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AUTOMORPHIC FORMS AND SINGULARITIES WITH C*-ACTION

BY

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Introduction

Suppose G is a finitely generated Fuchsian group of the first kind and $A(G)$ is the (graded) ring of automorphic forms relative to G (cf., 1.0). Then we shall see (4.4) that $A(G)$ is a finitely generated algebra over $\mathbf{C}$. For example, if G is the modular group, then $A(G)$ is a polynomial ring generated by a form of weight 2 and a form of weight 3, g_2 and g_3. That is $A(G) \approx \mathbf{C}[g_2, g_3]$. Any form of weight k is a linear combination of monomials of weight k in g_2 and g_3. For example, any form of weight 6 is a linear combination of g_2^3 and g_3^2. Of course for most groups $A(G)$ requires many generators. In fact we showed in [21] that there are only three groups for which $A(G)$ is generated by two elements and classified all groups for which $A(G)$ is generated by three elements. Knopp [8] obtained the same results independently using analytic methods. Those groups G for which $A(G)$ is generated by three elements and H_+/G is compact were previously classified by Doglacev [4] and Scherbak [18]. Milnor [12] studied the structure of certain rings of automorphic forms with fractional weights.

In this paper we determine a minimal set of generators for $A(G)$, for any group G as above (3.3). We then apply this to classify all groups for which $A(G)$ is generated by four elements (6.1). We also determine the number and degrees of the generators of the ideal of relations when $A(G)$ has four generators. We note that the ideal of relations always has either two or three generators in this case.

The paper is organized as follows. In Section 1 we recall some "classical" theorems of Castelnuovo and Mumford. In Section 2 we treat the case of a covering group G. The main theorem is stated and proved in Section 3 while the proof of several lemmas is left for Section 4. The relation between the algebra of automorphic forms and singularities of complex surfaces with $\mathbf{C}^*$-action is given in Section 5. Finally, we classify all groups with $A(G)$ generated by four elements in Section 6, and study the structure of $A(G)$ in that case.

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1. Two "classical" theorems

1.0. Suppose G is a finitely generated Fuchsian group of the first kind. Let $A(k)$ be the vector space of entire automorphic forms of weight k and define

$$A(G) = \bigoplus_{k=0}^{\infty} A(k)$$

the graded ring of automorphic forms. Note that $A(0) = \mathbf{C}$.

Now G acts on the upper half plane H_+ and the orbit space $X_0 = H_+/G$ is a Riemann surface with a finite number of punctures. We let X denote the compact Riemann surface obtained by filling in the punctures. Then $X - X_0$ is a finite number of points which we denote by $q_1, \ldots, q_s$. We shall call these points *parabolic* points. The orbit map $\pi: H_+ \to X_0$ is branched over a finite number of points $p_1, \ldots, p_r \in X_0$ which we shall call *elliptic* points. For each i we let $e_i \geq 2$ be the ramification degree of π at p_i.

The *signature* of G is the set of invariants $\langle g; s; e_1, \ldots, e_r \rangle$ where g is genus X and, by convention, $e_1 \leq e_2 \leq \cdots \leq e_r$. The signature contains all the topological information about G. A signature is said to be *admissible* if it is the signature of some G as above. A necessary and sufficient condition for admissibility [9] is that

$$2g - 2 + s + \sum_{i=1}^{r} \left(1 - \frac{1}{e_i}\right) > 0. \tag{1.1}$$

If D is a divisor on X we use $|D|$ to denote the linear system of effective divisors D' such that $D' \sim D$. We use $L(D)$ to denote the set of function f so that $(f) \geq -D$.

We begin by stating two classical theorems. These will be used later to show that certain specified elements generate $A(G)$.

Recall that by the Riemann-Roch theorem, if degree $D \geq 2g - 1$, then

$$\dim L(D) = \text{degree } D + 1 - g.$$

THEOREM 1.1 (Castelnuovo's lemma). *Suppose $|D_1|$ has no base points and either*

(a) degree $D_2 \geq$ degree $D_1 + 2g - 1$ *or*
(b) degree $(D_2 - D_1) = 2g - 2$ and $D_2 - D_1 \neq K$ *or*
(c) $D_2 - D_1 = K$ and degree $D_1 \geq \sup(2g + 1, 2)$.

Then

$$L(D_1) \otimes L(D_2) \to L(D_1 + D_2), \quad f \otimes g \mapsto fg,$$

is onto

Proof. Saint-Donat [16, 2.11] proves that the conclusion of the theorem holds if $|D_1|$ has no base points and $H^1(\mathcal{O}(D_2 - D_1)) = \{0\}$. This is guaranteed by the hypotheses (a) and (b). Now to prove (c) we note that there is a commutative diagram

$$\begin{tikzcd} L(D_1 - p) \otimes L(D_1 + K) \arrow[r] \arrow[d] & L(D_1 + D_2 - p) = L(2D_1 + K - p) \arrow[d] \\ L(D_1) \otimes L(D_1 + K) \arrow[r, "\phi"] & L(D_1 + D_2) \quad = L(2D_1 + K) \end{tikzcd}$$

for any point $p \in X$. For simplicity choose p so that it is not in the support of any of the divisors above. Now by (a) the top horizontal map is onto since degree $D_1 - p \geq 2g$ and therefore $D_1 - p$ has no base points. There exist

$$f_1 \in L(D_1) - L(D_1 - p) \quad \text{and} \quad f_2 \in L(D_1 + K) - L(D_1 + K - p)$$

since D_1 and $D_1 + K$ have degree $\geq 2g$. Then $f = \phi(f_1 \otimes f_2)$ lies in $L(2D_1 + K) - L(2D_1 + K - p)$. But

$$L(2D_1 + K) \supset L(2D_1 + K - p)$$

and

$$\dim L(2D_1 + K) = \dim L(2D_1 + K - p) + 1.$$

Hence we get the desired equality. ▮

THEOREM 1.2 (Mumford [13]). *If* degree $D \geq 2g + 1$ *and* $r, s \geq 1$ *then the product map* $L(rD) \otimes L(sD) \to L((r + s)D)$ *is onto.*

Proof. By [13] the map $S^*L(D) \to \bigoplus_{n \geq 0} L(nD)$ where S^* denotes the symmetric algebra functor, is onto. Now we have a commutative diagram

$$\begin{tikzcd} S^rL(D) \otimes S^sL(D) \arrow[r] \arrow[d] & S^{r+s}L(D) \arrow[d] \\ L(rD) \otimes L(sD) \arrow[r] & L((r + s)D) \end{tikzcd}$$

with the top map and the vertical maps being onto. Hence the bottom map is onto. ▮

The reason that assertions about $L(D)$ are relevant to studying $A(G)$ is that we can show that $A(k)$ is isomorphic to $L(D)$ for a suitable divisor D.

DEFINITION 1.3. If we are as above, and k is an integer, define a divisor on X by

$$D^{(k)} = kK + k(q_1 + \cdots + q_s) + \sum_{i=1}^{r} [k(1 - 1/e_i)]p_i$$

where $[x]$ denotes the greatest integer $\leq x$. Define

$$L(k) = L(D^{(k)}) \quad \text{and} \quad L(G) = \bigoplus_{k=0}^{\infty} L(k).$$

Then $L(G)$ has a natural structure of a graded algebra since $f \in L(k)$, $g \in L(k')$ implies $fg \in L(k + k')$.

PROPOSITION 1.4. *There are natural isomorphisms of vector spaces* $\phi_k\colon A(k) \to L(k)$ *which induce an isomorphism of graded algebras* $\phi\colon A(G) \to L(G)$.

Proof. The idea for this appears in Gunning [7]. Let h be a meromorphic 1-form for G and let $k(X)$ denote the field of meromorphic functions on X. Then we can define $\phi_k\colon A_k \to k(X)$ by $\phi_k(f) = f/h^k$. By Gunning [7, II, Section 8] ϕ_k maps A_k isomorphically onto $L(k)$. One easily checks that the induced map ϕ is an algebra homomorphism. ▮

Pinkham has generalized the above result [15, Section 5].

2. Covering groups

We shall first consider the case when G is the covering group of X, i.e., signature $G = \langle g \rangle$, $g \geq 2$. In this case $A(G)$ is isomorphic to the canonical ring of X.

