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ON COLORING MANIFOLDS

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1. Introduction

If K is a finite simplicial complex then we define $\mathrm{ch}_i(K)$ to be the smallest positive integer such that one can assign to each i-simplex of K one of $\mathrm{ch}_i(K)$ labels in such a way that not all the faces of an $i+1$ simplex have the same label. If X is a compact triangulable space we define $\mathrm{ch}_i(X) = \sup_K \mathrm{ch}_i(K)$ as K runs over all triangulations of X. Note that $\mathrm{ch}_i(X)$ is a positive integer or ∞. If the topological space X is homogenously of dimension m we will denote $\mathrm{ch}_{m-2}(X)$ by $\mathrm{ch}(X)$ and it will be called the *chromatic number of* X.

FINITENESS THEOREM. *For any closed triangulable manifold X of dimension $m \geq 2$ one has*

$$\mathrm{ch}(X) \leq \left\{\frac{m(m+1)}{m-1}[1 + b_{m-1}(X; \mathbf{Z}_2)]\right\}$$

where $b_{m-1}(X; \mathbf{Z}_2)$ is the dimension of the homology group $H_{m-1}(X; \mathbf{Z}_2)$ and $\{x\}$ denotes the "smallest integer containing x".

This result in fact holds even for all pseudomanifolds and will be proved in Section 3 below. One has in particular for the m-sphere

$$\mathrm{ch}(S^m) \leq \left\{\frac{m(m+1)}{m-1}\right\}.$$

For $m = 2$ this is a classical result of Heawood and was recently improved to the equality $\mathrm{ch}(S^2) = 4$ by Appel and Haken [1]. With this one knows how to calculate the chromatic number of any 2-dimensional manifold. See Ringel [2].

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2. Some inequalities

It will be assumed throughout that m is an integer such that $m \geq 2$. All simplicial sets and complexes will be assumed *finite*. Given a simplicial set K we will denote by K^n the subset of all simplices of dimensions $\leq n$. If σ is a simplex of K, $\mathrm{St}_K\,\sigma$ will denote the subset of K consisting of all simplices of K contain-

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ing σ. We will denote the number of i-simplices in K by $\alpha_i(K)$. For each pair $K \subseteq L$ of simplicial complexes we will denote by $H_i(L, K)$ the ith unaugmented homology group with *coefficients* $\mathbf{Z}_2$; this is in fact a vector space over the field $\mathbf{Z}_2$ and its dimensions will be denoted by $b_i(L, K)$. If K is empty we will use instead $H_i(L)$ and $b_i(L)$ respectively. For a topological space X the notations $H_i(X)$, $b_i(X)$ would have an analogous sense.

We recall that a topological space X is called a *m-dimensional pseudomanifold* if X can be triangulated by a simplicial complex L such that:

(i) L is homogenously of dimension m, i.e. each simplex is contained in an m-simplex.

(ii) Any $(m - 1)$-simplex of L is incident to precisely two m-simplicies.

(iii) Any two m-simplices η, ξ of L can be *joined through $m - 1$ simplices*, i.e. there exists a finite sequence of m-simplices

$$\eta = \sigma_1, \sigma_2, \ldots, \sigma_r = \xi$$

such that σ_i and σ_{i+1} are incident to the same $m - 1$ simplex.

It is known that then every triangulation of X has these three properties.

It is known that all (closed, connected) triangulable topological manifolds are pseudomanifolds in the above sense; the converse is true for dim 1 but the sphere with 2 points identified and the four dimensional space obtained by suspending Poincaré's dodecahedral manifold are pseudomanifolds which are not manifolds. In the following essential use is made only of (ii) and (iii).

LEMMA. *Let K be a subcomplex of a triangulation L of an m-dimensional pseudomanifold. Then at least one of the following holds:*

$$b_m(L, K) \leq \frac{2}{m + 1}\alpha_{m-1}(K), \tag{1}$$

$$b_m(L, K) = 1. \tag{2}$$

Proof. We will assume $b_m(L, K) \geq 1$ since otherwise (1) holds.

