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AUTOMORPHIC FUNCTIONS WITH GAP POWER SERIES

BY

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1. Introduction

L. A. Rubel [1, p. 136] has raised the following question. "What kind of gaps can the Taylor expansion of a non-constant automorphic function have? For example, can it have Hadamard gaps? (Presumably the sharp answer would depend on the group concerned.) This is closely related to a theorem of C. Renyi that a non-constant periodic entire function cannot have more than half of its coefficients zero."

In this paper we prove some theorems which partially answer this question. Throughout, G will denote a Fuchsian group acting in the unit disc $\Delta = \{z\colon |z| < 1\}$. We will consider only groups which are of the first kind, finitely generated and possess a parabolic element. These are precisely the groups for which the quotient surface Δ/G is obtained from a compact Riemann surface by deleting a finite, positive, number of points. Clearly we must require Δ/G to be non-compact otherwise no non-constant analytic automorphic function can exist.

We suppose $f(z)$ is an analytic automorphic function with respect to a group G as above and write

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \quad \text{where } c_k \neq 0. \tag{1.1}$$

We consider a function of the non-vanishing coefficients of f defined by

$$N_0(t) = \max\{k\colon n_k < t\} \quad \text{for any } t > 0. \tag{1.2}$$

THEOREM 1. *Let G be a Fuchsian group as above and let f be an analytic automorphic function for G which is of the form* (1.1). *Then*

$$N_0(t) \neq o(\log \log t) \quad \textit{as } t \to \infty.$$

If we ask for some regularity of spacing in the gaps we can do better than Theorem 1.

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THEOREM 2. *Let G be a Fuchsian group as above and let f be an analytic automorphic function for G which is of the form* (1.1). *Then*

$$\frac{(n_{k+1} - n_k)}{n_k} \log n_k$$

remains bounded as $k \to \infty$.

We note in particular that Hadamard gaps cannot occur. If we restrict the growth of the function we may also obtain, using the method of proof of Theorem 1, theorems of the following form:

THEOREM 3. *Let G be a Fuchsian group as in Theorem* 1 *and let f be an analyric automorphic function for G which is of the form* (1.1). *Let*

$$\limsup_{r \to 1} \frac{\log M(r)}{-\log (1-r)} = \beta$$

where $M(r)$ is the maximum modulus of f on $\{|z| = r\}$ and we assume that $\beta > 1$. Let $\psi(n)$ be the number of non-vanishing coefficients of f in the interval

$$(n^{a(3/\beta)}, n^{3/\beta}) \text{ where } a = (\beta - 1)/4\beta.$$

Then $\psi(n) \neq o(\log \log n)$ as $n \to \infty$.

We shall omit the proof of Theorem 3.

With a similar restriction on the growth of the function it can be shown, using the method of proof of Theorem 2, that an automorphic function cannot have gaps of a type previously considered by the authors [4, Theorem 6].

In proving these theorems essential use is made of three properties of automorphic functions. The first property is a lower bound for the growth of the maximum modulus of such a function. Defining $n_G(r, a)$ to be the cardinality of the set $\{V \in G: |V(a)| < r\}$ and $n(r, f)$ to be the number of zeros of f in $\{|z| < r\}$, we observe that $n(r, f) \geq n_G(r, a)$ if $f(a) = 0$. Our bound on the maximum modulus follows from a lower bound on $n_G(r, a)$ obtained by Tsuji.

The second property used concerns the covering of circles $\{|z| = r\}$ by group images of a disc around the origin. For many values of r the images of such a disc cover a fixed positive proportion of the circle $\{|z| = r\}$—this is a weak form of a mixture property of geodesic flows on the quotient space and was derived in this form by Tsuji. Thus our automorphic function will be bounded by some constant k on a fixed positive proportion of a large number of circles. This idea is the essential ingredient of Theorem 1.

