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# FIBERING COMPLEX MANIFOLDS OVER $S^3$ AND $S^4$

BY

STEVEN M. KAHN[1]

## 1. Introduction

For any closed manifold $N$, there are obstructions to the smooth fibering over $N$, of a stably almost complex manifold $M$. This paper examines those obstructions which are given by the Stiefel-Whitney, Pontrjagin and Chern numbers of $M$.

A complete solution, for a given base space $N$, consists of finding a set of obstructions whose vanishing on a manifold $M$ guarantees the existence of a fibration over $N$, with total space complex cobordant to $M$. Thus, we are trying to find those cobordism classes $\omega \in \Omega_*^u$ which contain a representative fibered over $N$. If such a representative exists, we say that the class $\omega$ fibers over $N$. Note that for fixed $N$, the set of $\omega$ which fiber over $N$ is an ideal in $\Omega_*^u$, which we denote by $\mathrm{Fib}_{\Omega_*^u}(N)$.

It was shown in [4, p. 68] that a class $\omega \in \Omega_*^u$ fibers over $S^1$ if and only if the signature $\sigma(\omega)$ is zero. Nelson [8, Theorem 3.8] proved that this signature condition is also the only obstruction to fibering a unitary class over $S^2$, while the general result for any connected surface $B^2$, is the following (see [1]):

A class $\omega \in \Omega_n^u\, n \geq 2$, fibers over $B^2$ if and only if

$$\sigma(\omega)\begin{cases} = 0 & \text{if } \chi(B) \geq 0 \\ \equiv 0 \pmod 4 & \text{if } \chi(B) < 0, \end{cases}$$

where $\chi$ denotes the Euler characteristic.

For fiberings over a manifold $N$ of dimension greater than two, the situation is quite different. The necessary signature conditions in this case are in general, no longer sufficient. There is an interplay that surfaces here, between fiber and total space cohomologies, involving actions of both stable and unstable operations on characteristic classes. The result is a more involved set of obstructions.

This paper completely determines the obstructions to fibering a class $\omega \in \Omega_*^u$ over $S^3$ and $S^4$ (the two solutions are precisely the same). As consequences we obtain results for some other 3 and 4-manifolds. We also get a partial description of the fibering ideal of $S^2 \times S^2$ and offer a conjecture as to what the complete answer should be. In addition, since almost all the fibrations we construct over the spheres actually fiber over CP(3), and since all obstruc-

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tions to fibering over $S^4$ apply to fiberings over CP(3) as well, we get for free a considerable amount of information about the fibering ideal of CP(3).

We now state the main results.

Let $c_i(M)$, $w_i(M)$ and $v_i(M)$ denote the $i$-th Chern, Stiefel-Whitney and Wu classes of $M$ respectively. Let $\sigma(M)$ denote the signature and let $s_n[M^{2n}]$ denote the "$s$-number" in the Chern classes detecting indecomposables in $\Omega_*^u$. Let

$$p\colon H^j(M; Z_2) \to H^{2j}(M; Z_4)$$

denote the Pontrjagin square operation (see [14, §1] for a general description) and let

$$\rho_m\colon H^*(M; Z) \to H^*(M; Z_m)$$

denote reduction mod $m$.

THEOREM 1. *A class $\omega \in \Omega_{2n}^u$ $(n \geq 2)$ fibers over $S^3$ or $S^4$ if and only if*

(i) $\omega = 0$ *for $n = 2$,*
(ii) $c_1^3(\omega) \equiv 0 \pmod 8$ *for $n = 3$*
(iii) $\sigma(\omega) = 0$ *and* $s_4(\omega) \equiv 0 \pmod 3$ *for $n = 4$*
(iv) $\sigma(\omega) = 0$ *if $n = 2k$*; $w_4 w_{2n-4}(\omega) = 0$ *and* $\rho_4 c_1 \cdot p(v_{2k})(\omega) = 0$ *if $n = 2k+1$* — *for $n \geq 5$*

*Remark* 1.1. Unlike the results for fiberings over surfaces and $S^1$ which involve only the signature, the result here involves a fairly complicated set of obstructions; seemingly unrelated and perhaps even a bit mysterious. There is, however, a unifying theme which can be seen clearly in the theorem's proof (see Section 4). With the exception of the signature, all obstructions emanate from one source; the *Wu* classes. Whether we are dealing with fiberings in $\Omega_8^u$ (part (iii) of Theorem 1), where a mod 3 *Wu* class comes into play, or in the other dimensions where we work in the usual $\mathbf{Z}_2$ setting, it is the fact that a *Wu* class in the cohomology of the total space restricts to zero on the fiber that leads to the appropriate obstruction(s).

Theorem 1 coupled with the previously known results for $S^1$ and $S^2$, leads to the conjecture that $\mathrm{Fib}_{\Omega_*^u}(S^{2k-1}) = \mathrm{Fib}_{\Omega_*^u}(S^{2k})$ for all $k$. The analysis of Theorem 1 in terms of the *Wu* classes supports it. We note that whether a stably almost complex manifold $M^{2n}$ fibers over $S^{2k}$ or over $S^{2k-1}$, the same exact *Wu* classes (specifically $v_i(M)$, $n - k + 1 \leq i \leq n$) restrict to zero on the fiber.

Let $M^3$ be any closed connected 3-manifold.

COROLLARY 1. *If a class $\omega \in \Omega_{2n}^u$ $(n \geq 2)$ satisfies the conditions of Theorem* 1, *then $\omega$ fibers over $M^3$.*

COROLLARY 2. *A class $\omega \in \Omega_{4k}^u$ $(k \geq 3)$ fibers over $M^3$ if and only if $\sigma(\omega) = 0$.*

The following corollary, although stated in general form, completely determines the fibering ideals of all Lens spaces $L_p$, for $p$ odd, $p \neq 3r$.

COROLLARY 3. *Let $M^3$ be any closed connected 3-manifold with $\pi_1(M)$ finite of odd order $k$.*

(i) *If $k \neq 3r$,* $\mathrm{Fib}_{\Omega^u_*}(M^3) = \mathrm{Fib}_{\Omega^u_*}(S^3)$.

(ii) *If $k = 3r$,* $\mathrm{Fib}_{\Omega^u_*}(M^3) = \mathrm{Fib}_{\Omega^u_*}(S^3)$ *ecept possibly in $\Omega^u_8$.*

The fibering ideals of certain 4-manifolds including $S^3 \times S^1$ are given by the following:

COROLLARY 4. *Let $M^4$ be any closed connected stably almost complex 4-manifold fibered over a 3-manifold $M^3$ with $\pi_1(M^3)$ finite of odd order $k$.*

(i) *If $k \neq 3r$,* $\mathrm{Fib}_{\Omega^u_*}(M^4) = \mathrm{Fib}_{\Omega^u_*}(S^4)$.

(ii) *If $k = 3r$,* $\mathrm{Fib}_{\Omega^u_*}(M^4) = \mathrm{Fib}_{\Omega^u_*}(S^4)$ *except possibly in $\Omega^u_8$.*

Before stating the result for fiberings over $S^2 \times S^2$, we give an idea of the size of the fibering ideal of $S^3$ or $S^4$ by describing the quotient of $\Omega^u_*$ by $\mathrm{Fib}_{\Omega^u_*}(S^3)$. Letting $G_4$ and $G_5$ denote indecomposable classes (to be described in Section 3) in $\Omega^u_8$ and $\Omega^u_{10}$ respectively, we have

$$\Omega^u_{2n}/\mathrm{Fib}_{\Omega^u_{2n}}(S^3) = \begin{cases} \mathbf{Z} \text{ generated by } \mathbf{C}P(1) & \text{for } n = 1 \\ \mathbf{Z} \oplus \mathbf{Z} \text{ generated by } \mathbf{C}P(1)^2 \text{ and } \mathbf{C}P(2) & \text{for } n = 2 \\ \mathbf{Z}_4 \text{ generated by } \mathbf{C}P(1)\mathbf{C}P(2) & \text{for } n = 3 \\ \mathbf{Z} \oplus \mathbf{Z}_3 \text{ generated by } \mathbf{C}P(2)^2 \text{ and } G_4 & \text{for } n = 4 \\ \mathbf{Z} \text{ generated by } \mathbf{C}P(2)^k & \text{for } n = 2k \geq 6 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \text{ generated by } \mathbf{C}P(1)\mathbf{C}P(2)^k & \\ \text{and } \mathbf{C}P(2)^{k-2}G_5 & \text{for } n = 2k + 1 \geq 5 \end{cases}$$

