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UNIVALENT FUNCTIONS MAXIMIZING Re $\{a_3 + \lambda a_2\}$

BY

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1. Introduction

Let S denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ analytic and univalent in the unit disk $|z| < 1$. Let L be a continuous linear functional defined on the space of functions analytic in $|z| < 1$. We call $f \in S$ a *support point* for L if L is nonconstant on S and

$$\text{(1)} \qquad \operatorname{Re} L(f) = \max_{g \in S} \operatorname{Re} L(g).$$

It is known [1], [5] that each support point of S maps the unit disk onto the complement of an analytic arc with monotonic modulus, whose tangent vector always makes an angle of less than $\pi/4$ with the radius vector. The Koebe function $k(z) = z/(1 - z)^2$ and its rotations are well-known support points of S.

In [2], we studied the point-evaluation functionals $L(g) = g(z_0)$ and gave some new examples of support points and extreme points of S. We found several geometric properties of the arcs omitted by these functions.

The purpose of the present paper is to investigate the support points for the linear functionals

$$\text{(2)} \qquad L(g) = a_3 + \lambda a_2, \quad \lambda \in \mathbf{C}.$$

More precisely, we prove that the arcs omitted by these support points lie in sectors and have monotonic arguments. We also show that only very special values of λ give rise to the Koebe function or a rotation as a support point for (2). In addition, we prove that there are only four rotations of the Koebe function arising as support points for (2). Finally we show that if $\lambda_1 \neq \lambda_2$ then the associated support points are distinct unless they are the same rotation of the Koebe function. Hence functionals of the form (2) generate a rich family of support points of S.

2. Reductions

Let $F(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ be a support point for the functional (2) and let Γ denote the arc omitted by F. Before stating and proving our main results, we establish some reductions. These serve to simplify our statements and proofs.

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In this investigation we require the Loewner representation and the Schiffer method of interior variation, which we shall now describe.

Loewner [4] showed that if a function in S is a slit mapping then its coefficients may be represented in terms of a control function $\kappa(t)$. Since F is known to be a slit mapping we have

$$A_2 = -2\int_0^\infty e^{-t}\kappa(t)\,dt \tag{3}$$

and

$$A_3 = -2\int_0^\infty e^{-2t}\kappa(t)^2\,dt + A_2^2, \tag{4}$$

where $\kappa(t) = e^{i\theta(t)}$ and $\theta(t)$ is a real-valued continuous function related to the geometric structure of Γ.

One of the most powerful variations is due to Schiffer [7], [8], [9]. His method of interior variation leads to a certain nonlinear differential equation which the extremal function F must satisfy. For our particular problem, this differential equation takes the form

$$\left\{\frac{zF'(z)}{F(z)}\right\}^2\left\{\frac{1 + BF(z)}{F(z)^2}\right\} = R(z), \tag{5}$$

where $R(z) = z^2 + \bar{B}z + B_0 + Bz^{-1} + z^{-2}$, $B = 2A_2 + \lambda$ and $B_0 = 2A_3 + \lambda A_2 > 0$ (cf. Schaeffer and Spencer [6, pp. 211–214]). Equation (5) may be integrated to obtain an implicit representation for $F(z)$. However, we will not make use of it here.

If we parameterize Γ by $w(t) = F(e^{it})$, then (5) becomes

$$\frac{1 + Bw}{-w^2}\left(\frac{w'}{w}\right)^2 > 0, \tag{6}$$

where $B = 2A_2 + \lambda$. This is the Schiffer differential equation for the omitted arc Γ. In terms of quadratic differentials, it is

