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HOLOMORPHIC FAMILIES OF COMPACT RIEMANN SURFACES WITH AUTOMORPHISMS

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Introduction

In this paper we study holomorphic families of Riemann surfaces defined by subgroups of the Teichmüller modular group. These families have as parameter space the fixed point set, in Teichmüller space, of the defining group H. The fibers are Riemann surfaces which admit an automorphism group isomorphic to H. Such families have been considered by several authors (Kuribayashi [17], Earle [7], Harvey [14], Earle and Kra [9]).

In the first section we give background information and set up our notation. For further details we refer to Bers [5], [6] and Earle and Kra [8], [9]. In the second section we look at equivalence of families. It is easily shown that an equivalence class of families corresponds (in general) to a conjugacy class of subgroups of the modular group. However, it is seen that there are some exceptional cases; most of the section is concerned with the classification of these exceptional cases. In Section 3 we introduce two types of invariants for our families. One type of invariant is given by the geometric nature of the action of the automorphisms near the fixed points (rotation constants). The other invariants are the characters of the representations of the automorphism groups on the spaces of holomorphic differentials and holomorphic quadratic differentials. The main result in this section is that these two sets of invariants determine each other. Finally in Section 4 we interpret, in the context of this paper, some results of Harvey [14] and Gilman [12]. The conclusion is that, in some special cases, the invariants defined in Section 3 determine the family (up to equivalence).

1. Preliminaries

Let Γ be a Fuchsian group acting on the upper half plane U. It will be assumed that U/Γ is compact. Denote by $Q(\Gamma)$ the group of all quasiconformal homeomorphisms ω of U such that $\omega\Gamma\omega^{-1} = \Gamma$. The set of Beltrami coefficients $M(\Gamma)$, with respect to the Fuchsian group Γ, is defined as the unit ball in the Banach space of all $\mu \in L^\infty(U, \mathbf{C})$ such that

$$\mu \circ \gamma \overline{\gamma'}/\gamma' = \mu.$$

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Given $\mu \in M(\Gamma)$ there is a unique homeomorphism $w_\mu \in Q(\Gamma)$, normalized by the conditions $w_\mu(0) = 0$, $\omega_\mu(1) = 1$ and $\omega_\mu(\infty) = \infty$, whose Beltrami coefficient is μ. Also there is a unique quasiconformal automorphism w^μ of $\mathbf{C} \cup \{\infty\}$, normalized by the same conditions, which has Beltrami coefficient μ on U and it is conformal on the lower half plane.

We say that μ is equivalent to ν if and only if $w_\mu = w_\nu$ on $\mathbf{R}$. The Teichmüller space $T(\Gamma)$ is the set of all equivalence classes in $M(\Gamma)$. We denote by Φ the natural projection of $M(\Gamma)$ onto $T(\Gamma)$. It is well known that $T(\Gamma)$ has a complex structure and the map Φ is holomorphic (see Bers [5]). If Γ has type (g, n), $\dim T(\Gamma) = 3g - 3 + n$ and $T(\Gamma)$ is biholomorphically equivalent to an open domain in $\mathbf{C}^{3g-3+n}$ (see Bers [5]).

The Bers fiber space is defined by

$$F(\Gamma) = \{(\Phi(\mu), z) \in T(\Gamma) \times \mathbf{C} \mid \mu \in M(\Gamma), z \in w^\mu(U)\};$$

we recall that $w^\mu(U)$ depends only on the equivalence class $\Phi(\mu)$ of μ. The group Γ acts on $F(\Gamma)$ as a discontinuous group of biholomorphic mappings (Bers [6]) by

$$\gamma(\Phi(\mu), z) = (\Phi(\mu), \gamma^\mu(z))$$

where $\mu \in M(\Gamma)$ and $\gamma^\mu = w^\mu \circ \gamma \circ (w^\mu)^{-1}$. The quotient space $V(\Gamma) = T(\Gamma)/\Gamma$ is a complex manifold and the natural projection $V(\Gamma) \to T(\Gamma)$ is holomorphic (Earle and Kra [9]). The inverse image of $\Phi(\mu)$ under this projection is the Riemann surface $w^\mu(U)/\Gamma^\mu$ where $\Gamma^\mu = w^\mu\Gamma(w^\mu)^{-1}$. Thus a holomorphic family of Riemann surfaces, in the sense of Kodaira and Spencer (see Morrow and Kodaira [19], p. 18), is defined.

