = 0$.

Again following the notation of [6], define $E(\mu, K)$ to be the set of those w in K that have a representing measure as in Proposition 3.1. We then let $\mathscr{E}(\mu, K)$ denote the set of those f in $L^1(\mu)$ for which there exist a point w in $E(\mu, K)$ and a complex representing measure $\lambda_w \ll \mu$, such that $\lambda_w(w) = 0$ and $d\lambda_w = f\, d\mu$. Observe that if $f \in \mathscr{E}(\mu, K)$, say $f\, d\mu = d\lambda_w$ for some $w \in E(\mu, K)$, then evaluation at w on $R(K)$ extends to a bounded linear functional Γ_f on $R^2(|f|\, d\mu, K)$. Consequently, Γ_f extends to a bounded linear functional on $R^2((1 + |f|)\, d\mu, K)$, which we also denote by Γ_f. In fact, for every r in $R(K)$,

$$\begin{aligned} |r(w)|^2 &= \left|\int r\, d\lambda_w\right|^2 \\ &= \left|\int rf\, d\mu\right|^2 \\ &\leq M \int |r|^2 |f|\, d\mu \\ &\leq M \int |r|^2 (1 + |f|)\, d\mu, \end{aligned}$$

where $M = \int |f|\, d\mu$.

Observe that if $\theta \in R^2((1 + |f|)\, d\mu, K)$, then

$$\Gamma_f(\theta) = \int \theta f \, d\mu. \tag{3.5}$$

The following theorem shows how these R^2 spaces determine $R^\infty(\mu, K)$.

THEOREM 3.6. $R^\infty(\mu, K) = \bigcap_{f \in \mathscr{E}(\mu,K)} [R^2((1 + |f|)\, d\mu, K) \cap L^\infty(\mu)]$.

Proof. Let $h \perp R^\infty(\mu, K)$ where $h \in L^1(\mu)$. Then choose a point z_0 in K as given by equation (3.3). Using (3.2) we see that

$$\int \frac{r(z)}{z - z_0} h \, d\mu = r(z_0) \int \frac{h(z)}{z - z_0} \, d\mu$$

for all $r \in R(K)$. Therefore, if $f = h/[(z - z_0)c]$ then $f \in \mathscr{E}(\mu, K)$. Hence, for every θ in $\bigcap_{g \in \mathscr{E}(\mu,K)} [R^2((1 + |g|\, d\mu, K) \cap L^\infty(\mu)]$,

$$\int \frac{\theta h}{z - z_0} \, d\mu = \Gamma_f(\theta) \int \frac{h}{z - z_0} \, d\mu.$$

(Use (3.5).) That is,

$$\int \frac{\theta - \Gamma_f(\theta)}{z - z_0} h \, d\mu = 0. \tag{3.6}$$

But if θ belongs to $\bigcap_{g \in \mathscr{E}(\mu,K)} [R^2((1 + |g|)\, d\mu, K) \cap L^\infty(\mu)]$, then so does $(z - z_0)\theta$. By (3.5) again, $\Gamma_f((z - z_0)\theta) = 0$. Hence, by (3.6), we see that

$$\int \left[\frac{(z - z_0)\theta - 0}{z - z_0}\right] h \, d\mu = \int \theta h \, d\mu = 0$$

That is, $h \perp [\bigcap_{g \in \mathscr{E}(\mu,K)} R^2((1 + |g|)\, d\mu, K) \cap L^\infty(\mu)]$. By duality it follows that the intersection set is contained in $R^\infty(\mu, K)$. Since the reverse inclusion is obvious, the proof of the theorem is finished.

Given a weak-star closed subalgebra $\mathscr{A}$ of $L^\infty(\mu)$, by [9, Prop. 3.4], there exists a (unique) measurable partition $\{\Delta_1, \Delta_2\}$ of spt μ such that $\chi_{\Delta_i} \in \mathscr{A}$ and

$$\mathscr{A} = L^\infty(\Delta_1, \mu|_{\Delta_1}) \oplus \mathscr{A}\chi_{\Delta_2}, \tag{3.7}$$

where $\mathscr{A}\chi_{\Delta_2}$ contains no ideal of $L^\infty(\mu)$. We shall call $L^\infty(\Delta_1, \mu|_{\Delta_1})$ the L^∞ summand of $\mathscr{A}$. By the further properties of (3.7) that are stated in [8], the following proposition is immediate.