PROPOSITION 2.1. *Suppose G is the covering group of X. If X is non-hyperelliptic, then $A(G)$ is generated by $A(1)$. If X is hyperelliptic and $g \geq 3$, then $A(G)$ is generated by g forms in $A(1)$ and g-2 forms in $A(2)$. If $g = 2$ then $A(G)$ is generated by two forms in $A(1)$ and one form in $A(3)$.*

Proof. For X non-hyperelliptic the result is due to Max Noether. For a proof, see [16, 2.10]. Now suppose X is hyperelliptic of genus $g \geq 3$. First recall that $A(k) = L(kK)$ and by Theorem 1.1,

$$L(K) \otimes L((n-1)K) \to L(nK)$$

is onto for $n \geq 4$. Thus $A(G)$ is generated by forms of degree 1, 2 and 3. We can write X in the form $y^2 = (z - \varepsilon_1) \cdots (z - \varepsilon_{2g+1})$ where the ε_i are distinct, non-zero complex numbers. Now $A(1)$ is just the space of holomorphic 1-forms on X. A basis for the space of 1-forms is given by

$$\frac{dz}{y}, \frac{z\,dz}{y}, \ldots, \frac{z^{g-1}\,dz}{y}$$

(cf. [19, Section 10.10, p. 293]). A basis for the 2-forms is

$$\frac{dz^2}{y^2}, \ldots, \frac{z^{2g-2}\,dz^2}{y^2}$$

$$\frac{dz^2}{y}, \ldots, \frac{z^{g-3}\,dz^2}{y}, \; g \geq 3,$$

and a basis for the 3-forms is

$$\frac{z^i\, dz^3}{y^3}, \frac{z^j\, dz^3}{y^2},$$

where $0 \le i \le 3g-3$, $0 \le j \le 2g-4$. One can easily see that in this case

$$A(1) \otimes A(2) \to A(3)$$

is onto. Moreover, a 2-form is in the image of $A(1) \otimes A(1)$ if and only if it is a linear combination of the $(z^i\, dz^2)/y^2$.

Finally, the case $g = 2$ was treated in [21, Theorem 4.6].

3. Statement of the main theorem

We now turn to the general case. Let $A = A(G)$. Then A is a graded algebra over the field of complex numbers $\mathbf{C}$. Let $\mathbf{m} = \bigoplus_{n=1}^{\infty} A(i)$. Then $\mathbf{m}$ is a maximal ideal of A. A set of elements $x_1, \ldots, x_d \in \mathbf{m}$ form a set of algebra generators if and only if the residue classes $\bar{x}_1, \ldots, \bar{x}_d \in \mathbf{m}/\mathbf{m}^2$ form a basis of $\mathbf{m}/\mathbf{m}^2$ as a $\mathbf{C}$-vector space.

DEFINITION 3.1. The embedding dimension of A, e.d. (A), is the dimension of $\mathbf{m}/\mathbf{m}^2$ as a $\mathbf{C}$-vector space.

This is equal to the number of elements in any minimal set of algebra generators for A.

Now $\mathbf{m}$ is a homogeneous ideal, hence $\mathbf{m}/\mathbf{m}^2$ is a *graded* vector space, i.e., $\mathbf{m}/\mathbf{m}^2 = \bigoplus_{i=1}^{\infty} (\mathbf{m}/\mathbf{m}^2)_i$ as a vector space, where $(\mathbf{m}/\mathbf{m}^2)_i$ is the subspace of $\mathbf{m}/\mathbf{m}^2$ generated by forms of degree i.

DEFINITION 3.2. The *Poincaré generating polynomial* of A is defined to be $g_A(t) = \sum_{i=1}^{\infty} a_i t^i$ where $a_i = \dim\, (\mathbf{m}/\mathbf{m}^2)_i$.

Thus the coefficient of t^i in g_A is just the number of generators of A in degree i. Now we can state our main theorem.

THEOREM 3.3. *Suppose G is a group with signature*

$$\langle g; s; e_1, \ldots, e_r \rangle.$$

Let $e = e_1 + \cdots + e_r$. Then

$$g_A(t) = f(t) + \sum_{i=1}^{r} (t^2 + \cdots + t^{e_i}) \tag{3.1}$$

where $f(t)$ is given in the table below. Note that e.d. $(A) = e - r + f(1)$.

signature	$f(t)$
$s \geq 3$ *or* $g = 0$ *and* $s = 2$	$(g + s - 1)t$
$s = 2$, $g \geq 2$ *and* $l(q_1 + q_2) = 1$	$(g + 1)t + t^2$
$s = 2$, $g \geq 1$ *and* $l(q_1 + q_2) = 2$	$(g + 1)t + gt^2$
$s = 1$, $g \geq 3$, X *non-hyperelliptic*	$gt + 2t^2 + t^3$
$s = 1$, $g \geq 1$, X *hyperelliptic and* $g + r \neq 1$	$gt + gt^2 + t^3$
$s = 1$, $g = 0$, $r \geq 2$	$-t^2 + (r - 2)t^3$
$s = 0$, $g \geq 3$, X *non-hyperelliptic*	gt
$s = 0$, $g \geq 2$, $g + r \geq 3$ *and* X *hyperelliptic*	$gt + (g - 2)t^2$
$s = 0$, $g = 1$ *and* $e - r \geq 3$	$t - t^2$
$s = 0$, $g = 0$, $r \geq 4$, $e \geq 11$	$-3t^2 + (r - 5)t^3$
$s = 0$, $g = 0$, $r = 3$, $e_i \geq 3$ *for all* i, $e \geq 12$	$-3t^2 - 2t^3 - t^4$
$s = 0$, $g = 0$, $r = 3$, $e_1 = 2$, $e_2, e_3 \geq 4$, $e \geq 13$	$-3t^2 - 2t^3 - t^4 - t^5$
$s = 0$, $g = 0$, $r = 3$, $e_1 = 2$, $e_2 = 3$, $e_3 \geq 9$	$-3t^2 - 2t^3 - t^4 - t^5 - t^7$

The finite number of signatures which do not appear above all have A generated by 2 or 3 elements. The generators and relations for these rings are listed in [21].

Proof. Recall that $D^{(k)} = k(K + q_1 + \cdots + q_s) + \sum [k(1 - 1/e_i)]p_i$ and, by (1.4), $A(k)$ is isomorphic to $L(D^{(k)})$. The subspace of $A(k)$ generated by forms of degree less than k is

$$B(k) = \sum_{i=1}^{k-1} \phi(L(D^{(i)}) \otimes L(D^{(k-i)}))$$

where ϕ denotes the product map $\phi(f \otimes g) = fg$ and $\sum$ denotes the sum (not necessarily direct) of sub-vector spaces of $L(D^{(k)})$. Of course $B(k)$ is just $(\mathbf{m}^2)_k$. To find generators for A we just construct, for each k, elements of $A(k)$ whose residues form a basis for $A(k)/B(k)$. Thus the coefficient of t^i in $g_A(t)$ is the dimension of $A(k)/B(k)$.

Let $g_A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $f(t) = \sum_{k=0}^{\infty} b_k t^k$. Recall that the e_i are in non-decreasing order. For any k define

$$\alpha_k = \begin{cases} 1 & \text{if } r = 0, \\ \text{the smallest } n \text{ such that } e_n \geq k & \text{if } k \leq e_r, \\ r + 1 & \text{if } k > e_r. \end{cases} \tag{3.2}$$

Then the conclusion of the theorem is equivalent to

$$a_k = b_k - \alpha_k + r + 1 \tag{3.3}$$

for all k.

Step I. We first verify the assertion for $k = 1, 2, 3$. If $k = 1$ then $\dim A(1)/B(1) = \dim A(1) = g + s - 1$ if $s > 0$, and equals g if $s = 0$. This shows that the coefficient of t is correct in the statement of the theorem. We calculate a_2 and a_3 on a case by case basis.

Suppose $s \geq 3$ or $g = 0$ and $s = 2$. Then, by Theorem 1.2, $A(1) \otimes A(1)$ maps onto $L(2(K + q_1 + \cdots + q_s))$, since

$$\text{degree } K + q_1 + \cdots + q_s \geq 2g + 1.$$

By Riemann-Roch, $a_2 = \dim A(2)/B(2) = r$, which is the desired result. Now

$$B(3) = L(3(K + q_1 + \cdots + q_s) + p_1 + \cdots + p_r).$$

By Riemann-Roch, $\dim A(3)/B(3) = r - \alpha_3 + 1$, the desired result.

Suppose $s = 2$, $g \geq 1$.

LEMMA 3.4. *If $s = 2$ and $g \geq 1$ then*

$$\dim A(2)/B(2) = \begin{cases} 1 & \text{if } \dim L(q_1 + q_2) = 1, \\ g & \text{if } \dim L(q_1 + q_2) = 2. \end{cases}$$

Note that $\dim L(q_1 + q_2) = 2$ *implies X is hyperelliptic.*

This is proven in Section 4 and gives the desired result. Now by Theorem 1.1,

$$B(3) = L(3(K + q_1 + q_2) + p_1 + \cdots + p_r)$$

and by Riemann-Roch, $\dim A(3)/B(3) = r - \alpha_3 + 1$.