$$Q(w)\,dw^2 > 0 \quad \text{where } Q(w) = -(1 + Bw)/w^4. \tag{7}$$

We shall now show that we can assume, with no loss of generality, that Re $B \geq 0$ and Im $B \leq 0$. Indeed, suppose F is a support point for (2) and Γ is parameterized by $w(t) = F(e^{it})$. Hence $w(t)$ satisfies (6) with $B = 2A_2 + \lambda$. If we let $w_1(t) = F_1(e^{it})$, where $F_1(z) = -F(-z)$, then $w_1(t)$ satisfies (6) with $B = -(2A_2 + \lambda)$. (Note that the functional (2) changes with λ being replaced by $-\lambda$.) Hence we may assume Re $B \geq 0$. If we now let $w_2(t) = F_2(e^{it})$, where $F_2(z) = \overline{F(\bar{z})}$, then $w_2(t)$ satisfies (6) with $B = 2\bar{A}_2 + \bar{\lambda}$. (In this case λ is replaced by $\bar{\lambda}$ in (2).) Thus, there is no loss in assuming Re $B \geq 0$ and Im $B \leq 0$. The following preliminary theorem states that if F is not a rotation of the Koebe function then Re $B \neq 0$ and Im $B \neq 0$.

THEOREM 1. *Suppose* $F(z) = z + A_2 z^3 + A_3 z^3 + \cdots$ *is a support point for* (2) *with* $\operatorname{Re} B \geq 0$ *and* $\operatorname{Im} B \leq 0$, *where* $B = 2A_2 + \lambda$. *Then:*

(a) $B > 0$ *if and only if the Koebe function is the unique extremal function;*
(b) $iB > 0$ *if and only if* $-ik(iz)$ *is the unique extremal function.*

Proof. From (6), it is clear that if $B \neq 0$ then the quadratic differential $Q(w)\,dw^2$ has a simple pole at infinity. (Pfluger [5] proved that the quadratic differential associated with any support point has a simple pole at infinity.) Thus, there is a unique trajectory terminating at infinity.

A brief calculation shows that if $w(t) = -te^{i\theta}$, $t > 1/4$, θ real, is a solution to (6) then $Be^{-i\theta} > 0$ and $e^{2i\theta} = \pm 1$. On the other hand, if $Be^{-i\theta} > 0$ and $e^{2i\theta} = \pm 1$ then for large positive values of t, $w(t) = -te^{i\theta}$ satisfies (6). By the remark above, $w(t) = -te^{i\theta}$ for $t > 1/4$ is the solution. Hence $w(t) = -t$ is a solution to (6) if and only if $B > 0$; while $w(t) = it$ is a solution if and only if $iB > 0$. The proof of the theorem is complete.

Thus if $F(z) = z + A_2 z^2 + \cdots$ is a support point for (2) and is not a rotation of the Koebe function, we may assume that $\operatorname{Re} B > 0$ and $\operatorname{Im} B < 0$.

3. Main results

Let $F(z) = z + A_2 z^2 + A_3 z^2 + \cdots$ be a support point for (2) that is not a rotation of the Koebe function. Then, we may assume, with no loss of generality, that $\operatorname{Re} B > 0$ and $\operatorname{Im} B < 0$, where $B = 2A_2 + \lambda$. Let Γ denote the arc omitted by F. Parameterize Γ by $w = w(t)$, with $w'(t) \neq 0$, and let $w(0) = \infty$ and $w(T)$ be the finite tip of Γ. Set $\tau_0 = \arg(-B)$ and $\alpha_0 = \arg w(T)$. Then we prove:

THEOREM 2. $\theta(t) = \arg w(t)$ *is a monotonic function of* t. *More precisely,* $\theta'(t) < 0$ *for* $0 < t < T$.

THEOREM 3. Γ *lies entirely in the sector*

$$\Omega = \{\rho e^{i\theta}\colon \alpha_0 \leq \theta \leq \tau_0,\ 0 < \rho < \infty\}.$$

To prove these theorems we get information about Γ by focusing on the local and global trajectory structure of the quadratic differential in (7). We then represent Γ as the image under the Koebe function of a certain arc. By examining this arc, using the known properties of support points, and exploiting the mapping properties of the Koebe function, we are able to deduce properties about Γ. We used this technique with success in [2]. There we could determine all parameters explicitly. This is not the case here, but we can still find some general properties of Γ. Thus, our theorems may be viewed as qualitative results about certain solutions to a one parameter family of differential equations.

LEMMA 1. Γ *lies in the sector* $\Omega_1 = \{\rho e^{i\theta}\colon \pi/2 \leq \theta \leq \tau_0,\ 0 < \rho < \infty\}$.