THEOREM 2. *A class $\omega \in \Omega^u_{2n}$ ($n = 2k \geq 2$ or $n = 3$) fibers over $S^2 \times S^2$ if and only if*

(i) $c_1^3(\omega) \equiv 0 \pmod 4$ *for $n = 3$*

(ii) $\sigma(\omega) = 0$ *for $n = 2k \geq 2$.*

*Conjecture.* A class $\omega \in \Omega^u_{2n}$, $n = 2k + 1 \geq 5$, fibers over $S^2 \times S^2$ if and only if

$$[2\rho_4(c_2 c_{n-2}) + \rho_4 c_1 \cdot p(v_{2k})](\omega) = 0 \tag{1.2}$$

In Section 2, we briefly discuss the two basic tools we use in the construction of fibrations. We recall, for reference, some standard facts about projective bundles and state two so-called "pullback" theorems. In Section 3, we actually construct manifolds and obtain a series of classes in $\Omega^u_*$ fibered over $S^4$. In Section 4, we prove Theorem 1 and its corollaries and in Section 5, we prove Theorem 2 and give evidence for the corresponding conjecture.

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## 2. Preliminaries

Let $\xi$ be an $n$-dimensional complex vector bundle over a manifold $B$. The complex projective bundle $\mathbf{C}P(\xi)$ is the manifold consisting of all lines in the fibers of $\xi$. $\mathbf{C}P(\xi)$ fibers over $B$ with fiber $\mathbf{C}P(n-1)$ and supports a canonical line bundle $\lambda$.

Let $b = c_1(\lambda) \in H^2(\mathbf{C}P(\xi))$. It is well known [10, p. 62] that $H^*(\mathbf{C}P(\xi))$ is a free $H^*(B)$ module on generators $1, b, \ldots, b^{n-1}$ with the relation

$$\sum_{i=0}^{n} b^i c_{n-i}(\xi) = 0 \tag{2.1}$$

Let $\tau(M)$ denote the tangent bundle of the manifold $M$. Szczarba [12] shows that for the fibration $p\colon \mathbf{C}P(\xi) \to B$,

$$\tau(\mathbf{C}P(\xi)) \oplus \theta^1 = p^*\tau(B) \oplus (p^*\xi \otimes \bar{\lambda}) \tag{2.2}$$

where $\theta^1$ denotes the trivial complex line bundle and $\bar{\lambda}$ denotes the conjugate bundle of $\lambda$. If $B$ is stably almost complex, we get an induced stably almost complex structure on $\mathbf{C}P(\xi)$.

For convenience, the "$p^*$" and the words "stably almost" will henceforth often be suppressed.

Now with actual fibrations in hand, we turn to "pullbacks". The first proposition and its corollary, due to Nelson [8, Lemma 2.1], are the complex analogues of two of a series of pullback results by Stong [11, Proposition 2.1]. (See also [1, Corollary 1.3] for a more general result). It is the corollary that gives $\mathrm{Fib}_{\Omega_*^u}(S^n)$ its essential significance in the overall problem of finding $\mathrm{Fib}_{\Omega_*^u}(N)$ for arbitrary.

Proposition 2.3. *If for $N$ and $N'$ stably almost complex, there exists a map $f\colon N' \to N$ complex bordant to the identity, then*

$$\mathrm{Fib}_{\Omega_*^u}(N) \subset \mathrm{Fib}_{\Omega_*^u}(N').$$

Corollary 2.4. *$\mathrm{Fib}_{\Omega_*^u}(S^n) \subset \mathrm{Fib}_{\Omega_*^u}(N^q)$ for all stably almost complex manifolds $N^q$ with $q \le n$.*

By pulling back covering spaces we get the following:

Proposition 2.5. *Let $\pi\colon \tilde{B} \to B$ be an $n$-fold covering of a closed manifold $B$. Then $n \cdot \mathrm{Fib}_{\Omega_*^u}(B) \subset \mathrm{Fib}_{\Omega_*^u}(\tilde{B})$.*

*Proof.* Let $\omega \in \mathrm{Fib}_{\Omega_*^u}(B)$ be represented by the fibration $f\colon M \to B$. Then $f^*(\tilde{B})$ is an $n$-fold cover of $M$ and fibers over $\tilde{B}$. A routine check of Chern numbers shows that $f^*(\tilde{B})$ represents $n\omega \in \Omega_*^u$. ∎

## 3. Constructing fibrations over $S^4$

In this section we prove, through a series of lemmas, the following technical result:

PROPOSITION 3. *There exist indecomposables $G_n \in \Omega^u_{2n}$, which fiber over $S^4$ for all $n$ except $n = 1, 2, 4$ and 5. Furthermore, all possible products of these indecomposables fiber over $S^4$ <u>except</u> for the following:*

$CP(1)^2$,
$aCP(1)CP(2)$ *where* $a \not\equiv 0 \pmod 4$,
$aG_4$ *where* $a \not\equiv 0 \pmod 3$,
$aCP(2)^k$ *where* $a \in \mathbf{Z}$,
$aCP(1)CP(2)^k$ $(k \geq 2)$ *where* $a \not\equiv 0 \pmod 2$,
$aCP(2)^k G_5$ *where* $a \not\equiv 0 \pmod 2$,

To put this result in better perspective, we note that the non-fibering classes listed above are precisely those classes detected by the set of obstructions given by Theorem 1.

The following notation will be used throughout this section:

Let $\lambda$ denote the canonical line bundle over $CP(n)$. Let $\eta$ and $\xi$ denote appropriate canonical line bundles over given projective bundles. Let $\gamma$ denote the quaternionic line bundle over $S^4 = HP(1)$. (Note that $\gamma$ is a complex 2-bundle over $S^4$). Let $\theta^n$ denote the trivial $n$-bundle. Also, let $\mathscr{F}$ denote $\mathrm{Fib}_{\Omega^u_*}(S^4)$. We will use no notation to distinguish between a manifold and its cobordism class. The context should make clear which we mean.

Before we begin construction, we note the following standard fact about complex bundles: Let $\alpha$ and

$$\beta = \beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n$$

be complex bundles whose underlying real bundles are isomorphic. We can equip $\alpha$ with different complex structures, induced by the real isomorphism, by conjugating summands of $\beta$. Let $\bar{\beta}$ denote the conjugate complex bundle of $\beta$.

Recall now, that the complex cobordism ring $\Omega^u_*$, is a polynomial algebra over $Z$ on generators $G_n$ of dimension $2n$, one for each positive integer $n$. The class of a stably almost complex manifold $G_n$, may be chosen as the $2n$-dimensional generator (see [10, p. 128]) if and only if

$$s_n[G_n] = \begin{cases} \pm p & \text{if } n+1 \text{ is a power of the prime } p \\ \pm 1 & \text{otherwise,} \end{cases}$$

where $s_n$ can be defined as follows: If the total Chern class $c$, of the complex bundle $E$ over $B$ is given by

$$c(E) = (1 + t_1)(1 + t_2) \cdots (1 + t_k) \quad \text{for } t_i \in H^2(M)$$

then

$$s_n(E) = t_1^n + t_2^n + \cdots + t_k^n \tag{3.1}$$

and $s_n$ is natural. If the stable tangent bundle $\tau$, of a manifold $M$, has a complex structure, then $s_n(M) = s_n(\tau)$.