Suppose $s = 1$ and $g > 0$. Then $A(1) = L(K + q_1) = L(K)$ and thus $B(2) = \phi(L(K) \otimes L(K))$. By Proposition 2.1,

$$\dim L(2K)/\phi(L(K) \otimes L(K)) = \begin{cases} 0, & X \textit{ non-hyperelliptic or } g = 1, \\ g - 2, & X \textit{ hyperelliptic, } g \geq 2. \end{cases} \tag{3.4}$$

Now

$$\dim A(2)/L(2K) = \dim L(2K + 2q_1 + p_1 + \cdots + p_r) - \dim L(2K),$$

which is $r + 2$ if $g > 1$, and $r + 1$ if $g = 1$. Hence

$$\dim A(2)/B(2) = \begin{cases} r + 2, & X \textit{ non-hyperelliptic} \\ g + r, & X \textit{ hyperelliptic.} \end{cases}$$

Now

$$\begin{aligned} B(3) &= \phi(L(K + q_1) \otimes L(2K + 2q_1 + p_1 + \cdots + p_r)) \\ &= L(3K + 2q_1 + p_1 + \cdots + p_r). \end{aligned}$$

by Theorem 1.1 and the fact that $L(K + q_1) = L(K)$. By Riemann-Roch, $\dim A(3)/B(3) = r - \alpha_3 + 2$, the desired result.

Suppose $s = 1$, $g = 0$. Then one can easily see that $A(1) = \{0\}$ and hence that $B(2) = B(3) = \{0\}$. The result follows from $\dim A(2) = r - 1$ and $\dim A(3) = 2r - \alpha_3 - 2$.

Suppose $s = 0$, $g \geq 2$. Then $A(1) = L(K)$ and by (3.4) and Riemann-Roch,

$$\dim A(2)/B(2) = \begin{cases} r & X \textit{ non-hyperelliptic}, \\ r + g - 2, & X \textit{ hyperelliptic} \end{cases}$$

If $r > 0$ then by Theorem (1.1), $B(3) = L(3K + p_1 + \cdots + p_r)$ and, by Riemann-Roch, $\dim A(3)/B(3) = r - \alpha_3 + 1$.

Suppose $s = 0$, $g = 1$. Here $A(1) = L(0)$ which consists of constants, and $A(2) = L(p_1 + \cdots + p_r)$. Thus $\dim A(2)/B(2) = r - 1$. Now $B(3) = A(2)$, hence $\dim A(3)/B(3) = r - \alpha_3 + 1$.

Suppose now that $s = g = 0$. Let p_0 be a point of X. Then $K = -2p_0$ and $A(1) = L(-2p_0) = \{0\}$. Hence $B(2) = B(3) = \{0\}$. Thus $a_2 = \dim A(2)$ and $a_3 = \dim A(3)$. Using Riemann-Roch we calculate that $\dim A(i) = r - 2i - \alpha_i + 1$, for $i \geq 2$. This gives the desired result.

Step II. We now verify the theorem for $k \geq 4$ and $2(g + s) + r \geq 5$.

DEFINITION. If $D_i = \sum_j n_{ij} p_j$ are divisors on X then

$$\sup (\{D_i\}) = \sum_j \sup_i \{n_{ij}\} p_j.$$

Note that

$$B(k) \subset \sum_{i=1}^{k-1} L(D^{(i)} + D^{(k-i)}) \subset L\left(\sup_{i=1,\ldots,k-1} \{D^{(i)} + D^{(k-i)}\}\right)$$
$$\subset L(D^{(k)} - p_{\alpha_k} - \cdots - p_r) \quad \text{for all } k \geq 2. \tag{3.5}$$

We shall use two lemmas here whose proof is postponed to Section 4.

LEMMA 3.5. *If $D_0, D_1, \ldots, D_k$ are divisors on X, degree $D_0 \geq 2g - 1$ and $D_i \geq 0$ for $i \geq 1$, then*

$$\sum_{i=1}^{k} L(D_0 + D_i) = L(D_0 + \sup \{D_i\}) = L\left(\sup_i \{D_0 + D_i\}\right)$$

where $\sum$ denotes the sum of subspaces of $L(D_0 + \sup \{D_i\})$.

LEMMA 3.6. *If $k \geq 4$ then*

$$\sup_{k=2,\ldots,k-2} \{D^{(i)} + D^{(k-i)}\} = D^{(k)} - p_{\alpha_k} - \cdots - p_r.$$

COROLLARY TO LEMMA 3.6. If $k \geq 7$ then

$$\sup_{i=3,\ldots,k-3} \{D^{(i)} + D^{(k-1)}\} = D^k - p_{\alpha_k} - \cdots - p_r.$$

Continuing with the proof of the theorem we recall that by (3.5),

$$B(k) \subset L(D^{(k)} - p_{\alpha_k} - \cdots - p_r). \tag{3.6}$$

On the other hand, $B(k) \supset \sum_{i=2}^{k-2} \phi(L(D^{(i)}) \otimes L(D^{(k-i)}))$ and Theorem 1.2 implies that

$$\phi(L(D^{(i)}) \otimes L(D^{(k-i)})) = L(D^{(i)} + D^{(k-i)}) \tag{3.7}$$

since degree $D^{(i)} \geq i(2g-2) + is + r \geq 2g+1$ whenever $i \geq 2$ and $2(g+s) + r \geq 5$. Then

$$\begin{aligned} B(k) &\supset \sum_{i=2}^{k-2} L(D^{(i)} + D^{(k-i)}) \\ &= L\left(\sup_{i=2,\ldots,k-2} \{D^{(i)} + D^{(k-i)}\}\right) \quad \text{(by Lemma 3.5)} \\ &= L(D^{(k)} - p_{\alpha_k} - \cdots - p_r) \quad \text{(by Lemma 3.6).} \end{aligned}$$

Hence we have equality in (3.6).

$$\text{Degree } D^{(k)} - p_{\alpha_k} - \cdots - p_r \geq 8g - 8 + 4s + r \geq 2g - 1$$

(using the fact (see 1.0) that $2g - 2 + s + \sum_{i=1}^{r} (1 - 1/e_i) > 0$) hence by Riemann-Roch,

$$\begin{aligned} \dim A(k)/B(k) &= \dim L(D^{(k)}) - \dim L(D^k - p_{\alpha_k} - \cdots - p_r) \\ &= r - \alpha_k + 1. \end{aligned}$$

This is the desired result in this case.

Step III. We now must consider the cases where $k \geq 4$ and $2(g+s) + r \leq 4$.

The signatures $\{2; 0\}$ and $\{1; 1\}$ are not included in the statement of the theorem. In these cases A is generated by 3 elements and is discussed in [21]. The signature $\{0; 2\}$ is not admissible.

Suppose $g = 1$, $s = 0$, $r = 2$. Then degree $D^{(2)} = 2$ and degree $D^{(i)} \geq 3$ for $i \geq 3$. By Theorems 1.1 and 1.2 we have equality (3.7) for $k \geq 5$ and as above we get $\dim A(k)/B(k) = r - \alpha_k + 1$ for $k \geq 5$. If $k = 4$ we do not have equality in (3.7). However it is sufficient to show that $B(4) = L(2p_1 + 2p_2)$. In what follows we let $\langle f_1, \ldots, f_s \rangle$ denote the vector space generated by the f_i and $(f)_\infty$ the divisor of poles of f. Now

$$A(1) = L(0) = \langle 1 \rangle, \quad A(2) = L(p_1 + p_2) = \langle 1, f \rangle,$$
$$A(3) \supset L(p_1 + 2p_2) = \langle 1, f, g \rangle$$

(since $e_2 \geq 3$) where $(f)_\infty = p_1 + p_2$ and $(g)_\infty = 2p_2$. Then

$$B(4) \supset \langle 1, f, g, f^2 \rangle = L(2p_1 + 2p_2).$$

The other inclusion always holds, so we get the desired equality.