Proof. Purely on the basis of (6) we can show that Γ has an asymptotic direction at infinity. In order to prove this lemma we need the sharper result due to Brickman and Wilken [1]. They showed that Γ is asymptotic to a certain line near infinity. In our case, this line is given by

$$l\colon\ -Bt + 1/3B, \quad t > 0.$$

Since B lies in the fourth quadrant, Γ must eventually lie in Ω_1. If Γ does not lie entirely in Ω_1, it must enter by intersecting either the positive imaginary axis or the line $\tilde{l}\colon -Bt$, $0 < t < \infty$. Thus one of the two points $w(t_1) = ir_1$ or $w(t_2) = -r_2 B$ ($t_1, t_2 \neq 0$ and $r_1, r_2 > 0$) must be a point of Γ. Since Γ has the $\pi/4$-property, if $w(t_1) \in \Gamma$ then we must have

$$-\pi/2 \leq \arg w'(t_1) \leq -\pi/4,$$

while if $w(t_2) \in \Gamma$ then

$$3\pi/4 \leq \arg\,(-w'(t_2)/B) \leq \pi.$$

From these we can conclude that either

$$\operatorname{Im}\left\{\left(\frac{w'(t_1)}{w(t_1)}\right)^2\right\} \geq 0 \quad \text{or} \quad \operatorname{Im}\left\{\left(\frac{w'(t_2)}{w(t_2)}\right)^2\right\} \leq 0.$$

Using (6) and the last two inequalities, we find that either $\operatorname{Re} B \leq 0$ or $\operatorname{Im}\{1/B^2\} \leq 0$. Both of these contradict the assumption that $\operatorname{Re} B > 0$ and $\operatorname{Im} B < 0$. This proves Lemma 1.

We now integrate the Schiffer differential equation (6) for Γ and show that Γ is the image under the Koebe function of a certain arc. Let us choose a parametrization $w = w(t)$, $w'(t) \neq 0$, $w(0) = \infty$ so that the right-hand side of (6) is the constant $1/4$. Then we may integrate the resulting differential equation to obtain

$$\log \frac{\sqrt{1 + Bw} - 1}{\sqrt{1 + Bw} + 1} - \frac{2\sqrt{1 + Bw}}{Bw} = \pm\frac{it}{B} + C, \quad 0 < t < T. \tag{8}$$

From Lemma 1, we see that $\operatorname{Im}(1 + Bw) > 0$. Consequently we may choose the principal branches of the square roots and logarithm in (8). If we let t tend to zero and recall that $w(0) = \infty$, then $C = 0$. Also, since $\operatorname{Im}(1 + Bw) > 0$, we see that the left-hand side of (8) has positive imaginary part. Thus $\operatorname{Im}\{\pm i/B\} > 0$ and because B lies in the fourth quadrant, we find that the plus sign must be chosen in (8).

We can conclude (since $\operatorname{Im} Bw > 0$) that there exists a unique arc $\gamma\colon s = s(t)$, $0 < t < T$, which satisfies

$$Bw(t) = 4k(s(t)), \quad s(0) = 1, \tag{9}$$

and which lies in the upper half of the unit disk. Next, using (9), we see that $\sqrt{1+Bw}=(1+s)/(1-s)$ and from (8) we find that each point of γ satisfies

$$(10) \qquad \log s + \frac{(s^2-1)}{2s} = \frac{it}{B}, \quad 0<t<T,$$

where $|s(t)|<1$, $s(0)=1$ and $B=2A_2+\lambda$.

Lemma 2. *Suppose* $s=s(t)$ *satisfies* (10) *and* $0<t<T$. *Then* $\operatorname{Re}\{B\log s\}<0$.

Proof. Let $u(t)=\operatorname{Re}\{B\log s(t)\}$. Then $u(t)$ is a bounded continuous function with $u(0)=0$. From (10) and an easy computation we get

$$u'(t)=\operatorname{Re}\{-2ik(-s)\}=2\operatorname{Im}\{k(-s)\}.$$

Using $\operatorname{Im} s(t)>0$ and the fact that the Koebe function preserves the lower half-plane, we can conclude that $u'(t)<0$. Thus, $u(t)<0$.