The extended modular group mod (Γ) is defined as the set of equivalence classes $[\omega]$, $\omega \in Q(\Gamma)$, with ω equivalent to ω' if and only if $\omega = \omega'$ on $\mathbf{R}$. There is a well defined action of mod (Γ) on $F(\Gamma)$:

$$[\omega](\Phi(\mu), z) = (\Phi(\nu), w^\nu \circ \omega \circ (w^\mu)^{-1}(z)).$$

Here ν is the Beltrami coefficient of $w_\mu \circ \omega^{-1}$.

The relation $[\omega](\Phi(\mu)) = \Phi(\nu)$, with ν defined as above, gives an action of mod (Γ) on $T(\Gamma)$. The group Γ can be identified with a subgroup of mod (Γ) via $\gamma \to [\gamma]$. One verifies easily that Γ, as a subgroup of mod (Γ), acts trivially on $T(\Gamma)$. The modular group Mod (Γ) of Γ is defined as the factor group mod $(\Gamma)/\Gamma$. Thus we have an action of Mod (Γ) on $T(\Gamma)$ which lifts to an action on $V(\Gamma)$. We will denote by $\langle\omega\rangle$ the class of $\omega \in Q(\Gamma)$ in Mod (Γ).

Now we assume that Γ uniformizes a closed Riemann surface of genus $g \geq 2$, i.e. Γ has type (g, 0) with $g \geq 2$. Let H be a subgroup of Mod (Γ) and let $\Phi(\mu) \in T(\Gamma)$ be fixed by H. If $h = \langle\omega\rangle \in H$ then $\langle\omega\rangle(\Phi(\mu)) = \Phi(\mu)$ therefore

$$\Phi(\text{Beltrami coefficient of } w_\mu \circ \omega^{-1}) = \Phi(\mu).$$

This is equivalent to

$$\alpha_\mu \circ w_\mu \circ \omega^{-1} = w_\mu \quad \text{on } \mathbf{R},$$

for some Möbius transformation $\alpha_\mu: U \to U$. The map $f_\mu = w^\mu \circ w_\mu^{-1}$ is a conformal bijection from U onto $w^\mu(U)$. Let $\alpha^\mu = f_\mu \circ \alpha_\mu \circ f_\mu^{-1}$, this map induces a conformal automorphism h^μ of the Riemann surface $w^\mu(U)/\Gamma^\mu$ by $\Gamma^\mu z \to \Gamma^\mu \alpha^\mu z$. It is easy to see that the induced automorphism h^μ depends only on $h = \langle \omega \rangle$. The correspondence $h \to h^\mu$ establishes an isomorphism of H onto a subgroup H^μ of the group Aut $(w^\mu(U)/\Gamma^\mu)$ of conformal automorphisms of $w^\mu(U)/\Gamma^\mu$. Note that h^μ is precisely the restriction of h (acting on $V(\Gamma)$) to the fiber over $\Phi(\mu)$.

The fixed point set of H will be denoted by $T(\Gamma)^H$ and the inverse image of $T(\Gamma)^H$ under the projection $V(\Gamma) \to T(\Gamma)$ will be denoted by $V(\Gamma)^H$. Thus for dim $T(\Gamma)^H \neq 0$ we have a holomorphic family of Riemann surfaces $V(\Gamma)^H \to T(\Gamma)^H$, with each fiber admitting a group of conformal automorphisms isomorphic to H. Obviously if $T(\Gamma)^H \neq \phi$, H must be a finite group.

The map $f_\mu: U \to w^\mu(U)$ induces a conformal bijection $U/\Gamma_\mu \to w^\mu(U)/\Gamma^\mu$. Corresponding to

$$H^\mu \subseteq \operatorname{Aut}\,(w^\mu(U)/\Gamma^\mu)$$

we have

$$H_\mu \subseteq \operatorname{Aut}\,(U/\Gamma_\mu)$$

(h_μ is determined by α_μ as h^μ was determined by α^μ). Define

$$\Gamma'_\mu = \{g_\mu: U \to U \mid g_\mu \text{ is the lift of } h_\mu \in H_\mu\}.$$

Γ'_μ is a Fuchsian group and we have an exact sequence

$$\{1\} \to \Gamma_\mu \to \Gamma'_\mu \to H_\mu \to \{1\}$$

(see Kra [15], p. 48). In this context we say that Γ'_μ is the lift of H_μ to U.