LEMMA 3.2. *There exist indecomposables, $G_n \in \Omega^u_{2n}$, which fiber over $CP(3)$ for $n = 3$, $n = 6$ and $n = 7$.*

*Proof.* (i) $n = 3$. Let $G_3 = CP(3)$ with the non-standard complex structure given by

$$\tau CP(3) = \lambda \oplus \bar{\lambda} \oplus \bar{\lambda} \oplus \bar{\lambda} \quad \text{stably.}$$

Then the total Chern class of $G_3$ is given by

$$c(G_3) = (1 - a)(1 + a)^3$$

where $a = c_1(\bar{\lambda})$. And so by (3.1), $s_3(G_3) = 2a^3$. Thus $s_3[G_3] = 2$.

(ii) $n = 6$. Let $G_6$ be the projective bundle

$$\begin{array}{c} CP(\bar{\lambda} \oplus \theta^3) \\ \downarrow \\ CP(3) \end{array}$$

with the non-standard structure given by

$$\tau CP(\bar{\lambda} \oplus \theta^3) = 2\bar{\lambda} \oplus 2\lambda \oplus (\bar{\lambda} \otimes \bar{\eta}) \oplus \bar{\eta} \oplus 2\eta \quad \text{stably.}$$

Letting $a = c_1(\bar{\lambda})$ and $b = c_1(\eta)$, we have that

$$c(G_6) = (1 + a)^2(1 - a)^2(1 + a - b)(1 - b)(1 + b)^2.$$

By (3.1), along with (2.1),

$$s_6(G_6) = 2a^6 + 2(-a)^6 + (a - b)^6 + (-b)^6 + 2b^6$$

and $s_6[G_6] = -7$.

(iii) $n = 7$. Consider the projective bundle

$$\begin{array}{c} CP(\bar{\lambda} \oplus \theta^4) \\ \downarrow \\ CP(3) \end{array}$$

Let $CP'(\bar{\lambda} \oplus \theta^4)$ be $CP(\bar{\lambda} \oplus \theta^4)$ with the non-standard complex structure given by

$$\tau CP(\bar{\lambda} \oplus \theta^4) = 2\bar{\lambda} \oplus 2\lambda \oplus (\bar{\lambda} \otimes \eta) \oplus \bar{\eta} \oplus 3\eta \quad \text{stably.}$$

and let $CP''(\bar{\lambda} \oplus \theta^4)$ be $CP(\bar{\lambda} \oplus \theta^4)$ with the non-standard structure given by

$$\tau CP(\bar{\lambda} \oplus \theta^4) = 2\bar{\lambda} \oplus 2\lambda \oplus (\bar{\lambda} \otimes \bar{\eta}) \oplus 2\bar{\eta} \oplus 2\eta \quad \text{stably.}$$

Let

$$G_7 = CP'(\bar{\lambda} \oplus \theta^4) - CP''(\bar{\lambda} \oplus \theta^4)$$

(the negation here and throughout this section is in the cobordism sense).

With $a = c_1(\bar{\lambda})$ and $b = c_1(\eta)$ we have that

$$cCP'(\bar{\lambda} \oplus \theta^4) = (1 + a)^2(1 - a)^2(1 + a - b)(1 - b)(1 + b)^3,$$

while

$$cCP''(\bar{\lambda} \oplus \theta^4) = (1 + a)^2(1 - a)^2(1 + a - b)(1 - b)^2(1 + b)^2.$$

Then, by (3.1) and (2.1), $s_7[G_7] = 22 - 20 = 2$. ∎

For $n \geq 8$, we obtain indecomposables fibered over $CP(3)$ through a general construction of indecomposables fibered over $CP(j)$. We note that Nelson [8] constructs such indecomposables $G_n \in \Omega^u_{2n}$ for $n$ "sufficiently large". However, here we require a precise bound.

LEMMA 3.3. *There exist indecomposables, $G_n \in \Omega^u_{2n}$, which fiber over $CP(j)$ for all $n \geq 2j + 2$.*

*Proof.* We first note the following algebraic fact:

$$\text{g.c.d.}\left\{\binom{n+1}{i} \,\middle|\, 1 \leq i \leq n-1\right\} = \begin{cases} p & \text{if } n+1 \text{ is a power of the prime } p \\ 1 & \text{otherwise.} \end{cases}$$

This reduces our problem to that of finding a set of fibering manifolds $M^n_i$, in each dimension $n \geq 2j + 2$, with

$$s_n[M^n_i] = \binom{n+1}{i} \quad \text{for } 1 \leq i \leq n-1.$$

Let $\bar{N}^n_0$ be the projective bundle

$$\begin{array}{c} CP(\bar{\lambda} \oplus \theta^{n-j}) \\ \downarrow \\ CP\,(j) \end{array}$$

and let $a = c_1(\bar{\lambda})$ and $b = c_1(\eta)$, where $\eta$ is the canonical line bundle over $CP(\bar{\lambda} \oplus \theta^{n-j})$. Then

$$c(\bar{N}^n_0) = (1 + a)^{j+1}(1 + a - b)(1 - b)^{n-j}.$$

So, by (3.1) and (2.1),

$$s_n[\bar{N}^n_0] = (-1)^n\left[\binom{n}{2} - \binom{n}{3} + \cdots + (-1)^j\binom{n}{j} - j + 1\right]$$

Let $\bar{N}^n_i$ $(1 \leq i \leq n - j - 1)$ be the iterated projective bundle

$$\begin{array}{c} CP(\eta \oplus \theta^{n-j-i}) \\ \downarrow \\ CP(\bar{\lambda} \oplus \theta^i) \\ \downarrow \\ CP(j) \end{array}$$

and let $a = c_1(\bar{\lambda})$, $b = c_1(\eta)$ and $c = c_1(\xi)$ where $\xi$ is the canonical line bundle over $CP(\eta \oplus \theta^{n-j-1})$. Then

$$c(\bar{N}_i^n) = (1 + a)^{j+1}(1 + a - b)(1 - b)^i(1 - c + b)(1 - c)^{n-j-i}.$$

So, by (3.1) and (2.1),

$$s_n[\bar{N}_i^n] = (-1)^n\left[\binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{i+j}\binom{n}{i+j} - (i+j) + 1\right].$$

Let

$$M_i^n = (-1)^{i+n}[\bar{N}_{i-j} - 2\bar{N}_{i-j-1} + \bar{N}_{i-j-2}] \quad \text{for } j + 2 \leq i \leq n - 1.$$

Then

$$s_n[M_i^n] = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i} \quad \text{for } j + 2 \leq i \leq n - 1$$

For $2 \leq i \leq j + 1$, let $M_i^n = M_{n+1-i}^n$. Note that for $2 \leq i \leq j + 1$, we have $n - j \leq n + 1 - i \leq n - 1$ and since $n \geq 2j + 2$, we have $j + 2 \leq n + 1 - i \leq n - 1$. Thus $M_i^n$ is well defined for $2 \leq i \leq n - 1$ and

$$s_n[M_i^n] = \binom{n+1}{i} \quad \text{for } 2 \leq i \leq n - 1.$$

All we need now is a fibering manifold $M_1^n$, with $s_n(M_1^n) = n + 1$. Let

$$\bar{M}_1^n = (\bar{N}_{n-j-2} - \bar{N}_{n-j-1}).$$

If $n$ is even, $s_n[\bar{M}_1^n] = n + 1$ and we let $M_1^n = \bar{M}_1^n$. If $n$ is odd, then $s_n[\bar{M}_1^n] = n - 1$ and we let

$$M_1^n = \bar{M}_1^n + \tilde{N}_0^n - \bar{N}_0^n$$

where $\tilde{N}_0^n$ is $\bar{N}_0^n$ with the non-standard complex structure given by

$$\tau CP(\bar{\lambda} \oplus \theta^{n-j}) = (j + 1)\bar{\lambda} \oplus (\bar{\lambda} \otimes \bar{\eta}) \oplus (n - j - 1)\bar{\eta} \oplus \eta \quad \text{stably.}$$

Note that $s_n[\tilde{N}_0^n] = s_n[\bar{N}_0^n] + 2$. So $s_n[M_1^n] = (n - 1) + 2 = n + 1$.