Suppose $g = 1$, $s = 0$, $r = 1$. If $i \geq 3$ then degree $D^{(i)} \geq 2g + 1 = 3$. Thus by

Theorem 1.2 we have equality in (3.7) for $k \geq 6$. Now

$$
\begin{aligned}
B(k) &\supset \sum_{i=3}^{k-3} \phi(L(D^{(i)}) \otimes L(D^{(k-i)})) \\
&= \sum_{i=3}^{k-3} L(D^{(i)} + D^{(k-i)}) \quad \text{(by (3.7))} \\
&= L\left(\sup_{i=3,\ldots,k-3} \{D^{(i)} + D^{(k-i)}\} \right) \quad \text{(by Lemma 3.5)} \\
&= L(D^{(k)} - p_{\alpha_k} - \cdots - p_r),\ k \geq 7 \quad \text{(by the corollary to Lemma 3.6)}
\end{aligned}
$$

Thus $\dim A(k)/B(k) = r - \alpha_k + 1$, $k \geq 7$ as above. Now, calculating these groups explicitly for $k \leq 6$, we let $\mathscr{P} \in L(2p_1)$ so that $(\mathscr{P})_\infty = 2p_1$ and $(\mathscr{P}')_\infty = 3p_1$ (the Weierstrass $\mathscr{P}$-function). Then $A(1) = L(0) = \langle 1 \rangle$, $A(2) = L(p_1) = \langle 1 \rangle$, $A(3) = L(2p_1) = \langle 1, \mathscr{P} \rangle$, $A(4) = L(3p_1) = \langle 1, \mathscr{P}, \mathscr{P}' \rangle$, since $e_1 \geq 4$. Then it follows that $B(4) = L(2p_1)$, $B(5) = L(3p_1)$ and $B(6) = L(4p_1)$. It follows that $\dim A(k)/B(k) = r - \alpha_k + 1$ for $k = 4, 5, 6$.

Remark. The restriction $e - r \geq 3$ when $g = 1$, $s = 0$ is necessary. For example if $r = 1$, $e = 3$ then we get

$$
\begin{aligned}
&A(1) = \langle 1 \rangle, A(2) = L(p_1) = \langle 1 \rangle, A(3) = L(2p_1) = \langle 1, \mathscr{P} \rangle, A(4) = L(2p_1) \\
&\quad = \langle 1, \mathscr{P} \rangle, A(5) = L(3p_1) = \langle 1, \mathscr{P}, \mathscr{P}' \rangle.
\end{aligned}
$$

Thus

$$\phi(L(p_1) \otimes L(2p_1)) \neq L(3p_1)$$

and hence $A(5) \neq B(5)$. In this case one can easily see that $g_A(t) = t + t^3 + t^5$.

In all the remaining cases $g = 0$. One can do a case by case analysis similar to the above to prove the result in these cases. Instead we shall use the geometric technique described more fully in Section 5. To apply these techniques we must first know that $A(G)$ is finitely generated as an algebra over $\mathbf{C}$. The following lemma will be proven in Section 4.

LEMMA 3.8. *$A(G)$ is finitely generated as a $\mathbf{C}$-algebra.*

PROPOSITION 3.9. *Suppose $g = 0$.*

(1) *If $s \geq 2$ then* e.d. $(A) = s + 1 + e - r$.
(2) *If $s = 1$ then* e.d. $(A) = e - 3$.

In the remaining cases $s = 0$.

(3) *If $r \geq 4$ then* e.d. $(A) = \max(3, e - 8)$.
(4) *If $r = 3$ and $e_i \geq 3$, for all i, then* e.d. $(A) = \max(3, e - 9)$.
(5) *If $r = 3$, $e_1 = 2$, $e_2, e_3 \geq 4$ then* e.d. $(A) = \max(3, e - 10)$.
(6) *If $r = 3$, $e_1 = 2$, $e_2 = 3$, $e_3 \geq 7$ then* e.d. $(A) = \max(3, e - 11)$.

This proposition will be proved in Section 5.6.

DEFINITION 3.10. Suppose $h(t)$ and $k(t)$ are polynomials with non-negative coefficients. If the coefficient of t^i in $h(t)$ is greater than or equal to the coefficient of t^i in $k(t)$ for all i we write $h \succ k$. Note that if $h \succ k$ and $h(1) = k(1)$ then $h = k$.

We shall now apply this to $h = g_A$ and $k = f_A(t) + \sum_{i=1}^r (t^2 + \cdots + t^{e_i})$.

Consider the case $s = 0$, $r \geq 4$. Let p_0 be a point on X. Then the canonical divisor $K = -2p_0$. Now $A(1) = \{0\}$.

$$A(2) = L\left(-4p_0 + \sum_{i=1}^r p_i\right), \quad A(3) = L\left(-6p_0 + \sum_{i=1}^r [3(1 - 1/e_i)]p_i\right).$$

Thus

$$\dim A(2)/B(2) = \dim A(2) = r - 3$$

and

$$\dim A(3)/B(3) = \dim A(3) = r + \alpha_3 - 5.$$

Let $h(t) = f(t) + \sum_{i=1}^r (t^2 + \cdots + t^{e_i})$ as in (3.1). Then the coefficients of $g_A(t)$ and $h(t)$ are the same in degrees 1, 2 and 3. If $i > 3$ the coefficient of t^i in $g_A(t)$ is $\dim A(i)/B(i) \geq r - \alpha_i + 1$. Thus $g_A \succ h$. But if $e \geq 11$ then $g_A(1) = e - 8 = h(1)$ by proposition (3.9). Thus by the remark above $g = h$. Similar considerations prove the other cases. ∎

Remark 3.11. The proof of the theorems actually tells us what the generators of A are. In particular if $g + s > 0$ and $k \geq 4$ one can construct generators as follows:

For each $i \geq \alpha_k$ let $f_{i,k}$ be a function in $L(D^{(k)})$ with a pole of order $k - 1$ at p_i and a pole of order $\leq k - 2$ at p_j for $\alpha_k \leq j \leq r, j \neq i$. Then the residue classes of the $f_{i,k}$, $i = \alpha_k, \ldots, r$ form a basis for $L(D^{(k)})/L(D^{(k)} - p_{\alpha_k} - \cdots - p_r)$, and hence the $f_{i,k}$ are a subset of a set of generators for A.

4. Proof of the lemmas

Proof of Lemma 3.5. It is sufficient to prove the assertion for $k = 2$. In addition we may assume the supports of D_1 and D_2 are disjoint. If one of the $D_i = 0$ then the assertion is trivial. Now proceed by induction on d = degree D_1 + degree D_2. The case $d = 1$ is done. Assume the assertion true for $d - 1$. We must show that

$$L(D_0 + D_1) + L(D_0 + D_2) = L(D_0 + D_1 + D_2)$$

in $L(D_0 + D_1 + D_2)$ (since supp $|D_1|$ and supp $|D_2|$ are disjoint). Now suppose $p \in$ supp (D_2). Then

$$L(D_0 + D_1) + L(D_0 + D_2 - p) = L(D_0 + D_1 + D_2 - p)$$

by the inductive hypotheses. Let m be the multiplicity of p in $D_0 + D_2$. Now by Riemann-Roch,

$$\dim L(D_0 + D_2 - p) = \text{degree } (D_0 + D_2) - g$$

and $\dim L(D_0 + D_2)$ is one larger. Hence there is an $f \in L(D_0 + D_2)$ which has a pole of order precisely m at p. But now

$$L(D_0 + D_1 + D_2) \supset L(D_0 + D_1) + L(D_0 + D_2) \supset L(D_0 + D_1 + D_2 - p)$$

and $f \notin L(D_0 + D_1 + D_2 - p)$. The dimensions if the spaces on either end differ by one, hence the space in the middle, by virtue of containing f, is equal to the space on the left. ∎

Proof of Lemma 3.6. It is sufficient to show that the coefficient of p_j is the same on both sides of the equation, for all j. Let $e = e_j$ and let

$$a_i = [i \cdot (1 - 1/e)].$$

If $e \geq k$ then $a_i + a_{k-i} = a_k - 1$, for all $i = 2, \ldots, k-2$. If $e < k$ then $a_e + a_{k-e} = a_k$. This gives the desired result. ∎

Proof of the corollary to Lemma 3.6. It is sufficient to show that if $k \geq 7$ and $e < k$, then there is an i so that $k - 3 \geq i \geq 3$ and

$$\left[i \cdot \frac{e-1}{e}\right] + \left[(k-i) \cdot \frac{e-1}{e}\right] = \left[k \cdot \frac{e-1}{e}\right].$$

Suppose first that $3 \leq e \leq k - 3$. Then $i = e$ will work. If $e = k - 2$ then $i = 3$ will work. If $e = k - 1$ then $i = 3$ works (using the fact that $e \neq 2$). If $e = 2$ then $i = 4$ works.