There is a natural biholomorphic map between $T(\Gamma)$ and $T(\Gamma_\mu)$. Under this map $T(\Gamma)^H$ corresponds precisely to $T(\Gamma'_\mu) \subseteq T(\Gamma_\mu)$ (Kravetz [16]). Thus, without loss of generality, it may be assumed that $\Phi(\mu) = \Phi(0)$ and $T(\Gamma)^H = T(\Gamma')$.

2. Equivalence of families

Two holomorphic families of Riemann surfaces, $V \to B$ and $V' \to B'$ are called equivalent if there are biholomorphic maps $f: B \to B'$ and $F: V \to V'$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d] & V' \arrow[d] \\ B \arrow[r, "f"'] & B'. \end{tikzcd}$$

In this section we consider two families determined by subgroups H and H' of Mod (Γ) (see Section 1). We will show that these families are equivalent if and only if H and H' are conjugate, except in some special cases.

Suppose $H' = fHf^{-1}$ for $f \in$ Mod (Γ). The action of f on $T(\Gamma)$ can be lifted to $V(\Gamma)$; denote by F the lifted map. Since f: $T(\Gamma)^H \to T(\Gamma)^{H'}$, F: $V(\Gamma)^H \to V(\Gamma)^{H'}$.

Conversely, suppose the families

$$V(\Gamma)^H \xrightarrow{\pi} T(\Gamma)^H \quad \text{and} \quad V(\Gamma)^{H'} \xrightarrow{\pi} T(\Gamma)^{H'}$$

are equivalent. Let F and f be the maps realizing the equivalence. For $\tau \in T(\Gamma)^H$,

$$F\Big|_{\pi^{-1}(\tau)} : \pi^{-1}(\tau) \to \pi^{-1}(f(\tau))$$

is holomorphic. Further, since the maps $F|_{\pi^{-1}(\tau)}$ depend smoothly on τ they are homotopic. It follows that there is $h \in \mathrm{Mod}(\Gamma)$ such that $h(\tau) = f(\tau)$ for all $\tau \in T(\Gamma)^H$ (For an interpretation of the basic definitions in Section 1 in terms of homotopy see Ahlfors [2], Chapter V). Now, let $H'' = hH'h^{-1}$, clearly $T(\Gamma)^{H'} = T(\Gamma)^{H''}$. We wish to investigate the relationship between H' and H''.

We let K to be the subgroup of Mod (Γ) generated by H' and H''. Then $T(\Gamma)^K = T(\Gamma)^{H'} = T(\Gamma)^{H''}$. We can assume, without loss of generality, that $\Phi(0)$ belongs to the common fixed set. Let X be the Riemann surface U/Γ. Aut (X) has subgroups isomorphic to H' and H'' (and therefore a subgroup isomorphic to K); denote these groups with the same letters H', H'' and K. Let Γ', Γ'' and G be the lifts of H', H'' and K to U. Then we have $\Gamma' \subseteq G$, $\Gamma'' \subseteq G$ and $T(\Gamma') = T(\Gamma'') = T(G)$. Now we break our analysis into two cases:

(1) Γ', Γ'' and G have the same type. In this case, using the Riemann-Hurwitz relation for the coverings $X \to X/H'$ and $X \to X/K$, it is easy to verify that $H' = K$. Similarly one verifies that $H'' = K$.

(2) The type of G as different from the type of say Γ'. By a theorem of Patterson [20] the types of G and Γ' are either (0, 6) and (2, 0) or (0, 5) and (2, 1) or (0, 4) and (1, 1) respectively. Γ'' must have the same type as Γ' or G. Case (2) now breaks into two subcases:

(a) The type of Γ'' is the same as the type of G. Again, an argument based on the Riemann-Hurwitz relation shows that $H'' = K$. Thus $H' \subseteq H''$ and $H' \neq H''$.