Proof of Theorem 2. Since $w(t)=4k(s(t))/B$, where $s(t)$ satisfies (10), a brief calculation shows that

$$\theta'(t)=\frac{d}{dt}\{\arg w(t)\}=\operatorname{Re}\left\{\frac{1}{B}\frac{2s}{(1-s^2)}\right\}.$$

Using the identity that $\operatorname{Re}(z)=|z|^2\operatorname{Re}(1/z)$ and (10), we get

$$\theta'(t)=p\operatorname{Re}\{B\log s\} \quad \text{where } p=\left|\frac{1}{B}\frac{2s}{(1-s^2)}\right|^2>0.$$

Applying Lemma 2 yields the result.

Proof of Theorem 3. This theorem is an immediate consequence of Lemma 1 and Theorem 2.

We conclude this section by proving some results that will show that only certain special values of λ can allow a rotation of the Koebe function as a support point for (2)

Lemma 3. *If* $F(z)=z/(1-xz)^2$, $|x|=1$, *is a support point for* (2), *then* $x^2=\pm 1$.

Proof. (cf. Schober [10, pp. 83–84]). Using two very elementary variations [10], it can be shown that if g is a support point for an arbitrary continuous linear functional L, then

$$\operatorname{Im}\{L(zg'(z)-g(z))\}=0$$

and

$$L((g(z)g''(0)-g'(z)+1)+\overline{L(z^2g'(z))}=0.$$

If $F(z) = z/(1 - xz)^2$ is a support point for the particular functional (2), the above equations become

$$\operatorname{Im}\{3x^2 + \lambda x\} = 0 \quad \text{and} \quad -4x^3 - \lambda x^2 + 4\bar{x} + \bar{\lambda} = 0.$$

This last equation implies that

$$4x^2 + \lambda x - 4\bar{x}^2 - \bar{\lambda}\bar{x} = 0$$

or

$$\{x^2 + (3x^2 + \lambda x)\} - \{\bar{x}^2 + (3\bar{x}^2 + \bar{\lambda}\bar{x}\} = 0.$$

Using the fact that $3x^2 + \lambda x$ is real, we can conclude that $x^2 = \pm 1$.

LEMMA 4. *Let $F(z) = z + A_2 z^2 + A_3 z^3 + \cdots$ be a support point for* (2) *and let $B = 2A_2 + \lambda$. Then the following statements hold:*

(a) *If* $\operatorname{Re} \lambda \neq 0$ *then* $\operatorname{Re} B \neq 0$.
(b) *If* $\operatorname{Re} \lambda = 0$ *and* $0 \leq |\lambda| < 6$ *then* $\operatorname{Re} B \neq 0$.
(c) *If* $\operatorname{Re} \lambda = 0$ *and* $8 \leq |\lambda| < \infty$ *then* $\operatorname{Re} B = 0$.

Proof. (a) We first show that if $\operatorname{Re} \lambda \neq 0$ then $(\operatorname{Re} \lambda)(\operatorname{Re} A_2) \geq 0$. If this were false then $(\operatorname{Re} \lambda)(2 \operatorname{Re} A_2) < 0$, and the identity

$$(\operatorname{Re} \lambda)(2 \operatorname{Re} A_2) = \operatorname{Re}\{\lambda(A_2 + \bar{A}_2)\}$$

would imply $\operatorname{Re}(\lambda A_2) < \operatorname{Re}\{\lambda(-\bar{A}_2)\}$. Define $F^* \in S$ by

$$F^*(z) = -\overline{F(-\bar{z})} = z - \bar{A}_2 z^2 + \bar{A}_3 z^3 + \cdots.$$

Next, observe that

$$\begin{aligned} \operatorname{Re} L(F) &= \operatorname{Re}\{A_3 + \lambda A_2\} \\ &= \operatorname{Re}\{\bar{A}_3\} + \operatorname{Re}\{\lambda A_2\} \\ &< \operatorname{Re}\{\bar{A}_3\} + \operatorname{Re}\{\lambda(-\bar{A}_2)\} \\ &= \operatorname{Re} L(F^*). \end{aligned}$$

This contradicts the extremality of F. We now have our result because if $\operatorname{Re} \lambda > 0$ then $\operatorname{Re} A_2 \geq 0$, while if $\operatorname{Re} \lambda < 0$ then $\operatorname{Re} A_2 \leq 0$. Both of these imply, in particular, that $\operatorname{Re} B = \operatorname{Re}(2A_2 + \lambda) \neq 0$.