Proof of Lemma 3.4. First let us assume X is not hyperelliptic. We may assume $K \geq 0$ and $q_1, q_2 \notin \operatorname{supp}(K)$. Choose a basis $f_1, \ldots, f_g$ for $L(K)$ so that $f_1 = 1$. Recall that $L(K + q_i) = L(K)$ and $l(K + q_1 + q_2) = g + 1$. Let $f_{g+1} \in L(K + q_1 + q_2)$ be so that $f_{g+1} \notin L(K)$. Thus f_{g+1} has poles of order 1 at q_1 and q_2. Consider the cummutative diagram

$$\begin{tikzcd}
L(K + q_1 + q_2) \otimes L(K + q_1 + q_2) \arrow[r, "\phi"] & L(2K + 2q_1 + 2q_2) \\
L(K) \otimes L(K) \arrow[u] \arrow[r, "\phi'"] & L(2K). \arrow[u]
\end{tikzcd}$$

Then ϕ' is onto because X is non-hyperelliptic (by Proposition 2.1). Now

$$l(2K) = 3g - 3 \quad \text{and} \quad l(2K + 2q_1 + 2q_2) = 3g + 1.$$

Then $B(2) = L(2K) + L(K)f_{g+1} + \langle f_{g+1}^2 \rangle \neq L(2K + 2q_1 + 2q_2)$ since there is no $f \in B(2)$ with a pole of order 1 at q_1 and order 2 at q_2. On the other hand

f_{g+1} and $f_{g+1}^2 \in B(2)$ are linearly independent mod $L(2K)$. Thus

$$3g - 1 \leq \dim B(2) \leq 3g.$$

Now $\dim B(2) = 3g$ if and only if there exists a function $f \in B(2)$ with a pole of order 1 at q_1 and a pole of order 0 at q_2. If we can find $h \in L(K - q_2)$ so that $h(q_1) \neq 0$ then $f = hf_{g+1}$ has the desired properties. But if no such h existed then q_1 would be a base point of $|K - q_2|$. So then

$$l(K - q_1 - q_2) = l(K - q_2) \geq g - 1,$$

but, by Riemann-Roch,

$$l(K - q_1 - q_2) - l(q_1 + q_2) = g - 3$$

which implies $l(q_1 + q_2) \geq 2$. This is impossible since X is non-hyperelliptic.

Now suppose X is hyperelliptic. We write a defining equation for X in the usual way:

$$y^2 = \prod_{i=1}^{2g+1} (x - \varepsilon_1).$$

Then $L(K + q_1 + q_2) = \langle 1, x, \ldots, x^{g-1}, f \rangle$ where f is as above. Now image ϕ is spanned by $1, x, \ldots, x^{2g-2}, f, \ldots, x^{g-1}f, f^2$. Let V be the subspace spanned by the functions above with f^2 deleted. Then $\dim V \leq 3g - 1$ and $f^2 \notin V$. We *claim* that $\dim V = 3g - 1$ if $l(q_1 + q_2) = 1$ and $\dim V = 2g$ if $l(q_1 + q_2) = 2$.

Suppose first that $l(q_1 + q_2) = 2$. Then we may choose $f \in L(q_1 + q_2)$. Now $l(q_1 + q_2 + \infty) \leq 3$ and hence $1, x, f, xf$ are linearly dependent. Thus V is spanned by $1, x, \ldots, x^{2g-2}, f$ which is the desired result. Now suppose $l(q_1 + q_2) = 1$. If $\dim V < 3g - 1$ then $1, x, \ldots, x^{2g-2}, f, \ldots, x^{g-1}f$ are linearly dependent and hence f can be written in the form $h(x)/k(x)$. We may assume degree $h \leq$ degree k and that h, k are relatively prime. Each is a product of linear factors. But the only finite poles of f are poles of order 1 at q_1 and q_2. This can only happen if q_1 and q_2 have the same x coordinate, a and $f = 1/(x - a)$. But then $f \in L(q_1 + q_2)$ contradicting $l(q_1 + q_2) = 1$.

The lemma follows immediately from the claim since

$$l(2K + 2q_1 + 2q_2) = 3g + 1 \quad \text{and} \quad \dim(\text{image } \phi) = \dim V + 1.$$

Proof of Lemma 3.8. We know by 1.0 that for an admissible signature,

$$2g - 2 + s + \sum_{j=1}^{r} (1 - 1/e_j) > 0.$$

Thus $\lim_{k\to\infty}$ degree $D^{(k)} = \infty$. Let k_0 be such that degree $D^{(k)} \geq 2g + 1$, for all $k \geq k_0$. We may assume $k_0 \geq e_j$, for all j. Now

$$\phi(L(D^{(i)}) \otimes L(D^{(k-i)})) \subset B(k) \tag{4.1}$$

for all $k \geq k_0$. Thus if we can show that for all $k \geq 3k_0$ there exists an i so that the left hand side of (4.1) is equal to $A(k)$, we have proven that A is generated by elements of degree $\leq 3k_0$. Now by Theorem 1.2,

$$\phi(L(D^{(i)}) \otimes L(D^{(k-i)})) = L(D^{(i)} + D^{(k-i)}), \quad k_0 \leq i \leq k - k_0,$$

since degree $D^{(i)}$ and degree $D^{(k-i)}$ are greater than or equal to $2g + 1$. We get the desired result if we can show that there exists an i, $k_0 \leq i \leq k - k_0$ so that $D^{(i)} + D^{(k-i)} = D^{(k)}$. For this one can easily see that it is sufficient to show that for each j, there exists an i, $k_0 \leq i \leq k - k_0$ such that

$$[i(1 - 1/e_j)] + [(k - i)(1 - 1/e_j)] = [k(1 - 1/e_j)].$$

The equation above is satisfied if i is a multiple of e_j. Since $k \geq 3k_0 \geq 3e_j$ there must be a multiple of e_j in the interval $[k_0, k - k_0]$.

5. Singularities of complex surfaces

We were led to many of the statements in Section 3 via the theory of singularities. Although the main theorem, 3.3 can be proven without using singularities we believe that others will benefit, as we have, from the use of singularity theory. Moreover, deeper results about the structure of A require these methods.

Recall that we have shown that $A(G)$ is a finitely generated $\mathbf{C}$-algebra.

DEFINITION 5.1. If A is an arbitrary finitely generated graded algebra we define the Poincaré power series of A by

$$p_A(t) = \sum_{i=0}^{\infty} a_i t^i$$

where $a_i = \dim A(i)$.

Now it is well known that $p_A(t)$ is a rational function and the dimension of A equals the order of the pole of $p_A(t)$ at $t = 1$ (see [2, Chapter 11] for one). When $A = A(G)$ we can easily calculate a_i from the signature of G and then show that $p_A(t)$ has a pole of order 2 at $t = 1$. This was done explicitly in [21]. Thus A is an algebra of dimension 2.

5.2. *The variety associated to A.* The fact that A is a finitely generated algebra over $\mathbf{C}$ is equivalent to saying that there is a surjective algebra homomorphism

$$\phi\colon \mathbf{C}[X_1, \ldots, X_n] \to A$$

for some n. Let I be the kernel of ϕ and let

$$V = \{(x_1, \ldots, x_n) \in \mathbf{C}^n \mid f(x_1, \ldots, x_n) = 0, \text{ for all } f \in I\},$$

the algebraic variety associated to A. Endow V with the topology induced by the metric topology on $\mathbf{C}^n$. The analytic structure on $\mathbf{C}^n$ induces an analytic structure on V and we denote the sheaf of analytic functions by $\mathcal{O}_V$. It is a standard result that $(V, \mathcal{O}_V)$ is independent of the choice of ϕ. We can choose generators $x_1, \ldots, x_n$ for A so that x_i is homogeneous of some degree q_i. We make $\mathbf{C}[X_1, \ldots, X_n]$ a graded ring by decreeing that degree $X_i = q_i$. Then the ϕ defined by $\phi(X_i) = x_i$ is a homomorphism of graded algebras.

A function $f \in \mathbf{C}[X_1, \ldots, X_n]$ is homogeneous of degree d if

$$f(t^{q_1}X_1, \ldots, t^{q_n}X_n) = t^d f(X_1, \ldots, X_n) \quad \text{for all } t \in \mathbf{C}.$$

The fact that ϕ is a graded homomorphism implies that I is a homogeneous ideal, i.e., I is generated by homogeneous polynomials.