Proposition. *H' is of index two in H''.*

Proof. Let N' and N'' be the orders of H' and H'' respectively and consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X/H' \arrow[dl, "\pi"] \\ X/H'' & \end{tikzcd}$$

We observe that if $p \in X/H''$ and there is no branching over $\pi^{-1}(p)$ then $N''/v'' = kN'$, where v'' is the ramification number of p and k the cardinality of the fiber $\pi^{-1}(p)$. In particular if N''/N' is prime, $v'' = N''/N'$. Now we use the Riemann-Hurwitz relation for the coverings $X \to X/H'$ and $X \to X/H''$ in the three different cases:

(i) Types (2, 0) and (0, 6).

$$N''\left[-2 + \sum_{i=1}^{6}\left(1 - \frac{1}{v_i''}\right)\right] = N' \cdot 2.$$

Here and in the sequel we will denote by v_i'' (resp. v_i') the branch numbers of $X \to X/H''$ (resp. $X \to X/H'$). Observing that $v_i'' \geq 2$ we obtain $N'' \leq N' \cdot 2$ or $N''/N' \leq 2$, thus $N''/N' = 2$.

(ii) Types (1, 2) and (0, 5).

$$(*) \qquad N''\left[-2 + \sum_{i=1}^{5}\left(1 - \frac{1}{v_i''}\right)\right] = N'\left[\sum_{i=1}^{2}\left(1 - \frac{1}{v_i'}\right)\right].$$

Since $v_i'' \geq 2$ and $v_i' \leq N'$ we obtain

$$N''\frac{1}{2} \leq N'\left(2 - \frac{2}{N'}\right) \quad \text{or} \quad \frac{N''}{N'} \leq 4 - \frac{4}{N'};$$

thus $N''/N' \leq 3$. If $N''/N' = 3$ we see that $v_i'' = 3$ for at least three values of i. Relation $(*)$ leads to a contradiction. Thus $N''/N' = 2$.

(iii) Types (1, 1) and (0, 4). In this case we first note that for at least one value of i, $v_i'' \geq 4$. The argument then is almost identical to (ii). Again we obtain $N''/N' = 2$.

(b) The type of Γ'' is the same as the type of G. Now, by the proof of the proposition, we see that H' and H'' are subgroups of index two in G.

Suppose that $H' \neq H''$. Since H' and H'' are normal in G we see that $G = H'H''$ and $H' \cap H''$ is normal in G, H' and H''. A standard result of elementary group theory says that $H'H''/H'' \simeq H'/H' \cap H''$. We conclude that $H' \cap H''$ has index two in H' (and H''). Thus we have the following diagram of double covers:

$$\begin{tikzcd}
 & X/H' \cap H'' \arrow[dl] \arrow[dr] \arrow[d, phantom, "g"] & \\
g' \quad X/H' \arrow[dr] & g_0 & X/H'' \quad g'' \arrow[dl] \\
 & X/H'H'' &
\end{tikzcd}$$

We have indicated in the diagram the notation for the genera of the Riemann surfaces involved. The covering $X/H' \cap H'' \to X/H'H''$ is a four-sheeted covering, the group of the covering is the noncyclic group of order four ($H'H''/H' \cap H''$). This group admits a partition into three subgroups of order 2 (two of them are $H'/H' \cap H''$ and $H''/H' \cap H''$). In this situation a theorem of Accola [1] applies and gives the relation

$$g + 2g_0 = g' + g'' + g'''.$$

Here g''' is the genus of the Riemann surface determined by the third subgroup of order 2. We now examine the three different cases:

(i) Types (2, 0) and (0, 6). In this case the covering $X/H' \cap H'' \to X/H'$ is unbranched and hence $g = 3$. Also $g' = g'' = 2$ and $g_0 = 0$, this is impossible.

(ii) Types (1, 1) and (0, 4). Now we have $\pi: X/H' \cap H'' \to X/H'$ and π is branched at most at one point. If π is branched at exactly one point, the Riemann-Hurwitz relation gives $2g - 2 = 1$ which is impossible. So π is unbranched and $g = 1$. Since $g' = g'' = 1$ and $g_0 = 0$ again we reach a contradiction.

(iii) Types (1, 2) and (0, 5). Here we see that $\pi: X/H' \cap H'' \to X/H$ is branched at two points, hence $g = 2$. A similar analysis does not lead to a contradiction in this case. In fact, this situation may occur. Let $S = X/H' \cap H''$, $T: S \to S$ an involution with 2 fixed points and J the hyperelliptic involution. The covering group is $\{1, J, T, JT\}$.

THEOREM. *Suppose two holomorphic families defined by subgroups H and H' of the modular group are isomorphic then either:*

(1) *H is conjugate to H'.*

(2) *H is conjugate to a subgroup of index two in H' (or vice versa). This case may occur only if the types of H and H' are* (2, 0) *and* (0, 6) *or* (1, 2) *and* (0, 5) *or* (1, 1) *and* (0, 4) *respectively.*

(3) *H has a subgroup K of index two which is conjugate to a subgroup K' of index two in H'. This case may occur only if H and H' are of type* (1, 2), *K and K' will be of type* (2, 0).