This completes the proof of Lemma 3.3. ∎

At this point we observe that there is the standard fibration $f$: $CP(3) \to S^4 = HP(1)$. Thus, anything that fibers over $CP(3)$, fibers over $S^4$. And so we have:

COROLLARY 3.4. *There exist indecomposables, $G_n \in \Omega_{2n}^u$, which fiber over $S^4$ for $n = 3$, and $n \geq 6$.*

We make the following choices of indecomposables for $n = 1$, 2, 4 and 5.

$$G_1 = CP(1), \qquad G_2 = CP(2) \quad \text{and} \quad G_4 = CP(4) - CP(2)^2.$$

Consider the iterated projective bundle

$$\begin{array}{c} CP(\eta \oplus \theta^3) \\ \downarrow \\ CP(\lambda \oplus \theta^1) \\ \downarrow \\ CP(1). \end{array}$$

Let $CP'(\eta \oplus \theta^3)$ be $CP(\eta \oplus \theta^3)$ with the non-standard complex structure given by

$$\tau CP(\eta \oplus \theta^3) = \bar{\lambda} \oplus \bar{\lambda} \oplus (\lambda \otimes \bar{\eta}) \oplus \bar{\eta} \oplus (\eta \otimes \bar{\xi}) \oplus \bar{\xi} \oplus 2\xi \quad \text{stably}$$

($\xi$ being the canonical line bundle over $CP(\eta \oplus \theta^3)$). Let

$$G_5 = CP'(\eta \oplus \theta^3) - 2CP'(5)$$

where $CP'(5)$ is $CP(5)$ with the non-standard structure given by $\tau CP(5) = 4\bar{\lambda} \oplus 2\lambda$ stably.

That $G_1$, $G_2$, and $G_4$ are indecomposables is clear.

LEMMA 3.5. *The class $G_5$, defined above, is an indecomposable in $\Omega^u_{10}$ with $w_4 w_6[G_5] = 1$.*

*Proof.* Letting $a = c_1(\bar{\lambda})$, $b = c_1(\eta)$ and $c = c_1(\xi)$, we have that

$$cCP'(\eta \oplus \theta^3) = (1 + a)^2(1 - a - b)(1 - b)(1 + b - c)(1 - c)(1 + c)^2$$

The result now follows by routine computation. ∎

Having dealt with the problem of finding indecomposables which fiber over $S^4$, we turn to the problem of which products of these indecomposables fiber over $S^4$.

Nelson [8, Theorem 3.12] determines that the classes $CP(1)^3$ and $4CP(1)CP(2)$ in $\Omega^u_6$ contain representatives which fiber over $CP(3)$. The result is obtained by determining all the possible stably almost complex structures that can be put on $CP(3)$. (See also [9, p. 19]). For our purpose, we have:

LEMMA 3.6. *In $\Omega^u_6$, $CP(1)^3$ and $4CP(1)CP(2) \in \mathscr{F}$.*

From here we move up one dimension at a time, trying to account for all possible decomposables. Recall that for each free abelian group $\Omega^u_{2n}$, we have a basis $\mathscr{B}$, consisting of $G_n$ and all products of lower dimensional indecomposables $G_i$, $i < n$. By expressing classes known to be in $\mathscr{F}$ in terms of this basis, we can hope to get information about the basis classes themselves.

We use this technique in investigating dimensions eight through sixteen.

LEMMA 3.7. *In $\Omega^u_8$, $CP(1)^2CP(2)$ and $3G_4 \in \mathscr{F}$.*

*Proof.* Consider

$$\begin{array}{c} CP(\eta \oplus \theta^1) \\ \downarrow \\ CP(\gamma) \\ \downarrow \\ S^4 \end{array}$$

where $S^4$ carries the trivial stable complex structure. Calculating by hand, we find that

$$CP(\eta \oplus \theta^1) = -5CP(1)^4 + 5CP(1)^2CP(2) - 5CP(1)G_3.$$

Since $CP(\eta \oplus \theta^1)$, $CP(1)^4$ and $CP(1)G_3 \in \mathscr{F}$, we get that $5CP(1)^2CP(2) \in \mathscr{F}$. But Lemma 3.6 yields $4CP(1)^2CP(2) \in \mathscr{F}$. Subtracting, we get that $CP(1)^2CP(2) \in \mathscr{F}$.

Now consider the complex projective bundle $CP(\gamma \oplus \theta^1)$ over $S^4$, where $S^4$ carries the trivial stable complex structure. Letting $a = c_2(\gamma)$ and $b = c_1(\eta)$, where $\eta$ is the canonical line bundle over $CP(\gamma \oplus \theta^1)$, we have that

$$cCP(\gamma \oplus \theta^1) = ((1 - b)^2 + a)(1 - b).$$

Expanding and using Girard's formula

$$s_4 = c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4$$

we find that $s_4[CP(\gamma \oplus \theta^1)] = -15$. Then

$$CP(\gamma \oplus \theta^1) = -3G_4 + aCP(1)^4 + bCP(1)^2CP(2) + cCP(1)G_3 + dCP(2)^2$$

where $a, b, c, d, \in Z$. Since the signature $\sigma[(CP(\gamma \oplus \theta^1)] = 0$, we have that $d = 0$. Then the fact that $CP(\gamma \oplus \theta^1)$, $CP(1)^4$, $CP(1)^2CP(2)$ and $CP(1)G_3 \in \mathscr{F}$, gives the result that $3G_4 \in \mathscr{F}$. ▮

LEMMA 3.8. *In $\Omega^u_{10}$, $2G_5$, $CP(1)G_4$ and $2CP(1)CP(2)^2 \in \mathscr{F}$.*

*Proof.* Consider the complex projective bundle $CP(\bar{\lambda} \oplus \theta^2)$, over $CP(3)$. Calculating by hand, we find that

$$CP(\bar{\lambda} \oplus \theta^2) = -106CP(1)CP(2)^2 - 25CP(1)G_4 + 2G_5 + Y$$

where $Y \in \mathscr{F}$. Let $CP'(\bar{\lambda} \oplus \theta^2)$ be $CP(\bar{\lambda} \oplus \theta^2)$ with the non-standard complex structure given by

$$\tau CP(\bar{\lambda} \oplus \theta^2) = 4\bar{\lambda} \oplus (\bar{\lambda} \otimes \bar{\eta}) \oplus \bar{\eta} \oplus \eta \quad \text{stably,}$$

where $\eta$ is the canonical line bundle over $CP(\bar{\lambda} \oplus \theta^2)$. We find that

$$CP'(\bar{\lambda} \oplus \theta^2) = -214CP(1)CP(2)^2 - 52CP(1)G_4 + 4G_5 + W$$

where $W \in \mathscr{F}$. Substracting, we get that

$$-108CP(1)CP(2)^2 - 27CP(1)G_4 + 2G_5 \in \mathscr{F}.$$

But $4CP(1)CP(2)$ and $3G_4 \in \mathscr{F}$. So $2G_5 \in \mathscr{F}$.

Then $-106CP(1)CP(2)^2 - 25CP(1)G_4 \in \mathscr{F}$ and so

$$2CP(1)CP(2)^2 - CP(1)G_4 \in \mathscr{F}.$$

Since $3CP(1)G_4 \in \mathscr{F}$, we get that $CP(1)G_4 \in \mathscr{F}$, which in turn gives us that $2CP(1)CP(2)^2 \in \mathscr{F}$. ∎

Moving on to the $12th$ dimension, we note that $\Omega^u_{12}$ has a basis consisting of eleven classes. At this stage, calculating by hand becomes just about impossible. Thus in an effort to maintain our sanity, we prove the next two lemmas by using a computer to solve our matrix equations.

LEMMA 3.9. *In $\Omega^u_{12}$, $CP(2)G_4$ and $CP(1)G_5 \in \mathscr{F}$.*

*Proof.* Consider

$$\begin{array}{c} CP(\eta \oplus \theta^3) \\ \downarrow \\ CP(\gamma) \\ \downarrow \\ S^4. \end{array}$$

Let $CP'(\eta \oplus \theta^3)$ be $CP(\eta \oplus \theta^3)$ with the non-standard complex structure given by

$$\tau CP(\eta \oplus \theta^3) = \theta^2 \oplus (\gamma \otimes \bar{\eta}) \oplus (\eta \otimes \bar{\xi}) \oplus \bar{\xi} \oplus 2\xi \quad \text{stably}$$

where $\xi$ is the canonical line bundle over $CP(\eta \oplus \theta^3)$. We find, by computer, that

$$CP'(\eta \oplus \theta^3) = -5CP(1)G_5 + Y$$

where $Y \in \mathscr{F}$. So $5CP(1)G_5 \in \mathscr{F}$ and thus, since $2G_5 \in \mathscr{F}$, we get that $CP(1)G_5 \in \mathscr{F}$.