To prove Theorem 2 we use a third property—namely that an automorphic function is bounded on any radius which ends at a hyperbolic fixed point of the group. This is a trivial consequence of the fact that the axis of any hyperbolic transform must lift to a closed loop on the quotient space. We then show that the gap condition

$$\frac{n_{k+1} - n_k}{n_k} \log n_k \to \infty \quad \text{as } k \to \infty$$

satisfies the hypotheses of an earlier result of the authors' [4, Theorem 5] and functions with such gaps cannot be bounded on a radius.

We mention in conclusion that J. Lehner and T. Metzger have recently obtained some results on gap series representations for certain classes of automorphic forms (personal communication to the authors).

The authors thank the referee for the proof of Lemma 4 given here which is simpler than the original proof of the authors.

2. Proof of Theorem 1

Let G be a Fuchsian group as described in Section 1. If $A \subset \Delta$ is a disc containing the origin we define

$$\mu(r, A) = \{e^{i\theta}: re^{i\theta} = z \quad \text{and} \quad V(z) \in A \text{ for some } V \in G\}.$$

We say that $2\pi l(r, A)$ is the euclidean linear measure of the set $\mu(r, A)$. Then for $\varepsilon > 0$ we write

$$A(\varepsilon) = \{r, 0 < r < 1: l(r, A) \geq \varepsilon\}.$$

LEMMA 1. *For some* $\varepsilon > 0$, *depending on* A *and* G, $\int_{A(\varepsilon)} dr/(1-r)$ *is unbounded.*

Proof. From a result of Tsuji [5, p. 555] we note that for some $\delta > 0$, depending on A and G,

$$\int_0^R \frac{l(r, A)}{1-r}\, dr > \delta \log \frac{1}{1-R} \quad \text{for } R > R_0 \text{ say.}$$

We choose $\varepsilon = \delta/2$ and note that

$$\int_0^R \frac{l(r, A)}{1-r}\, dr = \int_{[0,R] \cap A(\varepsilon)} \frac{l(r, A)}{1-r}\, dr + \int_{[0,R]\setminus A(\varepsilon)} \frac{l(r, A)}{1-r}\, dr$$

$$< \int_{[0,R] \cap A(\varepsilon)} \frac{dr}{1-r} + \varepsilon \log \frac{1}{1-R}.$$

Thus, for $R > R_0$,

$$\int_{[0,R] \cap A(\varepsilon)} \frac{dr}{1-r} > \frac{\delta}{2} \log \left(\frac{1}{1-R}\right)$$

and the result follows.

LEMMA 2. *Let* G, f *be given as in the statement of Theorem* 1 *and let* $M(r)$ *denote the maximum modulus of* f *on the circle* $\{|z| = r\}$. *There exist positive constants* α, K *so that*

(i) $$\frac{\log M(r)}{-\log(1-r)} \geq \alpha \quad \text{for } r > r_0 \text{ say;}$$

and, outside a set E_0 of r such that $\int_{E_0} dr/(1-r) < K$ *we have*

(ii) $M(r + (1-r)/eM(r)^{1/\alpha}) < 2M(r)$ *and*

(iii) $\sum_{n_k \geq M(r)^{3/\alpha}} |c_k| r^{n_k} \leq 1$.

Proof. Without loss of generality we may assume that we can find an $a \in \Delta$ so that $f(a) = 0$. Clearly $n(r, f) \geq n_G(r, a)$ for all r, $0 < r < 1$. Tsuji has shown [5, p. 518] that there is a positive constant A, depending on G and a, so that

$$n_G(r, a) \geq A/(1-r) \quad \text{for } r \geq r_1 \text{ say.}$$

Thus $n(r, f) \geq A/(1-r)$ for $r \geq r_1$ and part (i) follows trivially from this.