By the type of a subgroup of the modular group is meant the type of the Fuchsian lift of the automorphism group corresponding to each fixed point.

3. Invariants

In this section we will study invariants of a family defined by a group $H \subseteq \text{Mod}(\Gamma)$. We will assume that $\Phi(0) \in T(\Gamma)^H$ and thus $T(\Gamma)^H = T(\Gamma') = \{\Phi(\mu) \mid \mu \in M(\Gamma')\}$, where Γ' is the lift of H (when regarded as a subgroup of $\text{Aut}(U/\Gamma)$).

Let $h \in H$ and $\alpha \in \Gamma'$ be a lift of h. Suppose h has a fixed point represented by the orbit Γz_0. This means that $h(\Gamma z_0) = \Gamma\alpha z_0 = \Gamma z_0$ or $z_0 = (\gamma\alpha)z_0$ for some (unique) $\gamma \in \Gamma$. But $\gamma\alpha$ is also a lift of h, therefore we can always choose a lift with a fixed point. Now we denote by α a lift of h with $\alpha z_0 = z_0$. Let k be the order of h, then $\alpha^k \in \Gamma$, but since Γ is fixed point free, $\alpha^k = 1$. Thus α is an elliptic transformation of order k; its multiplier is a primitive kth root of 1, and it will be denoted by ε. Note that ε depends only on the orbit Γz_0. The rotation constant of h at the fixed point will be by definition $1/\varepsilon$, thus locally $h^{-1}: z \to (1/\varepsilon)z$.

Let $\Phi(\mu) \in T(\Gamma)^H$; then $\alpha_\mu = w_\mu \circ \alpha \circ w_\mu^{-1}$ is conformal and therefore it is an elliptic transformation of order k fixing $w_\mu(z_0)$. It is not hard to verify that α_μ is a lift of $h_\mu \in \operatorname{Aut}(U/\Gamma_\mu)$.

LEMMA. *The multiplier ε_μ of α_μ does not depend on μ.*

Proof. The function $\mu \to \varepsilon_\mu$ is continuous (see Bers [3]). But $\varepsilon_\mu^k = 1$, this clearly implies that ε_μ is constant.

We conclude that for every $h \in H$ of order k we can define $\lambda(h) = (n_1, n_2, \ldots, n_{k-1})$ where n_j is the number of fixed points of h_μ with rotation constant $\exp(2\pi ij/k)$.

Another set of invariants is obtained by considering representations of the automorphism groups in spaces of holomorphic differentials. We denote by $A_q(\Gamma) \to T(\Gamma)$, $q \geq 1$, the vector bundle of holomorphic q-differentials. The fiber over $\Phi(\mu)$ can be identified with the space of holomorphic automorphic forms $A_q(w^\mu(U), \Gamma^\mu)$ of weight $(-2q)$ (Bers [4], Kra [15]). Let $A_q(\Gamma)^H$ be the preimage of $T(\Gamma)^H$ under the projection $A_q(\Gamma) \to T(\Gamma)$. There is a natural action of the group H on $A_q(\Gamma)^H$. If $h \in H$ and $\Phi(\mu) \in T(\Gamma)^H$ the conformal map $\alpha^\mu: w^\mu(U) \to w^\mu(U)$ induces a linear map

$$\rho_q^\mu(h): A_q(w^\mu(U), \Gamma^\mu) \to A_q(w^\mu(U), \Gamma^\mu)$$

(recall that $\alpha^\mu \Gamma^\mu (\alpha^\mu)^{-1} = \Gamma^\mu$ and $\Phi(0) \in T(\Gamma)^H$). One can consider the maps ρ_q^μ as a family of representations of the finite group H.

THEOREM. *The representations $\{\rho_q^\mu \mid \Phi(\mu) \in T(\Gamma)^H\}$ are all equivalent.*

Proof. It follows immediately from Eichler trace formulas (Eichler [10], Guerrero [13]) that the characters χ_q^μ of ρ_q^μ depend only on the rotation constants. A standard result of group representation theory says that representations are equivalent if and only if they have the same character.