Let $CP'(\lambda_1 \oplus \lambda_2 \oplus \theta^1)$ be the complex projective bundle $CP(\lambda_1 \oplus \lambda_2 \oplus \theta^1)$ over $CP(3) \times CP(1)$, (where $\lambda_1$ and $\lambda_2$ are the pullbacks of the appropriate canonical line bundles), with the non-standard structure given by

$$\tau CP(\lambda_1 \oplus \lambda_2 \oplus \theta^1) = 2\bar{\lambda}_1 \oplus 2\lambda_1 \oplus \bar{\lambda}_2 \oplus \lambda_2 \oplus (\lambda_1 \otimes \bar{\eta}) \oplus (\lambda_2 \otimes \bar{\eta}) \oplus \eta \quad \text{stably}$$

where $\eta$ is the canonical line bundle over $CP(\lambda_1 \oplus \lambda_2 \oplus \theta^1)$. Then we find, by computer, that

$$CP'(\lambda_1 \oplus \lambda_2 \oplus \theta^1) = 2CP(2)G_4 + W$$

where $W \in \mathscr{F}$. Then, $2CP(2)G_4 \in \mathscr{F}$, and, with $3G_4 \in \mathscr{F}$, we have $CP(2)G_4 \in \mathscr{F}$. ∎

LEMMA 3.10. *In* $\Omega^u_{16}$, $G_4^2 \in \mathscr{F}$.

*Proof.* Consider

$$\begin{array}{c} CP(\eta \oplus \theta^2) \\ \downarrow \\ CP(\gamma \oplus \theta^3) \\ \downarrow \\ S^4. \end{array}$$

Let $CP'(\eta \oplus \theta^2)$ be $CP(\eta \oplus \theta^2)$ with the non-standard complex structure given by

$$\tau CP(\eta \oplus \theta^2) = \theta^2 \oplus (\gamma \otimes \bar{\eta}) \oplus \bar{\eta} \oplus 2\eta \oplus (\eta \otimes \bar{\xi}) \oplus 2\xi \quad \text{stably.}$$

We find, by computer, that

$$CP'(\eta \oplus \theta^2) = 35G_4^2 + Y$$

where $Y \in \mathscr{F}$. Thus $35G_4^2 \in \mathscr{F}$. Since $3G_4 \in \mathscr{F}$, we have $36G_4^2 \in \mathscr{F}$, and so $G_4^2 \in \mathscr{F}$. ∎

LEMMA 3.11. *In* $\Omega^u_{18}$, $G_4 G_5 \in \mathscr{F}$.

*Proof.* Since both $3G_4 \in \mathscr{F}$ and $2G_5 \in \mathscr{F}$, we have that both $3G_4G_5 \in \mathscr{F}$ and $2G_4G_5 \in \mathscr{F}$. The result follows trivially. ∎

Proceeding to an investigation of $\Omega^u_{20}$, we are now faced with a basis consisting of forty-two classes. At this point, with even the computer being driven nuts by the quantity and size of the numbers involved in our matrix computations, we turn to a new method for determining the fibering status of $G_5^2$. Its use in the next proof will serve to describe it.

LEMMA 3.12. *In* $\Omega^u_{20}$, $G_5^2 \in \mathscr{F}$.

*Proof.* As usual, we consider the basis $\mathscr{B}$ of $\Omega^u_{20}$, consisting of all appropriate products of the indecomposables $G_n$, constructed earlier. We claim the following fact: With the possible exception of $CP(2)^5$, $G_5^2$ is the only class in $\mathscr{B}$ for which the characteristic number

$$c_4c_6 + c_3c_7 + c_2c_4^2 + c_{10} + s_{10} \equiv 1 \pmod 2.$$

Letting $d$ denote the above characteristic class, it is clear that for any class $x_{20}$ in $\mathscr{B}$, containing a factor of $CP(1)$ or $G_3$, we have $d[X_{20}] \equiv 0 \pmod 2$. This is because both $CP(1)$ and $G_3$ are unoriented corbordant to zero. The numbers of the other classes are checked by routine computation.

Now consider

$$\begin{array}{c} CP(\eta \oplus \theta^5) \\ \downarrow \\ CP(\bar{\lambda} \oplus \theta^2) \\ \downarrow \\ CP(3). \end{array}$$

Let $CP'(\eta \oplus \theta^5)$ be $CP(\eta \oplus \theta^5)$ with the non-standard complex structure given by

$$\tau CP(\eta \oplus \theta^5) = 2\bar{\lambda} \oplus 2\lambda \oplus (\bar{\lambda} \otimes \eta) \oplus \bar{\eta} \oplus \eta \oplus (\eta \otimes \bar{\xi}) \oplus 2\bar{\xi} \oplus 3\xi \quad \text{stably},$$

where $\xi$ is the canonical line bundle over $CP(\eta \oplus \theta^5)$. A direct computation shows that $d[CP'(\eta \oplus \theta^5)] \equiv 1 \pmod 2$. This, together with the observation that

$$\mathscr{B} - \{CP(2)^5, G_5^2\} \subset \mathscr{F},$$

yields

$$CP'(\eta \oplus \theta^5) = (2k + 1)G_5^2 + aCP(2)^5 + Y,$$

where $k$, $a \in Z$ and $Y \in \mathscr{F}$. But since $\sigma[CP'(\eta \oplus \theta^5)] = 0$, we have $a = 0$, and consequently $(2k + 1)G_5^2 \in \mathscr{F}$. Since we already know that $2G_5^2 \in \mathscr{F}$ we have that $G_5^2 \in \mathscr{F}$. ∎

## 4. Proof of Theorem 1

*Proof.* We first note that all obstructions are shown to hold for fibrations over $S^3$, while all sufficient conditions pertain to fibrations over $S^4$. Since the two sets of conditions are identical and since $\mathrm{Fib}_{\Omega_*^u}(S^4) \subset \mathrm{Fib}_{\Omega_*^u}(S^3)$ by Corollary 2.4, it follows that $\mathrm{Fib}_{\Omega_*^u}(S^4) = \mathrm{Fib}_{\Omega_*^u}(S^3)$.

(i) ($\Rightarrow$) Let $p$: $M^4 \to S^3$ be a fibration. Since $\pi_1(S^3) = 0$, we have (see [2]) $\sigma(M^4) = \sigma(S^3) \cdot \sigma(F) = 0$. A trivial inspection of the Serre spectral sequence shows that $H^2(M^4) = 0$. Then $c_1(M^4) = 0$ and so $c_1^2(M^4) = 0$. Now by the Hirzebruch Signature Theorem [3], we know that $\sigma(M^4) = \frac{1}{3}p_1[M^4]$, where $p_1 \in H^4(M^4)$ is the first Pontrjagin class, and so $p_1[M^4] = 0$. Since $p_1 = c_1^2 - 2c_2$, we get that $(c_1^2 - 2c_2)[M^4] = 0$ and so $c_2[M^4] = 0$. Thus $M^4$ is $\Omega^u$-corbordant to zero.

($\Leftarrow$) $\omega = 0$ fibres over $S^4$ by merely choosing the empty manifold as a representative.

(ii) ($\Rightarrow$) Let $p$: $M^6 \to S^3$ be a fibration with fiber $F^3$. Consider the Wang sequence in $Z_2$-coefficients:

$$H^{-1}(F^3; Z_2) \to H^2(M^6; Z_2) \xrightarrow{i^*} H^2(F^3; Z_2) \to H^0(F^3; Z_2) \cdots$$

where $i$: $F^3 \to M^6$ is the inclusion. Clearly, $i^*$ is a monomorphism on $H^2(M^6; Z_2)$ and since $i^*(v_2(M^6)) = v_2(F^3) = 0$, we see that $v_2(M^6) = 0$. But by the Wu formula and the fact that $M$ is stably almost complex (so all of its odd Stiefel-Whitney classes vanish), we see that $w_2(M^6) = v_2(M^6) = 0$. Hence $c_1(M_6) = 2z$ for some $z \in H^2(M^6)$. And so $c_1^3[M^6] \equiv 0 \pmod 8$.