PROPOSITION 3.8. *$R^\infty(\mu, K)$ has no L^∞ summand if and only if there exists f in $\mathscr{E}(\mu, K)$ with $|f| > 0$ almost everywhere. Consequently, when $R^\infty(\mu, K)$ has no L^∞ summand, there exists a complex representing measure for $R(K)$ that is mutually absolutely continuous with respect to μ.*

The following theorem will be the key to our first counterexample to a question asked by Abrahamse and Douglas [1, Problem 1]. Before stating the theorem, we need some additional notation. First, let $\sigma(T)$ denote the spectrum of an operator T. For a fixed nonreductive normal operator N, let $\mathscr{S}(N)$ denote the collection of all subnormal operators for which N is the minimal normal extension. (See [8] for related results.)

THEOREM 3.9. *Let N be the normal operator M_z on $L^2(\mu)$. Then given a pure subnormal operator S in $\mathscr{S}(N)$, there exists a pure subnormal S_1 in $\mathscr{S}(N)$ for which $\sigma(S) = \sigma(S_1)$ and the residual spectrum of S_1 is nonempty.*

Proof. Let $S \in \mathscr{S}(N)$. It is well known that $\sigma(S) \supseteq \sigma(N)$ and if $\sigma(S) \supsetneq \sigma(N)$ then the residual spectrum of S is nonempty. So we may assume that $\sigma(S) = \sigma(N)$.

Since S is pure and $\sigma(S) = \sigma(N) = \operatorname{spt} \mu$, it follows that $R^\infty(\mu, \operatorname{spt} \mu)$ has no L^∞ summand. (If $R^\infty(\mu, \operatorname{spt} \mu)$ had an L^∞ summand, let χ_Δ belong to this latter space with $\mu(\Delta) > 0$. By using the same arguments that appear in Section 7 of [9], one can argue that χ_Δ induces a nontrivial projection P in the ultraweakly closed algebra generated by $\{r(S): r \in R(\sigma(S))\}$. Then the range of this projection would be a nonzero invariant subspace on which S is normal.)

Therefore, by Proposition 3.8, there exists f in $\mathscr{E}(\mu, \operatorname{spt} \mu)$ such that $|f|\, d\mu \equiv d\mu$. Consider the subnormal operator $\tilde{S}_1$ that is multiplication by z on $R^2(|f|\, d\mu, \operatorname{spt} \mu)$. By our remarks following the proof of Proposition 3.1, $R^2(|f|\, d\mu, \operatorname{spt} \mu)$ has a bounded point evaluation, say at $w \in \operatorname{spt} \mu$. Therefore $\tilde{S}_1$ has residual spectrum. (See [6].)

Let θ in $L^\infty(\mu)$ be such that $|\theta| = 1\mu$ almost everywhere and $f = |f|\theta$. Then define $g = \bar{\theta}(\bar{z} - \bar{w})$ and observe that for all $r \in R(\operatorname{spt} \mu)$,

$$\int r\theta(z-w)|f|\, d\mu = \int r(z-w) f\, d\mu = 0.$$

Therefore $g \perp R^2(|f|\, d\mu, \operatorname{spt} \mu)$. Since $\mu(w) = 0$, it follows that $|g| > 0\mu$ a.e. By the next proposition it now follows that $\tilde{S}_1$ is pure.

Define the map $V\colon L^2(\mu) \to L^2(|f|\, d\mu)$ by $g \to g/\sqrt{f}$. By using the same techniques as in Theorem 2.4 it follows that $\mathscr{M} \equiv V^{-1}(R^2(|f|\, d\mu, \operatorname{spt} \mu))$ belongs to the lattice of N. Then $S_1 \equiv N|_{\mathscr{M}}$ is the desired subnormal operator.

The following proposition was used in the proof of the previous proposition and will be used again when we give our second counterexample.

PROPOSITION 3.10. *Let $N = M_z$ on $L^2(\mu)$ and let $\mathscr{M}$ be an invariant subspace for N. Then $N|_{\mathscr{M}}$ is a pure subnormal operator if and only if there exists a function f in $L^2(\mu)$ such that $|f| > 0$ a.e. μ and $f \perp \mathscr{M}$.*

Proof. Assume there exists f in $L^2(\mu)$ as claimed. Let $\mathscr{N} \subset \mathscr{M}$ be such that $N|_{\mathscr{N}}$ is normal. Fix $h \in \mathscr{N}$. Since $\mathscr{N}$ reduces N, this implies $\bar{z}^n z^m h \in \mathscr{N}$ for all nonnegative integers n and m. For each continuous function g it follows that $gh \in \mathscr{N}$. Now since $f \perp \mathscr{M}$, $\int \bar{f}gh \, d\mu = 0$ for each continuous function g. Therefore, $\bar{f}h = 0$ a.e. μ. Hence, $f = 0$ on the set of positive μ measure where $|h| > 0$, a contradiction to our assumption.