5.3. *The* $\mathbf{C}^*$*-action on* V. If we are as above, define an action of the multiplicative group of complex numbers $\mathbf{C}^*$, on $\mathbf{C}^n$ by

$$t \cdot (z_1, \ldots, z_n) = (t^{q_1}z_1, \ldots, t^{q_n}z_n).$$

The fact that I is generated by homogeneous polynomials implies that if $z \in V$ then $t \cdot z \in V$. Thus $\mathbf{C}^*$ acts on V. It is not hard to prove that conversely, if V is an affine algebraic variety with (algebraic) $\mathbf{C}^*$-action, then the ring of polynomial functions on V is graded, i.e., $A = \bigoplus_{i=-\infty}^{\infty} A(i)$ where $A(i) = \{f \mid f(t \cdot z) = t^i f(z)\}$, the homogeneous functions of degree i.

5.4. *Resolution of the singularity of* V. A $\mathbf{C}^*$-action on an affine variety V is said to be a *good* $\mathbf{C}^*$-action [14], if there is a point $v \in V$ so that v is in the closure of every orbit. This is the case if all the q_i above are $\succ 0$.

It was shown in [14] that an affine surface V with an isolated singularity v and a good $\mathbf{C}^*$-action has a canonical equivariant resolution $\pi\colon \tilde{V} \to V$ i.e., there exists $\tilde{V}$, a non-singular surface with $\mathbf{C}^*$-action, π a proper map, so that $\pi(t \cdot v) = t \cdot \pi(v)$ for all $t \in \mathbf{C}^*$ and so that

(1) $\pi\colon V - \pi^{-1}(v) \to V - \{v\}$ is an isomorphism,
(2) $\pi^{-1}(v) = \bigcup_{i=1}^{l} X_i$, X_i non-singular Riemann surfaces meeting transversely.

Given a resolution as above one can construct a labeled graph Γ as follows:

(a) One vertex ε_i, for each X_i.
(b) One edge joins ε_i to ε_j for each point in $X_i \cap X_j$.
(c) Each vertex ε_i is labeled

$$\underset{[g_i]}{-n_i}$$

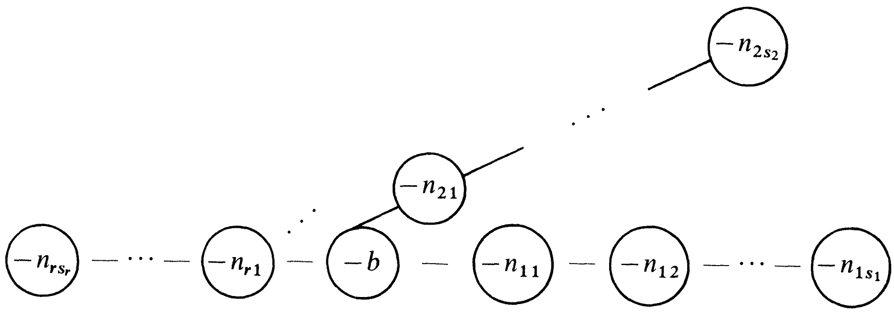
where $-n_i$ is the self-intersection number of X_i and g_i is the genus of X_i.

THEOREM 5.4.1. *Suppose* X *is a Riemann surface,* D_0 *is a divisor on* X, $p_1, \ldots,$ $p_r \in X$, $0 < \beta_i < \alpha_i$, $i = 1, \ldots, r$, *and we define*

$$A(k) = L\left(kD_0 + \sum_{i=1}^{r} \left[k \cdot \frac{\alpha_i - \beta_i}{\alpha_i}\right] p_i\right).$$

Let $A = \bigoplus_{k\geq 0} A(k)$ and suppose A is finitely generated. Let V be the algebraic variety associated to A, as above. Then V is a complex surface (i.e., V has complex dimension 2).

The graph of the resolution of the (isolated) singularity of V is of the form indicated in the following diagram:



where $n_{ij} \geq 2$,

$$\frac{\alpha_i}{\beta_i} = n_{i1} - \cfrac{1}{n_{i2} - \cfrac{\ddots}{-\cfrac{1}{n_{is_i}}}}$$

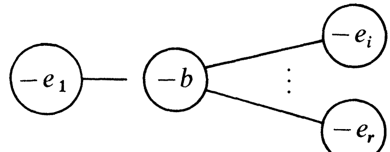
and $b = \text{degree } D_0 + r$.

Proof. This follows easily from Pinkham Theorem 5.1.

COROLLARY 5.4.2. *If G is a Fuchsian group with signature*

$$\langle g;\, s;\, e_1, \ldots, e_r\rangle$$

then the graph of the resolution of $V = \text{Spec }(A(G))$ is



where $b = 2g - 2 + s + r$.

Proof. degree $D_0 =$ degree $(K + q_1 + \cdots + q_s)$.

5.5. There are two classes of singularities which are rather well understood, the rational and minimal elliptic ones. In particular, the embedding dimension can be calculated from the graph using results of Artin and Laufer [1], [10]. To define these classes and state the results we must introduce the fundamental cycle. Let $\pi\colon \tilde{V} \to V$ be a resolution as in 5.4. A divisor $Z = \sum_{i=1}^{l} n_i X_i$ is called a *fundamental cycle* if the following hold:

(i) $Z > 0$.
(ii) $Z \cdot X_i \leq 0$ *for all* $i = 1, \ldots, l$.
(iii) If Z' satisfies (i) and (ii), then $Z \leq Z'$.

A fundamental cycle exists and is unique. Let K be the divisor of a meromorphic 2 form on $\tilde{V}$. Then the *arithmetic genus* of a divisor D is defined to be

$$p_a(D) = \tfrac{1}{2}(D \cdot D + D \cdot K) + 1.$$

A singularity is *rational* if $p_a(Z) = 0$, where Z is the fundamental cycle. A singularity is *minimal elliptic* if $p_a(Z) = 1$ and any connected proper subvariety of the exceptional set is the exceptional set of a rational singularity.

PROPOSITION 5.5.1. *V as in Corollary 5.4.2 has a rational singularity if and only if $g = 0$ and $s > 0$. If $g = s = 0$ then V has a minimal elliptic singularity.*

Proof. By Pinkham [15, 5.8], the singularity is rational if and only if $g = 0$ and

$$k(s-2) + \sum_{i=1}^{r} [k(1 - 1/e_i)] > -2 \quad \text{for all } k > 0. \tag{5.1}$$

We claim this holds if and only if $s > 0$. Clearly if $s = 0$, then the inequality fails for $k = 1$. If $s \geq 2$, the inequality obviously holds. Now when $s = 1$, recall that the signature is realizable if and only if $(-1) + \sum_{i=1}^{r} (1 - 1/e_i) > 0$. Thus $r \geq 2$ and if $r = 2$ then $e_1 = e_2 = 2$ does not occur. Now equation (5.1) becomes

$$\begin{aligned} -k + \sum_{i=1}^{r} [k(1 - 1/e_i)] &> -k + r \cdot \frac{k}{2} > 0, && k \text{ even}, \\ &\geq -k + r \cdot \frac{k-1}{2} \geq -1, && k \text{ odd}. \end{aligned}$$

If $s = 0$ then Dolgacev has shown [6, 4.4.14] that the singularity is elliptic. The proposition can also be proven directly by calculating $p_a(Z)$ using the calculation of Z below. ∎

Artin [1] has shown that if $v \in V$ is an isolated *rational* singularity then the embedding dimension of v is $-(Z \cdot Z) + 1$, and Laufer has shown that the embedding dimension is max $(3, -Z^2)$ for minimal elliptic singularities. We are now in a position for:

Proof of Proposition 3.9. Let X_0 be the exceptional curve corresponding to the center vertex of Γ (as in Corollary 5.4.2). Let $X_1, \ldots, X_r$ be the other curves.