($\Leftarrow$) Nelson [8, Theorem 3.12] proves that $c_1^3(\omega) \equiv 0 \pmod 8$ implies that $\omega$ fibers over $CP(3)$. (See also [9, p. 19]). But $CP(3)$ fibers over $S^4$.

(iii) ($\Rightarrow$) Let $p: M^8 \to S^3$ be a fibration with fiber, $F^5$. As in part (i), $\sigma(M^8) = 0$.

Now, using standard formulas for the $s$-class $s_4$, and the Pontrjagin classes $p_1$ and $p_2$, in terms of the Chern classes of $M$, we proceed as follows:

$$s_4[M_8] = (c_1^4 - 4c_1^2 c_2 + 4c_1 c_3 + 2c_2^2 - 4c_4)[M^8],$$

$$\begin{aligned}\sigma(M^8) &= \tfrac{1}{45}(7p_2 - p_1^2)[M^8] \quad \text{(by the Hirzebruch Signature Theorem)}\\ &= \tfrac{1}{45}(7(c_2^2 - 2c_1 c_3 + 2c_4) - (c_1^2 - 2c_2)^2)[M^8]\\ &= \tfrac{1}{45}(c_1^4 + 4c_1^2 c_2 - 14c_1 c_3 + 3c_2^2 + 14c_4)[M^8]\end{aligned}$$

Since $\sigma(M^8) = 0$, $s_4[M^8] = s_4[M^8] + 45\sigma(M^8)$. Then, by simply adding the two expressions above, we have

$$s_4[M^8] = (5c_2^2 - 10c_1 c_3 + 10c_4)[M^8] = 5p_2[M^8].$$

But $\sigma(M^8) = 0$ implies that $p_2[M^8] = \frac{1}{7}p_1^2[M^8]$. Thus

$$s_4[M^8] = \tfrac{5}{7}p_1^2[M^8].$$

We claim that $p_1^2[M^8] \equiv 0 \pmod 3$. For, let

$$\rho_3: H^*(\quad; Z) \to H^*(\quad; Z_3)$$

be reduction mod 3. Letting $\mathscr{P}^i: H^j(X; Z_3) \to H^{j+4i}(X; Z_3)$ be the Steenrod 3rd power operation, we have a mod 3 Wu-class, $\bar{v}_i$, characterized by the identity

$$\langle \mathscr{P}^i x, \mu \rangle = \langle x \cup \bar{v}_i, \mu \rangle \quad \text{for all } x \in H^{n-4i}(X; Z_3)$$

where $\mu$ denotes the fundamental class of the manifold $X$. Wu [13] shows that $\rho_3 p_1(X) = \bar{v}_1(X)$. Note that if $X = F^5$, we have that $\bar{v}_1(F^5) = 0$ for dimensional reasons. (If $x \in H^1(F^5; Z_3)$, then $\mathscr{P}^1 x = 0$). So $\rho_3 p_1(F^5) = 0$. Now consider the (mod 3) Wang sequence

$$H^1(F^5; Z_3) \xrightarrow{j^*} H^4(M^8; Z_3) \xrightarrow{i^*} H^4(F^5; Z_3) \to H^2(F^5; Z_3) \cdots$$

Since $i^*(\rho_3 p_1(M^8)) = \rho_3 p_1(F^5) = 0$, we have $\rho_3 p_1(M^8) = j^*(x)$ for some $x \in H^1(F^5; Z_3)$. Then $\rho_3 p_1^2(M^8) = [j^*(x)]^2 = 0$. Hence $p_1^2(M^8) \equiv 0 \pmod 3$. And so $s_4[M^8] \equiv 0 \pmod 3$.

($\Leftarrow$) By Lemmas 3.4–3.12, we know that

$$M^8 = aG_4 + bCP(2)^2 + Y$$

where $a = 0$, 1 or 2; $b \in Z$, and $Y \in \mathscr{F}$. By definition of $G^4$, we have $\sigma(G_4) = 0$. Thus, if $\sigma(M^8) = 0$, we get $b = 0$. Furthermore, we've just shown that $s_4[Y] \equiv 0 \pmod 3$. Thus if $s_4[M^8] \equiv 0 \pmod 3$, we get $a = 0$. Given both conditions, we have $M^8 = Y \in \mathscr{F}$.

(iv) (a) Let $n = 2k$.

($\Rightarrow$) Same as in the proofs of parts (i) and (iii).

($\Leftarrow$) By Lemmas 3.4–3.12, for $n \geq 10$, we have $M^{4k} = aCP(2)^k + Y$, where $a \in Z$ and $Y \in \mathscr{F}$. Then $\sigma(M^{4k}) = 0$ implies that $a = 0$ which gives $M^{4k} = Y \in \mathscr{F}$.

(b) Let $n = 2k + 1$.

($\Rightarrow$) Let $p\colon M^{4k+2} \to S^3$ be a fibration, with fiber $F^{4k-1}$. Consider the exact sequence

$$H^{2k}(M, F; Z_2) \xrightarrow{j^*} H^{2k}(M^{4k+2}; Z_2) \xrightarrow{i^*} H^{2k}(F^{4k-1}; Z_2) \to \cdots.$$

Note that

$$H^{2k}(M, F; Z_2) = H^{2k}(M, F \times D^3; Z_2) = H^{2k}(F \times D^3, F \times S^2; Z_2),$$

by excision. So

$$H^{2k}(M, F; Z_2) = H^{2k-3}(F; Z_2) \otimes H^3(D^3, S^2; Z_2).$$

Now, since $i^*(v_{2k}(M^{4k+2})) = v_{2k}(F^{4k-1}) = 0$, we have

$$v_{2k}(M^{4k+2}) = j^*(x \otimes a) \quad \text{for some } x \in H^{2k-3}(F; Z_2) \tag{4.1}$$

where $a \in H^3(D^3, S^2; Z_2)$ denotes the generator. By the Wu formula and the fact that for an orientable manifold, all odd Wu classes vanish,

$$w_{2n-4}(M) = w_{4k-2}(M) = Sq^{2k-2} v_{2k}.$$

So

$$w_{2n-4}(M) = j^*(Sq^{2k-2} x \otimes a) = 0$$

since $2k - 2 > 2k - 3$. Hence $w_4 w_{2n-4}(M) = 0$. (We note that for $M^{4k+2}$ orientable, $w_2^2 w_{4k-2}(M) = 0$ automatically).

Returning to (4.1) we have that $v_{2k}(M^{4k+2}) = j^*(x \otimes a)$. Then, applying the Pontrjagin square, we get

$$p(v_{2k}(M^{4k+2})) = j^* p(x \otimes a) = 0$$

by the product formula [14, Theorem 1]. Hence

$$\rho_4 c_1 \cdot p(v_{2k})[M^{4k+2}] = 0.$$

(We note that, since $v_{2k}$ is a universal class, $p(v_{2k})$ is the mod 4 reduction of some (not unique) characteristic Chern class, and so $\rho_4 c_1 \cdot p(v_{2k})[M]$ is indeed an $\Omega^u$-cobordism invariant.)

($\Leftarrow$) By Lemmas 3.4–3.12, we know that

$$M^{4k+2} = a \cdot CP(2)^{k-2} \cdot G_5 + b \cdot CP(1)CP(2)^k + Y,$$

where $a \in Z_2$; $b \in Z_2$ and $Y \in \mathscr{F}$. Noting that

$$w_4 w_{4k-2}[CP(2)^{k-2} \cdot G_5] = w_4 w_6[G_5],$$

we get, by application of Lemma 3.5, $w_4 w_{4k-2}[CP(2)^{k-2} \cdot G_5] = 1$.