Inequality (ii) is a result of Hayman [3, p. 38] applied to the function $M(r)^{1/\alpha}$. To prove (iii), for each r for which (i) and (ii) are valid, let

$$\bar{r} = r + (1-r)[eM(r)^{1/\alpha}]^{-1}$$

and for $|z| = r$ let

$$R(z) = \sum |c_k| z^{n_k}$$

where the summation is taken over all n_k such that $n_k \geq M(r)^{3/\alpha}$. From Cauchy's estimate we have, for each k,

$$|c_k| r^{nk} \leq M(\bar{r})(r/\bar{r})^{n_k},$$

and so

$$R(r) \leq M(\bar{r})(r/\bar{r})^{M(r)^{3/\alpha}} \bar{r}(\bar{r} - r)^{-1}.$$

For $|z| = r$ we have

$$\log |R(z)| \leq \log M(\bar{r}) + M(r)^{3/\alpha} \log (r/\bar{r}) + \log \left(\frac{\bar{r}}{\bar{r} - r}\right).$$

Using (i), (ii) and the definition of $\bar{r}$ we see that

$$\log |R(z)| \leq (1 + 2/\alpha) \log M(r) + C_1 - \frac{M(r)^{3/\alpha}}{6} M(r)^{-2/\alpha}$$

for all admissible r close enough to 1 where C_1 is a constant. Thus, for such r,

$$\log |R(z)| < 0$$

and the proof is complete.

Our next lemma is due to Gaier [2].

LEMMA 3. Let γ be a closed subset of $\partial\Delta$ of measure $2\pi\beta$ where $0 < \beta < 1$. If P is a polynomial with N terms then

$$\max_{z \in \partial\Delta} |P(z)| \leq \exp (2\beta^{-N}) \max_{z \in \gamma} |P(z)|.$$

We now prove Theorem 1. Choose a disc A as in Lemma 1, let $\varepsilon > 0$ and $A(\varepsilon)$ be as in Lemma 1. Note that f is bounded on A and, since it is automorphic with respect to G, we have, for some Q,

$$|f(z)| \leq Q \quad \text{for all } z \text{ in } G(A). \tag{2.1}$$

By Lemma 1 we find a sequence of numbers s approaching 1 from below which lie in the set $A(\varepsilon)$ and outside the set E_0 of Lemma 2. We write, for $|z| = s$,

$$\begin{aligned} f(z) &= \sum_{n_k < M(s)^{3/\alpha}} c_k z^{n_k} + \sum_{n_k \geq M(s)^{3/\alpha}} c_k z^{n_k} \\ &= P_s(z) + g_s(z). \end{aligned} \tag{2.2}$$

Note, from Lemma 2, that

$$|g_s(z)| \leq 1 \tag{2.3}$$

for all s in our sequence and $|z| = s$. We now apply Lemma 3 to the polynomial $P_s(sz)$ taking γ to be the set $\mu(s, A)$. Thus

$$\max_{|z|=s} |P_s(z)| \leq \exp\{2\varepsilon^{-N_0(M(s)^{3/\alpha})}\} \max_{z \in \lambda} |P_s(z)|, \tag{2.4}$$

where $N_0(t)$ is defined in (1.2) and $\lambda = \{z\colon |z| = r \text{ and } V(z) \in A \text{ for some } V \in G\}$. From (2.2), (2.3) and (2.4) it follows that

$$M(s) \leq \exp\{2\varepsilon^{-N_0(M(s)^{3/\alpha})}\}\{Q + 1\} + 1.$$

For some constant k, and for all s in our sequence,

$$M(s) \leq k \exp(2\varepsilon^{-N_0(M(s)^{3/\alpha})})$$

from which

$$\log M(s) \leq \log k + 2\varepsilon^{-N_0(M(s)^{3/\alpha})}. \tag{2.5}$$

If $N_0(t) = o(\log\log t)$ then, for s large enough on our sequence,

$$N_0(M(s)^{3/\alpha}) < \delta \log\log(M(s)^{3/\alpha}) \quad \text{where } \delta = [2\log(1/\varepsilon)]^{-1}.$$

For such s it follows from (2.5) that

$$\log M(s) \leq \log k + 2[\log(M(s)^{3/\alpha})]^{1/2}$$

and we have a contradiction.

3. Proof of Theorem 2

We need two lemmas.