Then

$$X_0^2 = 2 - r - s, \quad X_i^2 = -e_i, \ i \geq 1,$$

$$X_i \cdot X_j = \begin{cases} 0, & i > j > 0, \\ 1, & i > j = 0. \end{cases}$$

If $s \geq 2$ then one can easily see that the fundamental cycle is

$$Z = X_0 + X_1 + \cdots + X_r,$$

Then the embedding dimension is

$$1 - Z^2 = 1 - 2 + r + s - 2r + e = s - 1 + e - r.$$

If $s = 1$ then $Z = 2X_0 + X_1 + \cdots + X_r$ (note that $r \geq 2$ by (1.1)) and hence

$$1 - Z^2 = 1 - 4(1 - r) - 4r + e = e - 3.$$

Now assume $s = 0$. In this case the singularity is minimal elliptic. The result follows easily from Laufer's theorem once we compute Z. The computation follows:

$$\begin{aligned} Z &= 2X_0 + X_1 + \cdots + X_r && \text{if } r \geq 4, \\ Z &= 3X_0 + X_1 + X_2 + X_3 && \text{if } r = 3, e_i \geq 3 \text{ for all } i, \\ Z &= 4X_0 + 2X_1 + X_2 + X_3 && \text{if } r = 3, e_1 = 2, e_2, e_3 \geq 4, \\ Z &= 6X_0 + 3X_1 + 2X_2 + X_3 && \text{if } r = 3, e_1 = 2, e_2 = 3, e_3 \geq 7. \end{aligned}$$

This completes the proof of the proposition. ∎

6. Groups for which $n \leq 4$

THEOREM 6.1. *The following is a list of all groups for which $A(G)$ is generated by 4 elements.*

Signature	Degrees of generators	Degrees of relations	Degrees of syzygies
$\langle 4 \rangle$, X non-hyperelliptic	1 1 1 1	2 3	5
$\langle 3 \rangle$, X hyperelliptic	1 1 1 2	2 4	6
$\langle 3; 0; 2 \rangle$, X non-hyperelliptic	1 1 1 2	3 3	6
$\langle 2; 0; 2, 2 \rangle$	1 1 2 2	3 4	7
$\langle 2; 0; 3 \rangle$	1 1 2 3	4 4	8
$\langle 2; 2 \rangle$ $q(1) + q(2)$ not linearly equivalent to K.	1 1 1 2	3 3 4	5 5
$\langle 2; 3 \rangle$	1 1 1 1	2 3 3	4 4
$\langle 1; 0; 2, 2, 2, 2 \rangle$	1 2 2 2	4 4	8
$\langle 1; 0; 2, 2, 3 \rangle$	1 2 2 3	4 5	9
$\langle 1; 0; 2, 4 \rangle$	1 2 3 4	5 6	11

Signature	Degrees of generators	Degrees of relations	Degrees of syzygies
$\langle 1; 0; 3, 3\rangle$	1 2 3 3	4 6	10
$\langle 1; 0; 5\rangle$	1 3 4 5	6 8	14
$\langle 1; 1; 2\rangle$	1 2 2 3	4 5 6	7 8
$\langle 1; 2; 2\rangle$	1 1 2 2	3 4 4	5 6
$\langle 1; 3; 2\rangle$	1 1 1 2	3 3 3	4 5
$\langle 1; 4\rangle$	1 1 1 1	2 2	4
$\langle 0; 0; 2, 2, 2, 2, 2, 2\rangle$	2 2 2 3	4 6	10
$\langle 0; 0; 2, 2, 2, 2, 4\rangle$	2 2 3 4	6 6	12
$\langle 0; 0; 2, 2, 2, 3, 3\rangle$	2 2 3 3	5 6	11
$\langle 0; 0; 2, 2, 2, 6\rangle$	2 4 5 6	8 10	18
$\langle 0; 0; 2, 2, 3, 5\rangle$	2 3 4 5	7 8	15
$\langle 0; 0; 2, 2, 4, 4\rangle$	2 3 4 4	6 8	14
$\langle 0; 0; 2, 3, 3, 4\rangle$	2 3 3 4	6 7	13
$\langle 0; 0; 3, 3, 3, 3\rangle$	2 3 3 3	6 6	12
$\langle 0; 0; 2, 3, 10\rangle$	6 8 9 10	16 18	34
$\langle 0; 0; 2, 4, 8\rangle$	4 6 7 8	12 14	26
$\langle 0; 0; 2, 5, 7\rangle$	4 5 6 7	11 12	23
$\langle 0; 0; 2, 6, 6\rangle$	4 5 6 6	10 12	22
$\langle 0; 0; 3, 3, 7\rangle$	3 5 6 7	10 12	22
$\langle 0; 0; 3, 4, 6\rangle$	3 4 5 6	9 10	19
$\langle 0; 0; 3, 5, 5\rangle$	3 4 5 5	8 10	18
$\langle 0; 0; 4, 4, 5\rangle$	3 4 4 5	8 9	17
$\langle 0; 1; 2, 2, 3\rangle$	2 2 3 3	5 6 6	8 9
$\langle 0; 1; 2, 5\rangle$	2 3 4 5	6 7 8	10 11
$\langle 0; 1; 3, 4\rangle$	2 3 3 4	6 6 7	9 10
$\langle 0; 2; 2, 2, 2\rangle$	1 2 2 2	4 4 4	6 6
$\langle 0; 2; 2, 3\rangle$	1 2 2 3	4 4 5	6 7
$\langle 0; 2; 4\rangle$	1 2 3 4	4 5 6	7 8
$\langle 0; 3; 2, 2\rangle$	1 1 2 2	3 3 4	5 5
$\langle 0; 3; 3\rangle$	1 1 2 3	3 4 4	5 6
$\langle 0; 4; 2\rangle$	1 1 1 2	2 3 3	4 4
$\langle 0; 5\rangle$	1 1 1 1	2 2 2	3 3

Proof. The groups for which $n = 3$ and $s = 0$ were found by Dolgacev [4]. Later, all groups with $n \leq 3$ were classified in [21]. The list of groups with $n = 4$ and the generators of $A(G)$ can be found easily using Theorem 3.3. All that remains is to verify the assertions about the degrees of the generators of the ideal of relations.

Suppose $A(G)$ is generated by 4 elements $f_1, \ldots, f_4$ so that $f_i \in A(q_i)$, $R = \mathbf{C}[X_1, X_2, X_3, X_4]$ and let $\phi\colon R \to A(G)$ be defined by $\phi(X_i) = f_i$. If we grade R by letting degree $X_i = q_i$ then ϕ is a graded morphism of degree 0. Let

$I = \text{kernel } \phi$. Then I is the (homogeneous) *ideal of relations* (relative to the chosen generators).

LEMMA 6.2. *If $s = 0$ then I is generated by two elements. If $g = 0$ and $s > 0$ then I is generated by 3 elements.*

Proof. If $s = 0$ then by [6, Theorem 3.3.12] $A(G)$ has a Gorenstein singularity (5.5) and the result follows from [23, Proposition 5]. If $g = 0$ and $s > 0$, we have seen (Proposition 5.5.1) that the singularity is rational. Since the embedding dimension is 4, by [22], I is generated by 3 elements.

There are six signatures not covered by the lemma; i.e., with $g > 0$ and $s > 0$. We shall deal with these later (6.8). Now we wish to determine the degrees of the generator of I in the cases above. We shall do this using the Poincaré power series of A (defined in 5.1).

We shall see that in the above cases the degrees of the generators of I can be determined from $p_A(t)$. This is useful since in [21] we calculated $p_A(t)$ explicitly.

PROPOSITION 6.3 [21, 2.4]. *If G has signature $\langle g; s; e_1, \ldots, e_r\rangle$ and $A = A(G)$ then*

$$p_A(t) = \frac{2g - 2 + s}{(1 - t)^2} + \frac{3 - 3g - s}{(1 - t)} + \sum_{i=1}^{r} p_{e_i}(t) + \delta_{s,0}\, t + g$$

where δ is the Kronecker delta and

$$p_e(t) = \frac{\sum_{k=2}^{e-1} (k - 1)t^k}{1 - t^e} + \frac{(e - 1)(\sum_{k=0}^{e-1} t^k)t^e}{(1 - t)^2}$$

PROPOSITION 6.4. *Suppose $A = A(G)$.*

(1) *If I is generated by two elements of degrees r_1, r_2 then*

$$p_A(t) = \frac{1 - t^{r_1} - t^{r_2} + t^{r_1 + r_2}}{(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})(1 - t^{q_4})}$$

(*recall that the q_i's are the degrees of the generators of A*).

(2) *If I is generated by three elements of degrees r_1, r_2, r_3 respectively, then there exist s_1, s_2 so that*

$$p_A(t) = \frac{1 - t^{r_1} - t^{r_2} - t^{r_3} + t^{s_1} + t^{s_2}}{(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})(1 - t^{q_4})}.$$

Proof. Let $R^{[d]}$ denote R with its grading shifted by d, i.e., $R^{[d]}_{(i)} = R(d - i)$, for all i. If I is generated by two elements g_1 and g_2 of degrees r_1 and r_2 respectively we have an exact sequence

$$0 \to R^{[r_1 + r_2]} \xrightarrow{\phi} R^{[r_1]} \oplus R^{[r_2]} \xrightarrow{\psi} R \to A \to 0$$

where $\psi(f_1, f_2) = f_1 g_1 + f_2 g_2$ and $\phi(h) = h \cdot (-g_2, g_1)$. Note that all morphisms are graded of degree 0. Using the basic properties of Poincaré power series we can now compute $p_A(t)$. From [2],

(a) $p_R(t) = \dfrac{1}{(1 - t^{q_1})(1 - t^{q_1})(1 - t^{q_2})(1 - t^{q_3})}$;

(b) $p_{R[d]}(t) = t^d p_R(t)$;

(c) if $0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence of graded R-modules (with degree 0 maps) then $\sum_{i=1}^{r} p_{M_i}(t) = 0$.