Since L is m-dimensional $H_m(L, K)$ is formed from all $\mathbf{Z}_2$ chains c of $L - K$ whose boundary ∂c is contained in K. From (ii) it follows that if c contains an m-simplex σ then it contains the chain c_σ of all m-simplices of $L - K$ which can be joined to σ through $(m - 1)$-simplices of $L - K$. We note that for each m-simplex σ of $L - K$ this m-chain c_σ is non-zero and that ∂c_σ is contained in K; further two such chains c_{σ_1} and c_{σ_2} are either disjoint or equal. Thus the set

$$\{c_\sigma \mid \sigma \text{ an } m\text{-simplex of } L - K\}$$

is a basis of $H_m(L, K)$.

If $\partial c_\sigma = 0$ for some σ then it follows from (iii) that c_σ contains all m-simplices of $L - K$; there is thus only one chain of type c_σ and $b_m(L, K) = 1$. So assume

$\partial c_\sigma \neq 0$ for all σ. Let ξ be an $(m-1)$-simplex contained in ∂c_σ; if the number of $(m-1)$-simplices contained in ∂c_σ was less than $m+1$ then one of the m faces of ξ would be incident to no other $(m-1)$-simplex of ∂c_σ. This contradicts $\partial\partial c_\sigma = 0$; so each ∂c_σ must contain at least $m+1$ simplices. On the other hand each $(m-1)$-simplex of K is incident to at most two m-simplices of L and so can occur in at most two chains of type ∂c_σ. This gives (1).

THEOREM 1. *Let K be a subcomplex of a triangulation L of an m-dimensional pseudomanifold X such that $L^{m-3} \subseteq K \subseteq L^{m-1}$ and K has at least one $m-1$ simplex. Then*

$$\frac{m-1}{m+1}\frac{\alpha_{m-1}(K)}{\alpha_{m-2}(K)} \leq 1 + \frac{b_{m-1}(X)-1}{\alpha_{m-2}(K)} \tag{A}$$

Proof. The long exact homology sequence of (L, K) is

$$\cdots \to H_i(K) \to H_i(L) \to H_i(L, K) \to H_{i-1}(K) \to \cdots \tag{3}$$

Its exactness implies that the alternating sum of the dimensions of its terms is zero; so

$$\begin{aligned} b_m(L, K) = {} & \{b_m(L) - b_{m-1}(L) + \cdots\} - \{b_m(K) - b_{m-1}(K) + \cdots\} \\ & + \{b_{m-1}(L. K) - b_{m-2}(L, K) + \cdots\}. \end{aligned} \tag{4}$$

Next we note that

$$\begin{aligned} \{b_m(K) - b_{m-1}(K) + \cdots\} &= \{\alpha_m(K) - \alpha_{m-1}(K) + \cdots\} \\ &\qquad \text{(both terms being } (-1)^m\chi(K)) \\ &= -\alpha_{m-1}(K) + \alpha_{m-2}(K) \\ &\qquad - \{\alpha_{m-3}(L^{m-3}) - \alpha_{m-4}(L^{m-3}) + \cdots\} \\ &\qquad \text{(since } L^{m-3} \subseteq K \subseteq L^{m-1}) \\ &= -\alpha_{m-1}(K) + \alpha_{m-2}(K) \\ &\qquad - \{b_{m-3}(L^{m-3}) - b_{m-4}(L^{m-3}) + \cdots\} \\ &\qquad \text{(using exp for } (-1)^{m-3}\chi(L^{m-3})) \\ &= -\alpha_{m-1}(K) + \alpha_{m-2}(K) - b_{m-3}(L^{m-3}) \\ &\qquad + \{b_{m-4}(L) - b_{m-5}(L) + \cdots\} \end{aligned} \tag{5}$$

because $H_i(L^{m-3}) \to H_i(L)$ is an isomorphism for $i \leq m-4$. Since $L^{m-3} \subseteq K$ the map $H_i(K) \to H_i(L)$ is surjective for $i \leq m-3$ and an isomorphism for $i \leq m-4$; so the exactness of (3) implies that

$$b_i(L, K) = 0 \quad \text{for } i \leq m-3. \tag{6}$$

Substituting (5) and (6) into (4) we get

$$b_m(L, K) = b_m(L) - b_{m-1}(L) + b_{m-2}(L) - b_{m-3}(L) + \alpha_{m-1}(K) - \alpha_{m-2}(K) + b_{m-3}(L^{m-3}) + b_{m-1}(L, K) - b_{m-2}(L, K). \quad (7)$$