(b) Let $\lambda = i|\lambda|$, say, and suppose that $\operatorname{Re} B = 0$ for $0 \leq |\lambda| < 6$. Then since $B = 2A_2 + \lambda$, $\operatorname{Re} A_2 = 0$. Since $\lambda = i|\lambda|$, the Loewner formulas (3) and (4) allows us to conclude (after a calculation) that

$$\operatorname{Re}(A_3 + \lambda A_2) = \left|1 - 4\int_0^\infty e^{-2t} \cos^2 \theta(t)\, dt + 4u^2\right| + g(v),$$

where $u + iv = \int_0^\infty e^{-t} \kappa(t)\, dt$ and $g(v) = 2|\lambda| v - 4v^2$. For convenience, let $M = \operatorname{Re}(A_3 + \lambda A_2)$. Observe first that we get the following elementary estimate on M:

$$M \geq 3 + |\lambda|^2/6.$$

Indeed, $M \geq m(\theta)$ where $m(\theta) = \text{Re } L(k_\theta)$ and $k_\theta(z) = e^{-i\theta}k(e^{i\theta}z)$. Maximizing $m(\theta)$ when $0 \leq |\lambda| < 6$ yields the estimate. By assumption, $\text{Re } A_2 = 0$ and hence $u = 0$ and

$$M \leq 1 + \max_{|v| \leq 1} g(v).$$

If $0 \leq |\lambda| \leq 4$, then $g(v) \leq g(|\lambda|/4)$ and

$$M \leq 1 + g(|\lambda|/4) = 1 + |\lambda|^2/4.$$

If $4 \leq |\lambda| < 6$, then $g(v) \leq g(1)$ and

$$M \leq 1 + g(1) = 2|\lambda| - 3.$$

Thus, if $0 \leq |\lambda| < 6$ then

$$3 + \frac{|\lambda|^2}{6} \leq M \leq \max\left\{1 + \frac{|\lambda|^2}{4}, 2|\lambda| - 3\right\}.$$

This is clearly false. Hence $\text{Re } A_2 \neq 0$ and, consequently, $\text{Re } B \neq 0$. The case $\lambda = -i|\lambda|$ is treated similarly.

(c) Let $\lambda = -i|\lambda|$ with $8 \leq |\lambda|$ and suppose that $\text{Re } A_2 \neq 0$. As in (b),

$$M = \text{Re }(A_3 + \lambda A_2) = \left\{1 - 4\int_0^\infty e^{-2t}\cos^2\theta(t)\,dt + 4u^2\right\} + g(v).$$

Let $v = 1 - \varepsilon$, $0 < \varepsilon < 2$. Then, using the fact that $u^2 + v^2 \leq 1$, we find that

$$M \leq h(\varepsilon) \quad \text{where } h(\varepsilon) = \{2|\lambda| - 3\} + \varepsilon\{16 - 2|\lambda|\} - 8\varepsilon^2.$$

It is easy to show that for $0 < \varepsilon < 2$,

$$M \leq h(\varepsilon) < h(0) = 2|\lambda| - 3.$$

However, $\text{Re } L(-ik(iz)) = 2|\lambda| - 3$. A contradiction arises and hence $\text{Re } A_2 = 0$. Moreover, $iB > 0$ and by Theorem 1, $-ik(iz)$ is the unique extremal function. The case $\lambda = i|\lambda|$ is treated similarly.