Now suppose $\mathscr{M}$ is an invariant subspace for N such that $N|_{\mathscr{M}}$ is a pure subnormal. By a lemma of Chaumat [6], there exists a function f_0 in $\mathscr{M}^\perp$ such that every measure $f \, d\mu$ is absolutely continuous with respect to $f_0 \, d\mu$ for every f in $\mathscr{M}^\perp$. An easy argument shows that $|f_0| > 0$ a.e. (Otherwise, let E be the zero set of f_0. Then $\chi_E L^2(\mu) \subset \mathscr{M}$ and $\chi_E L^2(\mu)$ reduces N.)

Remark. The lemma quoted from [6] in the last proof has the assumption that the given convex set lies inside $L^1(\mu)$. However, by a very slight modification, the result holds in $L^p(\mu)$ for $1 \le p < \infty$.

We are now ready to present the first counterexample to problem 1 in [1]. The notation here is consistent with that of [1]. Recall their question:

"If for $i = 1, 2$ the pure subnormal operator S_i has N_i as its minimal normal extension with $\sigma(N_i) \subset \partial\sigma(S_i)$, $\sigma(S_1) = \sigma(S_2)$, and N_1 is unitarily equivalent to N_2, then are S_1 and S_2 similar?"

Example 1. Let K be an arbitrary Swiss cheese obtained by removing from the closed unit disk $\bar{D}$ the open disks $D_j = \{z: |z - a_j| < r_j\}$, $j = 1, 2, \ldots$, where the D_j have mutually disjoint closures and $\sum_1^\infty r_j < \infty$. Let μ denote arc length on $\bigcup_1^\infty \partial D_j \cup \partial D$. Let $N = M_z$ on $L^2(\mu)$, let $\mathscr{M} = R^2(\mu, K)$, and let $S_1 = N|_{\mathscr{M}}$. If f is a rational function with poles off K, then $\int f(dz/d\mu) \, d\mu = \int f \, dz = 0$ by Cauchy's theorem. Thus $dz/d\mu \perp \mathscr{M}$. Since $|dz/d\mu| = 1$ a.e. μ, it follows from Proposition 3.10 that S_1 is pure. It is easy to see that $\sigma(S_1) \subseteq K$ and since S_1 is pure, the reverse inclusion follows from [3]. The fact that $R^2(\mu, K)$ has no bounded point evaluations [4, p. 290] implies that S_1 has no residual spectrum [7]. Now by Proposition 3.9, there exists S_2 in $S(N)$ such that $\sigma(S_2) = \sigma(S_1)$ and the residual spectrum of S_2 is nonempty. The operators S_1 and S_2 satisfy the hypotheses of the question of Abrahamse and Douglas, but they are not similar since the residual spectrum of an operator is preserved under similarity.

The preceding example is not entirely satisfactory, for the spectrum of the subnormal operators has no interior. We shall now present a second counterexample where the spectrum of the minimal normal extensions is the countable union of disjoint analytic Jordan curves and one point.

4. The second counterexample

Example 2. Let $K = \bar{D} \backslash \bigcup_2^\infty D(2^{-n}, n^{-2}2^{-n})$ where $D(z, r) = \{w: |w - z| < r\}$. The point zero is not a peak point for $R(K)$ (see Zalcman [19]), so by [19, p. 5] there exists a representing measure μ for evaluation at zero with $\mu(\partial K) = 1$ and $\mu(0) = 0$. It follows easily that $R^2(\mu, K)$ has a bounded point evaluation at zero. Let $N_1 = M_z$ on $L^2(\mu)$ and let $S_1 = N|_{R^2(\mu, K)}$. Note that S_1 is pure by Proposition 3.10. ($\bar{z}\perp R(K)$ in $L^2(\mu)$ and $|\bar{z}| > 0$ a.e. μ.) We also point out that zero is in the residual spectrum of S_1 since $R^2(\mu, K)$ has a bounded evaluation at zero.

We shall now construct a measure ν that is mutually absolutely continuous with respect to μ and such that $R^2(\nu, K)$ does not have a bounded point evaluation at zero although $R^2(\nu, K) \neq L^2(\nu)$. Once this is done, we shall show that multiplication by z on $R^2(\nu, K)$ and S_1 provide a counterexample to the question of Abrahamse and Douglas.

Let C_1 denote the unit circle and let $C_j = \{z: |z - 2^{-j}| = j^{-2}2^{-j}\}$ for $j \geq 2$. Let $\nu_1 = \mu|_{C_1}$ and $\nu_j = j^{-4}2^{-2j}\mu|_{C_j}$ and let $\nu = \sum \nu_j$. Let $N_2 = M_z$ on $L^2(\nu)$ and let $S_2 = N_2|_{R^2(\nu,K)}$. Let

$$f(z) = \begin{cases} j^4 2^{2j}\bar{z}^2 & \text{for } z \in C_j,\ j \geq 2, \\ \bar{z}^2 & \text{for } z \in C_1. \end{cases}$$

Straightforward computations show that $f \in L^2(\nu)$ and that $f \perp R(K)$. Since $|f| > 0$ a.e. ν, it follows by Proposition 3.10 that S_2 is a pure subnormal operator.