Suppose that (1) holds; then on substituting (7) into (1) we get

$$\frac{m-1}{m+1}\alpha_{m-1}(K) \le -b_m(L) + b_{m-1}(L) - b_{m-2}(L) + b_{m-3}(L) + \alpha_{m-2}(K) - b_{m-3}(L^{m-3}) - b_{m-1}(L, K) + b_{m-2}(L, K). \quad (8)$$

We note that $H_{m-3}(L^{m-3}) \to H_{m-3}(K)$ is onto; so

$$-b_{m-3}(L^{m-3}) \le -b_{m-3}(K). \quad (9)$$

Using (6) we see that (3) has the following exact subsequence:

$$H_{m-2}(L) \xrightarrow{f} H_{m-2}(L, K) \to H_{m-3}(K) \to H_{m-3}(L) \to 0. \quad (10)$$

This implies that

$$b_{m-3}(L) - b_{m-3}(K) + b_{m-2}(L, K) = \dim(\mathrm{Imf}) \le b_{m-2}(L). \quad (11)$$

Substituting (9) and (11) into (8) we get

$$\frac{m-1}{m+1}\alpha_{m-1}(K) \le \alpha_{m-2}(K) + b_{m-1}(X) - 1. \quad (12)$$

Here we have used the fact that $-b_{m-1}(L, K) \le 0$ while $b_{m-1}(L) = b_{m-1}(X)$ and $b_m(L) = b_m(X) = 1$.

When (2) holds we substitute (7) into (2) and again use (9) and (11) to get

$$\alpha_{m-1}(K) \le \alpha_{m-2}(K) + b_{m-1}(X). \quad (13)$$

But K contains at least one $(m-1)$-simplex; so

$$\alpha_{m-2}(K) \ge m. \quad (14)$$

But (13) and (14) again imply (12) which is the desired inequality.

3. Coloring theorems

The finiteness theorem follows as an easy corollary:

THEOREM 2. *The chromatic number of an m-dimensional pseudomanifold X is less than or equal to*

$$\left\{\frac{m(m+1)}{m-1}[1 + b_{m-1}(X; \mathbf{Z}_2)]\right\}.$$

Proof. Let us denote the above integer by $n(X)$. For any triangulation L of X we have to show that $\mathrm{ch}_{m-2}(L) \le n(X)$. But $\mathrm{ch}_{m-2}(L) = \mathrm{ch}_{m-2}(L^{m-1})$ and thus it will be sufficient if we can show $\mathrm{ch}_{m-2}(K) \le n(X)$ for all subcomplexes K of L such that $L^{m-3} \subseteq K \subseteq L^{m-1}$. We do this by induction on $\alpha_{m-2}(K)$. When $\alpha_{m-2}(K) = 0$, $L^{m-3} = K$ and $\mathrm{ch}_{m-2}(K) = 1 \le n(X)$. So assume $\alpha_{m-2}(K) > 0$. If K contains no $(m-1)$-simplex it is clear that $\mathrm{ch}_{m-2}(K) = 1 \le n(X)$. Otherwise Theorem 1 applies. From the inequality (A) we see that

$$\frac{m-1}{m+1}\frac{\alpha_{m-1}(K)}{\alpha_{m-2}(K)} < 1 + b_{m-1}(X) \tag{15}$$

and so

$$\frac{m\alpha_{m-1}(K)}{\alpha_{m-2}(K)} < n(X). \tag{16}$$

But in (16) the left hand side denotes the average number of $(m-1)$-simplices incident to an $(m-2)$-simplex. So there exists a $\sigma^{m-2} \in K$ incident to less than $n(X)$ simplices of dimension $m-1$. Put $K' = K - \mathrm{St}_K\, \sigma$; now $L^{m-3} \subseteq K' \subseteq L^{m-1}$ and the subcomplex K' obeys $\alpha_{m-2}(K') < \alpha_{m-2}(K)$. Using the inductive hypothesis we can color K' with $n(X)$ labels and then, since σ is incident to less than $n(X)$ simplices of dimension $m-1$, extend it to a coloring of K using the same $n(X)$ labels.