Turning to $CP(1)CP(2)^k$, we have

$$v_{2k}(CP(1) \cdot CP(2)^k) = \rho_2(x_1 \cdots x_k)$$

where $x_i$ denotes the generator of $H^2(CP(2))$ and

$$\rho_2 \colon H^*(\ \ ; Z) \to H^*(\ \ ; Z_2)$$

denotes reduction (mod 2). Now $p(v_{2k}) = \rho_4(x_1^2 \cdots x_k^2)$, by the product formula for $p$ and the fact that $p(\rho_2 x_i) = \rho_4 x_i^2$. Furthermore

$$c_1(CP(1)CP(2)^k) = 2z + x_1 + \cdots + x_k$$

where $z$ denotes the generator of $H^2(CP(1))$. Then

$$\rho_4 c_1 \cdot p(v_{2k})[CP(1)CP(2)^k] = 2.$$

Hence, if $w_4 w_{2n-4}(M) = 0$ and $\rho_4 c_1 \cdot p(v_{2k})[M] = 0$, then $a = 0$ and $b = 0$. Then $M = Y \in \mathscr{F}$. ∎

*Remark* 4.2. Although the obstruction $\rho_4 c_1 \cdot p(v_{2k})(\omega) = 0$ is given in $Z_4$-coefficients, it is actually a "$Z_2$ condition" in the sense that for any s.a.c. manifold $M^{4k+2}$,

$$\rho_4 c_1 \cdot p(v_{2k})(M) \equiv 0 \pmod 2.$$

Observe that

$$\rho_2[\rho_4 c_1 \cdot p(v_{2k})](M) = v_2 \cdot v_{2k}^2(M) = Sq^2(v_{2k}^2)(M) = 0.$$

Similarly, the obstruction to fibering in $\Omega_6^u$, $c_1^3(\omega) \equiv 0 \pmod 8$, is actually a "$Z_4$ condition" in the sense that for any s.a.c. manifold $M^6$, $c_1^3[M^6] \equiv 0 \pmod 2$. Observe that

$$\rho_2(c_1^3)(M) = v_2^3(M) = Sq^2(v_2^2)(M) = 0.$$

Alternatively, we note that $\rho_4(c_1^2) = p(v_2) \in H^4(M^6; Z_4)$, and so we can simply refer back to the above paragraph.

*Remark* 4.3. The proof that shows that if $M^{4k+2}$ fibers over $S^3$, then $p(v_{2k})(M) = 0$, actually holds for manifolds of dimension $4k$ as well. The reason it doesn't appear as an obstruction in that case is that

$$p(v_{2k})[M^{4k}] \equiv \sigma(M^{4k}) \pmod 4.$$

For a proof, see [6].

*Remark* 4.4. The condition that $\rho_4 c_1 \cdot p(v_{2k})[M^{4k+2}] = 0$, can be expressed in terms of the Chern numbers of $M$. In that obtaining a general expression, would seem to serve only curiosity, we restrict ourselves to working out the following example:

$$\rho_4 c_1 \cdot p(v_4) = \rho_4(c_1 c_2^2 + c_1^5 + 2c_1^3 c_2).$$

Note first that $v_4 = w_4 + w_2^2$. Then,

$$p(v_4) = p(w_4 + w_2^2) = p(w_4) + p(w_2^2) + \theta_2(w_4 w_2^2)$$

where $\theta_2$ is the map induced by the inclusion of $Z_2$ into $Z_4$. According to Wu [14], we have

$$p(w_4) = \rho_4 p_2 - \theta_2(w_8) - \theta_2(w_2 w_6) \quad \text{and} \quad p(w_2) = \rho_4 p_1 - \theta_2(w_4).$$

So

$$\begin{aligned} p(v_4) &= \rho_4(p_2 + p_1^2) - \theta_2(w_8) - \theta_2(w_2 w_6) + \theta_2(w_2^2 w_4) \\ &= \rho_4[(c_2^2 - 2c_1 c_3 + 2c_4 + c_1^4 - 4c_1^2 c_2 + 4c_2^2) \\ &\quad - 2c_4 - 2c_1 c_3 + 2c_1^2 c_2] \\ &= \rho_4[c_2^2 + c_1^4 + 2c_1^2 c_2] \end{aligned}$$

Thus $\rho_4 c_1 \cdot p(v_4) = \rho_4(c_1 c_2^2 + c_1^5 + 2c_1^3 c_2)$. We can then state Theorem 2, for $\Omega_{10}^u$, as follows:

*$\omega \in \Omega_{10}^u$ fibers over $S^3$ or $S^4$ if and only if*

$$w_4 w_6(\omega) = 0 \quad \text{and} \quad c_1 c_2^2 + c_1^5 + 2c_1^3 c_2(\omega) \equiv 0 \pmod 4.$$

We turn now to the study of other 3 and 4-dimensional manifolds as base space. In particular, we prove Corollaries 1–4.

*Proof of Corollary* 1. Let $M^3$ be an arbitrary closed connected 3-manifold. Let $\tilde{M}^3$ be its orientable double cover. Clearly

$$\mathrm{Fib}_{\Omega_*^u}(\tilde{M}) \subset \mathrm{Fib}_{\Omega_*^u}(M).$$

It is a standard fact that every compact orientable 3-manifold is parallelizable [5, p. 148] and thus also s.a.c. Then by Corollary 2.4, we get $\mathrm{Fib}_{\Omega_*^u}(S^3) \subset \mathrm{Fib}_{\Omega_*^u}(\tilde{M})$ and the result follows.

Corollary 2 is simply a consequence of Corollary 1 and the fact that any manifold fibered over a 3-manifold has signature equal to zero.

*Proof of Corollary* 3. Let $\pi_1(M^3)$ be finite of odd order $k$ and let $\tilde{M}^3$ be the $k$-fold universal cover of $M^3$. An easy argument shows $\tilde{M}^3$ to be a homotopy 3-sphere. We now note (simply by checking Chern numbers (see [10, p. 144])) that if $f: S^3 \to \Sigma^3$ is a homotopy equivalence, then $f$ is complex bordant to the identity. Thus by Proposition 2.3,

$$\mathrm{Fib}_{\Omega_*^u}(\tilde{M}^3) \subset \mathrm{Fib}_{\Omega_*^u}(S^3)$$

and so $\mathrm{Fib}_{\Omega_*^u}(\tilde{M}^3) = \mathrm{Fib}_{\Omega_*^u}(S^3)$. Then by Proposition 2.5,

$$k \cdot \mathrm{Fib}_{\Omega_*^u}(M^3) \subset \mathrm{Fib}_{\Omega_*^u}(S^3). \tag{4.5}$$

But all torsion in $\Omega_*^u/\mathrm{Fib}_{\Omega_*^u}(S^3)$ is either even or 3-torsion (the 3-torsion occurring in $\Omega_8^u/\mathrm{Fib}_{\Omega_8^u}(S^3)$). Thus, (4.5) implies that

$$\mathrm{Fib}_{\Omega_{2n}^u}(M^3) \subset \mathrm{Fib}_{\Omega_{2n}^u}(S^3)$$

for all $n$ if $k \neq 3r$ and for all $n$ except $n = 8$ if $k = 3r$. This along with Corollary 1 gives the result. ∎

Corollary 4 is obtained simply by combining Corollaries 3 and 2.4.

## 5. Fiberings over $S^2 \times S^2$

*Proof of Theorem 2.* (i) ($\Rightarrow$) Let $p: M^6 \to S^2 \times S^2$ be a fibration with fiber $F^2$. Let $a$ and $b$ denote the generators of $H^2(S^2 \times S^2; Z_2)$. Since

$$i^*(v_2(M^6)) = v_2(F^2) = 0$$

(where $i: F^2 \to M^6$ is the inclusion and $v_2$ denotes the 2nd Wu class), we have

$$v_2(M^6) = r_1 p^* a + r_2 p^* b \quad \text{for some } r_1, r_2 \in Z_2.$$

Noting that $\rho_2 c_1(M^6) = v_2(M^6)$, we see that

$$c_1(M^6) = \bar{r}_1 p^* \bar{a} + \bar{r}_2 p^* \bar{b} + 2z$$

where $\bar{r}_1, \bar{r}_2 \in Z$, $\bar{a}$ and $\bar{b}$ are the generators of $H^2(S^2 \times S^2)$ and $z \in H^2(M^6)$. Then $c_1^3[M^6] \equiv 0 \pmod 4$.