We next show that there does not exist a bounded point evaluation at zero for $R^2(\nu, K)$, and thus zero is not in the residual spectrum of S_2. Let $r_n(z) = (z - 2^{-n})^{-1}$. Using the facts that $|z| \geq 2^{-j-1}$ and

$$|r_n - 1/z| \leq j^2 2^j + 2^{j+1} \text{ on } C_j \quad \text{for } j \geq 2,$$

one can show that $1/z \in L^2(\nu)$ and that $r_n \to 1/z$ in $L^2(\nu)$. Thus the r_n's are uniformly bounded in $L^2(\nu)$ norm, but $r_n(0) \to \infty$.

It is easy to see (via the Stone-Weierstrass theorem) that the minimal normal extensions of S_1 and S_2 are N_1 and N_2. The fact that N_1 and N_2 are unitarily equivalent follows from the mutual absolute continuity of μ and ν. Thus we have established that S_1 and S_2 satisfy the hypotheses of the Abrahamse and Douglas question. (It is obvious that $\sigma(S_i) \subset K$ for $i = 1, 2$. Since $\sigma(N_i) = \partial K$ and S_i is pure, it follows from [3] that $\sigma(S_i) \supseteq \partial K$. It is well known that if the spectrum of a subnormal operator has zero planar measure, then the operator is normal. Therefore, using [3] again, it follows that $\sigma(S_1) = \sigma(S_2) = K$.) Since zero is in the residual spectrum of S_1 but not that of S_2, it follows that S_1 and S_2 are not similar.

Bibliography

1. M. B. Abrahamse and R. G. Douglas, *A class of subnormal operators related to multiply connected domains*, Advances in Mathematics, vol. 19 (1976), pp. 106–148.
2. W. B. Arveson, *A density theorem for operator algebras*, Duke Math. J., vol. 34 (1967), pp. 635–647.
3. J. Bram, *Subnormal operators*, Duke Math. J., vol. 22 (1955), pp. 75–94.
4. J. Brennan, *Invariant subspaces and rational approximation*, J. Func. Anal., vol. 7 (1971), pp. 285–310.
5. J. Chaumat, *Adherence Faible étoile d'algebres de fractions rationnelles*, Ann. Inst. Fourier, vol. 24 (1974), pp. 93–120.

6. ———, *Adherence faible ètoile d'algebres de fractions rationnelles*, Publications Mathématique d'Orsay, No. 147, pp. 75–40, U.E.R. Mathématique, Université Paris X1, Orsay, 1975.
7. K. F. CLANCEY AND B. B. MORREL, *The essential spectrum of some Toeplitz operators*, Proc. Amer. Math. Soc., vol. 44 (1974), pp. 129–134.
8. J. B. CONWAY AND R. F. OLIN, *A functional calculus for subnormal operators*, Bull. Amer. Math. Soc., vol. 82 (1976), pp. 259–261.
9. ———, *A functional calculus for subnormal operators, II*, Mem. Amer. Math. Soc., vol. 184 (1977), pp. vii and 61.
10. J. A. DEDDENS AND W. R. WOGEN, *On operators with the double commutant property*, Duke Math. J., vol. 43 (1976), No. 2, pp. 359–363.
11. R. G. DOUGLAS, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
12. T. GAMELIN, *Uniform algebras*, Prentice Hall, Englewood Cliffs, 1969.
13. H. HELSON, *Lectures on invariant subspaces*, Academic Press, New York, 1964.
14. K. HOFFMAN, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, 1962.
15. H. RADJAVI AND P. ROSENTHAL, *Invariant subspaces*, Springer Verlag, New York, 1973.
16. D. E. SARASON, *Invariant subspaces and unstarred operator algebras*, Pacific J. Math., vol. 17 (1966), pp. 511–517.
17. B. SZ. NAGY AND C. FOIAS, *Harmonic analysis of operators on Hilbert space*, American Elsevier, New York, 1970.
18. T. YOSHINO, *Subnormal operator with a cyclic vector*, Tohoku Math. J., vol. 21 (1969), pp. 47–55.
19. L. ZALCMAN, *Analytic capacity and rational approximation*, Lecture Notes in Mathematics, no. 50, Springer Verlag, Berlin, 1968.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VIRGINIA