The above results can be summarized in the following theorem:

THEOREM 4. *Let $L(g) = a_3 + \lambda a_2$ and $\lambda \in \mathbf{C}$.*

(a) *If $\lambda \geq 0 (\lambda \leq 0)$, then $k(z)(-k(-z))$ is the unique support point for L.*

(b) *If $8 \leq |\lambda| < \infty$ and $\lambda = i|\lambda|$ $(\lambda = -i|\lambda|)$, then $ik(-iz)$ $(-ik(iz))$ is the unique support point for L.*

(c) *If* Re $\lambda \neq 0$ *and* Im $\lambda \neq 0$, *or if* Re $\lambda = 0$ *and* $0 < |\lambda| < 6$, *then no rotation of the Koebe function is extremal.*

The case when $\lambda = \pm i|\lambda|$ and $6 \leq |\lambda| < 8$ is undetermined at the present time.

4. Remarks

(a) Using the method of Schaeffer and Spencer [6, Lemma XXXI], we can show that the functionals (2) generate a rich family of support points. Indeed, we show below that if $F \in S$ is a support point for $L_1(g) = a_3 + \lambda_1 a_2$ and for $L_2(g) = a_3 + \lambda_2 a_2$, then $\lambda_1 = \lambda_2$ unless F is one of four rotations of the Koebe function.

Assume that F is a support point for L_1 and L_2. Then the arc omitted by F satisfies two differential equations

$$\frac{1 + B_n w}{-w^2}\left(\frac{w'}{w}\right)^2 > 0,$$

where $B_n = 2A_2 + \lambda_n$, $n = 1, 2$. If we divide the two differential equations then

$$0 < \frac{1 + B_1 w}{1 + B_2 w}.$$

If we now let w tend to infinity we conclude that $B_1 = pB_2$, for some $p > 0$. Consequently, we get

$$0 < \frac{1 + B_1 w}{1 + pB_1 w}.$$

Hence either $p = 1$ or $B_1 w$ is real for all $w \in \Gamma$. This proves that either $\lambda_1 = \lambda_2$ or F is a rotation of the Koebe function. In the latter case Lemma 3 shows F must be one of the four rotations with $x^2 = \pm 1$.

(b) In a sense, the problem of maximizing Re $\{a_3 + \lambda a_2\}$ has long been solved, since Schaeffer and Spencer [6] and recently Haario [3] described the coefficient body V_3. Among other things, they proved that each point on the boundary of V_3 that supports a hyperplane corresponds to one or more functions in S which map the unit disk onto the complement of a single analytic arc. Indeed, such boundary points (the set of which they call K_3) correspond to support points for the functionals $L(g) = Aa_3 + Ba_2$, A, $B \in \mathbf{C}$. Our results now give additional qualitative results about functions corresponding to points in K_3. In particular, each point of K_3 corresponds to a unique function in S whose omitted arc lies in a sector and has monotonic argument.

References

1. L. Brickman and D. R. Wilken, *Support points of the set of univalent functions*, Proc. Amer. Math. Soc., vol. 42 (1974), pp. 523–528.

2. J. E. Brown, *Geometric properties of a class of support points of univalent functions*, Trans. Amer. Math. Soc., vol. 256 (1979), pp. 371–382.

3. H. Haario, *On coefficient bodies of univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, No. 22, (1978), 49 pp.

4. C. Loewner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann., vol. 89 (1923), pp. 103–121.

5. A. Pfluger, *Lineare Extremalprobleme bei schlichten Funktionen*, Ann. Acad. Sci. Fenn. Ser. AI, No. 489 (1971), 32 pp.

6. A. C. SCHAEFFER and D. C. SPENCER, *Coefficient regions for Schlicht functions*, Amer. Math. Soc. Colloq. Publ., vol. 35, Amer. Math. Soc., Providence, R.I., 1950.
7. M. SCHIFFER, *A method of variation within the family of simple functions*, Proc. London Math. Soc. (2), vol. 44 (1938), pp. 432–449.
8. ———, "Some recent developments in the theory of conformal mappings" Appendix to R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, New York, 1950.
9. ———, *Applications of variational methods in the theory of conformal mapping*, Proc. Sympos. Appl. Math., vol. 8, Amer. Math. Soc., Providence, R.I., 1958, pp. 93–113.
10. G. SCHOBER, *Univalent functions—selected topics*, Lecture Notes in Math., vol. 478, Springer-Verlag, Berlin-New York, 1975.

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