For $q \geq 2$ a stronger result is true:

THEOREM. *If $q \geq 2$ there exists holomorphic frames for $A_q(\Gamma)^H$ such that the matrices of $\rho_q^\mu(h)$, $h \in H$, are constant.*

Proof. Again we assume that $\Phi(0) \in T(\Gamma)^H$. Bers [4] has constructed isomorphisms

$$L_q^\mu: A_q(U, \Gamma) \to A_q(w^\mu(U), \Gamma^\mu).$$

It is sufficient to show that $L_q^\mu \circ \alpha_* = \alpha_*^\mu \circ L_q^\mu$ for every $\alpha \in \Gamma'$ (the lift of H), where α_* and α_*^μ are the induced maps on automorphic forms. Explicitly L_q^μ is given by

$$(L_q^\mu \psi)(t) = \int_L \frac{(2y)^{2q-2}\psi(\bar{z}) w_{\bar{z}}^\mu(z)^q \, dx \, dy}{(w^\mu(z) - t)^{2q}}$$

where L is the lower half plane and $z = x + iy$ (see [4]). Therefore for every $\psi \in A_q(U, \Gamma)$ we have to show that

$$L_q^\mu[\psi(\alpha(z))\alpha'(z)^q](t) = (L_q^\mu\psi)(\alpha^\mu(t))(\alpha^\mu)'(t)^q.$$

The left hand side is

$$\int_L \frac{(2y)^{2q-2}\psi(\alpha(\bar{z}))\overline{\alpha'(z)^q}w_{\bar{z}}^\mu(z)^q\,dx\,dy}{(w^\mu(z)-t)^{2q}}.$$

We set $z = \alpha^{-1}(s)$ and observe that

(1) $dx\,dy = |(\alpha^{-1})'(s)|^2\,du\,dv,\ s = u + iv$
(2) $y = |(\alpha^{-1})'(s)|u,$
(3) $w_{\bar{z}}^\mu(\alpha^{-1}(s))(\alpha^{-1})'(s) = ((\alpha^\mu)^{-1})'(w^\mu(s))w_{\bar{z}}^\mu(s);$

the last equation follows from $w^\mu \circ \alpha = \alpha^\mu \circ w^\mu$ (recall that $\alpha \in \Gamma'$ and $\mu \in M(\Gamma')$). We obtain

$$\int_L \frac{(2u)^{2q-2}\psi(\bar{s})w_{\bar{z}}^\mu(s)^q((\alpha^\mu)^{-1})'(w^\mu(s))^q\,du\,dv}{(w^\mu(\alpha^{-1}(s))-t)^{2q}}.$$

Finally we note that

$$\begin{aligned} w^\mu(\alpha^{-1}(s)) - t &= (\alpha^\mu)^{-1}(w^\mu(s)) - (\alpha^\mu)^{-1}(\alpha^\mu(t)) \\ &= (w^\mu(s) - \alpha^\mu(t))[((\alpha^\mu)^{-1})'(w^\mu(s))((\alpha^\mu)^{-1})'(\alpha^\mu(t))]^{1/2}. \end{aligned}$$

Substitution in the above integrals completes the proof.

We have seen that associated to H we have characters χ_q, $q \geq 1$. These characters are determined if the function λ is known (i.e. if rotation constants are known). The converse also holds:

THEOREM. *λ is determined by χ_1 and χ_2.*

COROLLARY. *χ_1 and χ_2 determine χ_q, $q > 2$.*

Proof. First we remark that the character χ_q determines the spectrum of $\rho_q(h)$, $h \in H$. In fact if we restrict the representation ρ_q to the cyclic group $\langle h \rangle$ we see that

$$\chi_q(h) = \sum_{j=1}^{k} m_j\chi_q^j(h), \quad k = \text{order of } h,\ \chi_q^j(h) = \exp(2\pi ij/k).$$

This is the canonical decomposition of the character χ_q as a linear combination of irreducible characters. The multiplicities m_j are given by the inner product of characters

$$m_j = \langle \chi_q, \chi_q^j \rangle$$

(see e.g. Ledermann [18]).