($\Leftarrow$) Consider the complex projective bundle $CP(\lambda_1 \oplus \lambda_2)$ over $S^2 \times S^2$ ($S^2 \times S^2$ carrying the trivial stable complex structure), where $\lambda_i$ is the pullback of the canonical line bundle over $CP(1)$. Direct computation shows that $c_1^3(CP(\lambda_1 \oplus \lambda_2)) = 4$. Choosing $G_3 \in \Omega_6^u$ as in Section 3, we have

$$CP(\lambda_1 \oplus \lambda_2) = aCP(1)^3 + bCP(1)CP(2) + cG_3$$

where $a, b, c \in Z$. Since $CP(1)^3$, $4CP(1)CP(2)$ and $G_3$ all fiber over $S^4$, they fiber over $S^2 \times S^2$. Thus

$$CP(\lambda_1 \oplus \lambda_2) = bCP(1)CP(2) + Y$$

where $b \in Z_4$ and $Y \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$. But $c_1^3(CP(1)CP(2)) \equiv 2 \pmod 4$, while $c_1^3[Y] \equiv 0 \pmod 4$. Since

$$c_1^3(CP(\lambda_1 \oplus \lambda_2)) \equiv 0 \pmod 4,$$

we have $b = 2$. Hence $2CP(1)CP(2) \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$. Now for any class $M^6$, we have

$$M^6 = aCP(1)CP(2) + W \quad \text{where } a \in Z_2 \quad \text{and} \quad W \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2).$$

If $c_1^3(M^6) \equiv 0 \pmod 4$, then $a = 0$ and $M^6 = W$, giving

$$M^6 \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2).$$

(ii) ($\Rightarrow$) Trivial.
($\Leftarrow$) Clearly $CP(1)^2$ fibers over $S^2 \times S^2$. Note that

$$M^4 = aCP(2) + bCP(1)^2$$

where $a, b \in Z$. Thus if $\sigma(M^4) = 0$, then $a = 0$ and $M^4 \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$.
In higher dimensions, the result is a consequence of Theorem 2 and pulling back. ▮

The evidence for our conjecture, as to what the complete description of $\mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$ should be, is two-fold. First consider the iterated projective bundle

$$\begin{array}{c} CP(\eta \oplus \lambda_2 \oplus \theta^1) \\ \downarrow \\ CP(\lambda_1 \oplus \theta^1) \times S^2 \\ \downarrow \\ S^2 \times S^2 \end{array}$$

where $\lambda_1$ is the canonical line bundle over $S^2$, and $\eta$ and $\lambda_2$ are the pullbacks of the canonical line bundles over $CP(\lambda_1 \oplus \theta^1)$ and $S^2$ respectively. Let $CP'(\eta \oplus \lambda_2 \oplus \theta^1)$ be $CP(\eta \oplus \lambda_2 \oplus \theta^1)$ with the non-standard complex structure given by

$$\tau CP(\eta \oplus \lambda_2 \oplus \theta^1) = \theta^2 \oplus (\lambda_1 \otimes \bar{\eta}) \oplus \bar{\eta} \oplus (\eta \otimes \bar{\xi}) \oplus (\lambda_2 \otimes \bar{\xi}) \oplus \xi \quad \text{stably,}$$

where $\xi$ is the canonical line bundle over $CP(\eta \oplus \lambda_2 \oplus \theta^1)$. By routine computations, we note that while

$$\rho_4 c_1 \cdot p(v_4)[CP'(\eta \oplus \lambda_2 \oplus \theta^1)] \neq 0$$

and

$$w_4 w_6[CP'(\eta \oplus \lambda_2 \oplus \theta^1)] \neq 0$$

we do have

$$(2\rho_4(c_2 c_3) + \rho_4 c_1 \cdot p(v_4))[CP'(\eta \oplus \lambda_2 \oplus \theta^1)] = 0.$$

Furthermore, for $G_5$ as in Section 2, we have

$$CP'(\eta \oplus \lambda_2 \oplus \theta^1) = G_5 + CP(1)CP(2)^2 + Y$$

where $Y \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$. Thus $G_5 + CP(1)CP(2)^2 \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$. This, together with the fact obtained by direct computation), that, for $n = 2k + 1 \geq 5$,

$$(2\rho_4(c_2 c_{n-2}) + \rho_4 c_1 \cdot p(v_{2k})[CP(2)^{k-2}G_5 + CP(1)CP(2)^k] = 2, \tag{5.1}$$

yields the following:

*If the condition* (1.2) *is necessary for a class* $\omega \in \Omega_{4k+2}^u$ *to fiber over* $S^2 \times S^2$, *then it is also sufficient.*

Simply note that (by pulling back fibrations over $S^4$)

$$\omega = aCP(2)^{k-2}G_5 + bCP(1)CP(2)^k + Y$$

where $a, b \in Z_2$ and $Y \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$. Then, assuming that (1.2) holds for $Y$, and using (5.1), we get

$$[2\rho_4(c_2 c_{2k-1}) + \rho_4 c_1 \cdot p(v_{2k})](\omega) = 0 \Rightarrow a + b = 0,$$

which gives that $\omega \in \mathrm{Fib}_{\Omega_*^u}(S^2 \times S^2)$.

The second piece of evidence in support of the conjecture is a proof of the necessity of (1.2) in the case of very "nice" fibrations over $S^2 \times S^2$. Specifically, let

$$p\colon M^{4k+2} \to S^2 \times S^2$$

be a fibration with fiber $F$, with the following property:

(5.2) The fact that $v_{2k}(M)$ pulls back to zero on the fiber implies that

$$v_{2k}(M) = r_1 a \cdot x_1 + r_2 a \cdot x_2 + r_3 ab \cdot y$$

where $r_i \in Z_2$; $a$ and $b$ generate $H^2(S^2 \times S^2; Z_2)$;

$$x_1, x_2 \in H^{2k-2}(M; Z_2) \quad \text{and} \quad y \in H^{2k-4}(M; Z_2).$$

(In particular, if the Serre spectral sequence of the fibration has no non-zero differentials, (e.g., any iterated projective bundle), property (5.2) holds.)

Then applying the Pontrjagin square $p$, and using its properties we get

$$p(v_{2k})(M) = \theta_2(r_1 r_2 abx_1 x_2)$$

where $\theta_2$ is the map induced by the inclusion of $Z_2$ into $Z_4$. Since $\rho_2 c_1(M) = v_2(M)$, where $\rho_2$ is reduction mod 2, we have that

$$\rho_4 c_1 \cdot p(v_{2k})(M) = \theta_2(r_1 r_2 v_2 ab \cdot x_1 x_2) = \theta_2[r_1 r_2 ab \cdot Sq^2(x_1 x_2)] \tag{5.3}$$

Now,

$$\begin{aligned} w_4 w_{4k-2}(M) &= v_{2k} Sq^2 v_{2k}(M) \\ &= r_1 a x_1 + r_2 b x_2 + r_3 aby)(r_1 a \cdot Sq^2 x_1 \\ &\quad + r_2 b \cdot Sq^2 x_2 + r_3 ab \cdot Sq^2 y) \\ &= r_1 r_2 ab x_1 Sq^2 x_2 + r_1 r_2 ab x_2 Sq^2 x_1 \\ &= r_1 r_2 ab Sq^2(x_1 x_2). \end{aligned}$$

Thus

$$2\rho_4(c_2 c_{2k-1})(M) = \theta_2[r_1 r_2 ab Sq^2(x_1 x_2)]. \tag{5.4}$$

Adding (5.3) and (5.4), we have

$$[2\rho_4(c_2 c_{2k-1}) + \rho_4 c_1 \cdot p(v_{2k})](M) = 0. \qquad \blacksquare$$

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Louisiana State University
Baton Rouge, Louisiana