Part (1) follows immediately. Now the ring A is integrally closed, hence Cohen-Macaulay [6]. Hence there is a resolution by free graded R-modules:

$$0 \to F_2 \to F_1 \xrightarrow{\phi_1} R \to A \to 0.$$

If I is generated by 3-elements of degrees r_1, r_2, r_3 we can choose

$$F_1 = R^{[r_1]} \oplus R^{[r_2]} \oplus R^{[r_3]}.$$

Then the kernel of ϕ_1 is free of rank 2. Let $F_2 = \text{kernel } \phi_1$. Then $F_2 = R^{[s_2]} \oplus R^{[s_1]}$, for some s_1, s_2. As above, the properties of Poincaré power series allows us to calculate $p_A(t)$.

Example 6.5. We give an example to show how we calculate the degrees of the generators of I. Suppose G has signature $\langle 0; 3; 2, 2 \rangle$. Then by Proposition 6.3,

$$p_A(t) = \frac{1}{(1-t)^2} + \frac{1}{(1-t)} + \frac{(1+t)t^2}{(1-t^2)^2}.$$

Now we write this as a rational function with denominator

$$\frac{1}{(1-t)^2(1-t^2)^2}$$

(since $q_1 = q_2 = 1$, $q_3 = q_4 = 2$) and we get

$$p_A(t) = \frac{1 - 2t^3 - t^4 + 2t^5}{(1-t)^2(1-t^2)^2}.$$

Thus $r_1 = r_2 = 3$, $r_3 = 4$ by Lemma 6.2 and Proposition 6.4.

Example 6.6 We give a second method for calculating the r_i. Assume we again have signature $\langle 0; 3; 2, 2 \rangle$. The dimensions of $R(i)$, $A(i)$, and $I(i)$, for $i = 1, \ldots, 4$ are given in the following table.

dim $R(i)$	2	5	8	14
dim $A(i)$	2	5	6	9
dim $I(i)$	0	0	2	5

Thus there are two generators for I, say g_1, g_2 in $I(3)$. Now if J is the ideal generated by g_1 and g_2, $J(4)$ is generated by $f_1 g_1, f_2 g_1, f_1 g_2, f_2 g_2$ and hence, $\dim J(4) \leq 4$. Therefore there is a third generator of I which is homogeneous of degree 4.

In either of the above ways one can determine the degrees of the generators of I in all cases.

All that remains is to find the *number* of generators of I when $g > 0$ and $s > 0$. If the signature is $\langle 1; 4\rangle$ then the graph of the resolution is

$$\underset{[-1]}{\textcircled{-4}}.$$

This is a minimal elliptic singularity and hence I is generated by two elements.

If the signature is $\langle 2; 3\rangle$ then $A(i) = L(iD)$ where

$$\text{degree } D = \text{degree } (K + q_1 + q_2 + q_3) = 5 = 2g + 1.$$

By a theorem of Saint-Donat [16, 2], I is generated by its elements of degree 2 and 3. Now one can verify directly that there is one generator of degree 2 and two generators of degree 3.

Now suppose the signature is $\langle 1; 1; 2\rangle$, Let $\langle f_1, \ldots, f_r\rangle$ denote the subspace generated by $f_1, \ldots, f_r$ and $(f)_\infty$ denote the divisor of poles of f. Then

$$\begin{aligned}
A(1) &\approx L(q_1) &&= \langle 1\rangle \\
A(2) &\approx L(2q_1 + p_1) &&= \langle 1, \mathscr{P}, f\rangle \\
A(3) &\approx L(3q_1 + p_1) &&= \langle 1, \mathscr{P}, \mathscr{P}', f\rangle \\
A(4) &\approx L(4q_1 + 2p_1) &&= \langle 1, \mathscr{P}, \mathscr{P}', \mathscr{P}^2, f, f^2\rangle \\
A(5) &\approx L(5q_1 + 2p_1) &&= \langle 1, \mathscr{P}, \mathscr{P}', \mathscr{P}^2, \mathscr{P}\mathscr{P}', f, f^2)
\end{aligned}$$

where $\mathscr{P}$ is the Weierstrass p-function with $(\mathscr{P})_\infty = 2q_1$, $(\mathscr{P}')_\infty = 3q_1$ and f is a function so that $(f)_\infty = q_1 + p_1$. Note that the functions spanning the subspaces above are linearly independent. Suppose I has a minimal set of homogeneous generators $F_1, \ldots, F_n$. Define $\phi_1 \colon R^{(n)} \to R$ by $\phi_1(e_i) = F_i$ where $e_i = (0, \ldots, 1, \ldots, 0)$, the standard basis element of the free R module $R^{(n)}$. The grading of $R^{(n)}$ is defined by degree $e_i =$ degree F_i. Now A is a Cohen-Macaulay ring, therefore kernel ϕ_1 is a projective module [11, p. 113]. A graded projective module of finite type is free, hence kernel ϕ_1 is isomorphic to $R^{(n-1)}$.

We claim that $n = 3$. By a thorem of Hilbert [3, Theorem 5] the ideal I is generated by the $(n-1) \times (n-1)$ minors of an $(n-1) \times n$ matrix whose entries are homogeneous polynomials of degree > 0. Now $\dim R(4) = 7$ and $\dim A(4) = 6$, hence $I(4) = \langle F\rangle$, $F \neq 0$, homogeneous of degree 4. Hence there is an $(n-1) \times (n-1)$ matrix M whose entries are homogeneous polynomials of degree > 0 so that $\det M = F$. Clearly $n \leq 5$. Suppose $n = 5$. The entries of M must be homogeneous polynomials of degree 1. But then F is a polynomial in X_1 only. This is impossible. Suppose $n = 4$. Then the entries of M must have degree less than or equal to 2, hence $F = F(X_1, X_2, X_3)$. Now the monomials in X_1, X_2, X_3 of degree 4 are $X_1^4, X_1^2 X_2, X_1^2 X_3, X_2^2, X_2 X_3, X_3^2$ and these are mapped to $1, \mathscr{P}, f, \mathscr{P}^2, f\mathscr{P}, f^2$ respectively. By considering the divisor of poles of these functions one can easily see they are linearly independent. Thus $\phi_1(F) \neq 0$, contradicting $\det M \in I$.

Now that we know that I is generated by 3 or fewer elements, we can calculate the degrees of those generators the same way we did in Examples 6.5 and 6.6.

The line of reasoning used for signature $\langle 1; 1; 2\rangle$ suffices to find the number and degrees of generators of I for all the remaining signatures. It would be useful to have a general argument to show that I is generated by 2 or 3 elements whenever A is generated by 4 elements.

Now we must find the degrees of the syzygies for A. In all the above cases we have a resolution

$$0 \to R^{(n-1)} \to R^{(n)} \to R \to A \to 0$$

where $n = 2$ or 3. The "degrees of the syzygies" mentioned in the statement of the theorem are defined to be the degrees, s_i, of the free generators of $R^{(n-1)}$. As in Proposition 6.4 we can read these off the Poincaré power series of A. If one wishes to avoid calculating the Poincaré power series, the following proposition can be used to determine the s_i in each of the cases above.

PROPOSITION 6.7. *If $n = 2$ then $s = r_1 + r_2$. If $n = 3$ then with the notation of Proposition* 6.4(2),

(i) $$s_1 + s_2 = r_1 + r_2 + r_3,$$

(ii) $$\frac{r_1(r_1 - 1) + r_2(r_2 - 1) + r_3(r_3 - 1) - s_1(s_1 - 1) - s_2(s_2 - 1)}{q_1 q_2 q_3 q_4} = -2\left(2g - 2 + s + \sum_{i=1}^{r} (1 - 1/e_i)\right).$$

Proof. The first assertion is in Theorem 6.1. Assertion (*i*) follows from Prop. 6.4(2) and the fact that $p(t)$ has a pole of order 2 at $t = 1$. To prove assertion (ii) we recall that $p(t)$ has a pole of order 2 at $t = 1$ and by (6.3) the coefficient of $1/(1 - t)^2$ in its Laurent series at $t = 1$ is

$$2g - 2 + s + \sum_{i=1}^{r} (1 - 1/e_i).$$

Calculating this same coefficient using the expression for $p(t)$ given in Prop. 6.4(2) we get 1/2 times the left hand side of equation (ii). ∎

This completes the proof of Theorem 6.1.

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