Next consider $h \in H$. We want to determine $\lambda(h)$. Let k be the order of h; p_j and q_j will denote the multiplicity of the eigenvalue $\exp(2\pi ij/k)$ for $\rho_1(h)$ and $\rho_2(h)$ respectively. The following formulas can be found in Guerrero [13]:

$$\begin{aligned}
p_0 &= g' \\
p_j &= g' - 1 + \sum_{l|k} x_l - \frac{1}{k} \sum_{l|k} \sum_{s=1}^{x_l} l\lambda_{sl}^{(j)} \\
q_0 &= 3g' - 3 + \sum_{l|k} x_l \\
q_j &= 3g' - 3 + 2 \sum_{l|k} x_l - \frac{1}{k} \sum_{l|k} \sum_{s=1}^{x_l} l\tilde{\lambda}_{sl}^{(j)}.
\end{aligned}$$

Now, we proceed to explain the notation in these formulas. Let X_l be the set of points fixed by h^l which are not fixed points of h^r, $r < l$. Then x_l is the cardinal number of the projection of X_l on $(U/\Gamma)/\langle h \rangle$. The rotation constants of h^l are denoted η_{sl}, $s = 1, 2, \ldots, x_l$. Note that for each s, h^l has l fixed points (on the same orbit) with rotation constant η_{sl}. Also note that $X_l = \phi$ if l is not a divisor of k. Now, $\lambda_{sl}^{(j)}$ and $\tilde{\lambda}_{sl}^{(j)}$ are defined as follows:

$$\eta_{sl}^{\lambda_{sl}(j)} = \mu^{jl}, \quad \mu = \exp(2\pi i/k).$$

$$\tilde{\lambda}_{sl}^{(j)} = \begin{cases} \lambda_{sl}^{(j)} & \text{if} \quad \lambda_{sl}^{(j)} = 1 \\ 1 + k/l & \text{if} \quad \lambda_{sl}^{(j)} = 1. \end{cases}$$

By the symbol $\sum_{l|k}$ is meant summation over all $l|k$ with $l < k$. Finally, g' denotes the genus of the factor surface $(U/\Gamma)/\langle h \rangle$.

Denote by m_{jl} the number of points in X_l with rotation constant μ^{lj} then we see that

$$q_j - p_j = q_0 - p_0 + 1 - \sum_{l|k} \frac{1}{l} m_{jl}.$$

In these equations, the multiplicities p_j, q_j, $j = 0, \ldots, k - 1$ are known. To determine $\lambda(h)$ it is sufficient to find m_{jl}, $j = 1, \ldots, k - 1$. The equations above can be written

$$\sum_{l|k} \frac{1}{l} m_{jl} = b_j, \quad j = 1, \ldots, k - 1.$$

It is easy to see that $m_{jl} = m_{sl}$ if $s \equiv j \pmod{k/l}$ and that $m_{jl} = 0$ for $(j, k/l) \neq 1$. Therefore the unknowns reduce to m_{jl}, $1 \leq j \leq k/l$, $(j, k/l) = 1$. Hence the number of unknowns is

$$\sum_{l|k} \phi(k/l) = k - 1,$$

where ϕ is Euler's function (see any introductory number theory book). Thus we have to solve a system of $k-1$ linear equations with $k-1$ unknowns. Unfortunately this system is often singular; in order to uniquely determine a solution we will need extra conditions.

We will see now that reordering the equations and unknowns we obtain a system of linear equations whose matrix of coefficients is quite simple. The unknowns are ordered as follows: m_{jl} is after m_{sd} if either $l < d$ or $l = d$ and $j < s$. We order the equations according to the rule: equation j is after equation s if $(j, k) < (s, k)$ or if $(j, k) = (s, k) = l$ and $[j] < [s]$ ($[n]$ denotes the residue modulo k/l). If $(j, k) = (s, k)$ and $[j] = [s]$ choose an arbitrary order.

In matrix form, the system of linear equations obtained will be written

$$Am = b.$$

We will show that the matrix A is lower triangular.

The set $\{1, 2, \ldots, k-1\}$ can be partitioned into subsets S_d, $d \mid k$, $S_d = \{j \mid (j, k) = d\}$. This partition induces a block partition of A. First we will prove that A is block triangular. Consider the equation

$$\sum_{l \mid k} \frac{1}{l} m_{jl} = b_j$$

and assume that $(j, k) = d$. We have to show that $m_{jl} = 0$ for $l < d$. If $m_{jl} \neq 0$ then $(j, k/l) = 1$, therefore $(jl, k) = l$. Since $(j, k) = d$ it follows that $l \geq d$. Now we examine the blocks along the main diagonal. Each of these blocks corresponds to a set S_d, we claim that such a block is either 0 or $(1/d)I$. Again we look at the equation