LEMMA 4. *Let n_k be an increasing sequence of positive integers satisfying*

$$\frac{(n_{k+1} - n_k)}{n_k} \log n_k \to \infty \quad \text{as } k \to \infty.$$

Then $k^{-1/2} \log n_k \to \infty$ as $k \to \infty$.

Proof. Let $x_k = \log n_k$. Then for each large number y, there is an integer m such that

$$\exp\{x_{k+1} - x_k\} \geq y/x_k \quad \text{for } k \geq m.$$

Hence there is an integer p such that

$$x_{k+1} - x_k \geq y/(2x_k) \quad \text{for } k \geq p.$$

Thus

$$x_{k+1}^2 \geq x_k^2 + y \quad \text{for } k \geq p,$$

and so, for $k \geq p$,

$$x_k^2 \geq (k-p)y + x_p^2.$$

Since y can be chosen arbitrarily large, the result follows.

LEMMA 5. *Let n_k be an increasing sequence of positive integers satisfying*

$$\frac{(n_{k+1} - n_k)}{n_k} \log n_k \to \infty \quad \text{as } k \to \infty.$$

Then

$$\sum_{\substack{k=0\\k\neq\rho}}^{\infty} \frac{n_\rho^2}{(n_k - n_\rho)^2} = o((\log n_\rho)^2) \quad \text{as } \rho \to \infty.$$

Proof. We write

$$h(\rho) = \min_{n_i \geq n_\rho/2} \frac{(n_{i+1} - n_i)}{n_i} \log n_i$$

and note that $h(\rho) \to \infty$ as $\rho \to \infty$. For $k > \rho$ we have

$$\begin{aligned} n_k - n_\rho &= (n_k - n_{k-1}) + (n_{k-1} - n_{k-2}) + \cdots + (n_{\rho+1} - n_\rho) \\ &\geq h(\rho)(k-\rho)\frac{n_\rho}{\log n_\rho}. \end{aligned}$$

Thus

$$\frac{n_\rho^2}{(n_k - n_\rho)^2} \leq \frac{(\log n_\rho)^2}{(k-\rho)^2 h(\rho)^2}. \tag{3.7}$$

Now we consider $k < \rho$ and observe that if $n_{\rho-k} \geq n_\rho/2$ then

$$(n_\rho - n_{\rho-k})^{-1} < h(\rho)^{-1}\left[\frac{n_{\rho-k}}{\log n_{\rho-k}} + \cdots + \frac{n_{\rho-1}}{\log n_{\rho-1}}\right]^{-1} \tag{3.8}$$

and so

$$(n_\rho - n_{\rho-k})^{-2} < \frac{4(\log n_\rho)^2}{k^2 h(\rho)^2 n_\rho^2}. \tag{3.9}$$

If $n_{\rho-k} < \rho/2$ then

$$(n_\rho - n_{\rho-k})^2 < 4/n_\rho^2. \tag{3.10}$$

Combining (3.9) and (3.10) we see that

$$\sum_{k=0}^{\rho-1} \frac{n_\rho^2}{(n_k - n_\rho)^2} \le 4(\rho - 1) + \frac{4(\log n_\rho)^2}{h(\rho)^2} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{(\log n_\rho)^2}{h(\rho)^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which, in view of lemma 4 and the fact that $h(\rho) \to \infty$ as $\rho \to \infty$, completes the proof of Lemma 5.

To prove Theorem 2 we suppose that G, f are as given in the theorem and

$$\frac{(n_{k+1} - n_k)}{n_k} \log n_k \to \infty \quad \text{as } k \to \infty.$$

In view of Lemma 5 we may deduce from a result of the authors' [4, p. 106] that for any θ in $[0, 2\pi]$ there is a sequence of numbers s tending to 1 for which

$$\log \left\{ \sup_{0 \le t \le 1} |f(ste^{i\theta})| \right\} \ge \tfrac{1}{2} \log M(s). \tag{3.11}$$

We take $e^{i\theta}$ to be a hyperbolic fixed point for G and, since f is automorphic, the left hand side of (3.11) remains bounded as $s \to 1$. This contradiction completes the proof of Theorem 2.

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