Remarks. (*a*) Some improvements on the above bounds can be easily made. Consider first the *case* $m = 2$. Then the inequality (8) becomes

$$\tfrac{1}{3}\alpha_1(K) \le -b_2(L) + b_1(L) - b_0(L) + \alpha_0(K) - b_1(L, K) + b_0(L, K)$$

Now here $b_0(L) = b_2(L) = 1$ while $b_0(L, K) = 0$ because K is a non-empty subcomplex of the connected subcomplex L. So instead of (12) we get the better inequality

$$\tfrac{1}{3}\alpha_1(K) \le \alpha_0(K) + b_1(X) - 2. \tag{12$'$}$$

Similarly instead of (13) one gets the better inequality

$$\alpha_1(K) \le \alpha_0(K) + b_1(X) - 1. \tag{13$'$}$$

However now (14), i.e. $\alpha_0(K) \ge 2$, does not ensure that (13′) implies (12′). We have to replace it with the stronger condition $\alpha_0(K) \ge 3$. We have thus reproved the following theorem of [2] (see pp. 55 and 59): *For any 1-dimensional subcomplex K of a triangulation of a closed 2-dimensional X one has*

$$\frac{1}{3}\frac{\alpha_1(K)}{\alpha_0(K)} \le 1 + \frac{b_1(X) - 2}{\alpha_0(K)}, \tag{A$'$}$$

provided $\alpha_0(K) \ge 3$. Using this inequality (A′) and a slight refinement of the argument of Theorem 2 one gets, for 2-manifolds X with $b_1(X) \ge 1$ the well known formula of Heawood:

$$\mathrm{ch}\,(X) \le \left\lfloor \frac{7 + \sqrt{1 + 24b_1(X; \mathbf{Z}_2)}}{2} \right\rfloor. \tag{17}$$

See p. 64 of [2]. We remark that the elementary theorems on pages 10–12 of [2] have obvious generalizations; so such a refinement can be made in the proof of Theorem 2 in general and one sees that *for any m-dimensional pseudomanifold X with* $b_{m-1}(X; \mathbf{Z}_2) \geq 1$,

$$\text{ch}\,(X) \leq \left\lfloor \frac{b + \sqrt{b^2 - 4c(X)}}{2} \right\rfloor, \tag{18}$$

where

$$b = 1 + \frac{m(m+1)}{m-1} \quad \textit{and} \quad c(X) = \frac{m(m+1)}{m-1}(1 - b_{m-1}(X; \mathbf{Z}_2)). \tag{19}$$

(*b*) *For any m-dimensional pseudomanifold X one has* $\text{ch}_{m-1}\,(X) \leq 3$. This can be proved analogously to Theorem 2 using the fact that in any subcomplex of a triangulation of X a $m-1$ simplex is incident to less than three m-simplices. On the other hand one can see that *for any piecewise linear manifold* Y *of dimension* $m \geq 3$, $\text{ch}_0\,(Y) = \infty$. To see this we will use the fact that the chromatic number of an orientable closed 2-manifold X is given by

$$\text{ch}\,(X) = \left\lfloor \frac{7 + \sqrt{1 + 24b_1(K)}}{2} \right\rfloor$$

Therefore given any $n > 0$ one can choose an orientable surface X_n with $\text{ch}\,(X_n) \geq n$. Next we note the fact there exists a piecewise linear imbedding of X_n in $\mathbf{R}^3$ and hence in any piecewise linear manifold Y of dimension ≥ 3. Hence $\text{ch}_0\,(Y) \geq \text{ch}_0\,(X_n) \geq n$ and since n is arbitrary we get $\text{ch}_0\,(Y) = \infty$.

References

1. K. Appel and W. Haken, *A proof of the four color theorem*, Discrete Mathematics, vol. 16 (1976), pp. 179–180.
2. G. Ringel, *Map color theorem*, Springer Verlag, New York, 1974.

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