$$\sum_{l \mid k} \frac{1}{l} m_{jl} = b_j \quad \text{with } (j, k) = d.$$

Corresponding to d we have the unknowns m_{sd}, $1 \leq s \leq k/d$, $(s, k/d) = 1$. At most one of these can appear in the equation, namely m_{rd} with $r \equiv j \pmod{k/d}$. We have to check whether $(r, k/d) = 1$ or not. Clearly $(r, k/d) = 1$ if and only if $(j, k/d) = 1$. Since $(j, k) = d$, $(j, k/d) = 1$ if and only if $(d, k/d) = 1$. Therefore the block corresponding to S_d is zero unless $(d, k/d) = 1$. In this case the rule given above determines a unique order for those equations corresponding to S_d $(j, k) = (s, k) = d$ and $[j] = [s]$ implies $j = s$). It is easily seen that then the block is $(1/d)I$.

If the condition $(d, k/d) = 1$ is satisfied for every $d \mid k$ then A is non-singular and we have a unique solution for $Am = b$. This occurs if and only if k is square free (i.e. product of distinct primes). If A is singular we have to introduce additional information.

Let n_{jl} be the number of fixed points of h^l with rotation constant μ^{jl}. We have

$$n_{jl} = m_{jl} + \sum_{d \mid l} \sum_{s} m_{sd}$$

where the second summation runs over all s such that $1 \leq s \leq k \mid d$, $(s, k \mid d) = 1$ and $s \equiv j \pmod{k/l}$. In matrix form we write

$$n = Cm.$$

Using the same ordering of the variables as before, we see that C is upper triangular with ones along the main diagonal. Therefore C has an inverse B which is also upper triangular. Using the new variables n, we write

$$ABn = b.$$

We will show that from these equations we can compute n_{j1} in terms of n_{sl}'s with $l > 1$. To calculate n_{sl} we apply the same procedure to h^l. If the order of h^l is square free then n_{sl} will be uniquely determined. Otherwise we continue recursively. This process will end after finitely many steps.

Since we are interested in variables n_{j1}, we partition our matrices into four blocks corresponding to the partition of $\{1, \ldots, k-1\}$ into S_1 and its complement. It is not hard to see that

$$A = \begin{pmatrix} A_1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ A_2 & \vdots & I \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \vdots & B_2 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I \end{pmatrix}$$

where I is a $p \times p$ identity matrix (p is the cardinal number of S_1). From this, using the fact that B is non-singular, it is easy to verify that the last p columns of AB are linearly independent. This proves the claim that the unknowns n_{j1} can be expressed in terms of n_{sl}'s with $l > 1$. This completes the proof.

Consider now a group $H' = fHf^{-1}$ conjugate to H in Mod (Γ). Let $\Phi(\mu) \in T(\Gamma)^H$ and $\Phi(\nu) \in T(\Gamma)^{H'}$, if $f = \langle \phi \rangle$ and $h = \langle \omega \rangle \in H$ we have

$$\alpha \circ w_\mu \circ \omega^{-1} = w_\mu \text{ on } \mathbf{R}, \qquad \beta \circ w_\nu \circ \phi \circ \omega^{-1} \circ \phi^{-1} = w_\nu \text{ on } \mathbf{R}$$

for Möbius transformations α, $\beta\colon U \to U$. As before, α and β induce automorphisms

$$h_\mu \in \operatorname{Aut}(U/\Gamma_\mu) \quad \text{and} \quad (fhf^{-1})_\nu \in \operatorname{Aut}(U/\Gamma_\nu).$$

The quasiconformal map $\tau = w_\nu \circ \phi \circ w_\mu^{-1}$ satisfies $\tau \circ \alpha \circ \tau^{-1} = \beta$ on $\mathbf{R}$ and $\tau\Gamma_\mu\tau^{-1} = \Gamma_\nu$. By Teichmüller theorem we can find τ_0 satisfying the same conditions and such that its maximal dilation is a minimum. By uniqueness of τ_0, $\tau_0 \circ \alpha = c \circ \tau_0$ for a conformal map c, i.e. $\tau_0 \circ \alpha \circ \tau_0^{-1}$ is conformal. But $\tau_0 \circ \alpha \circ \tau_0^{-1} = \beta$ on $\mathbf{R}$, therefore $\tau_0 \circ \alpha \circ \tau_0^{-1} = \beta$ everywhere. Since τ_0 is an orientation preserving homeomorphism, the elliptic transformations α and β must have the same multiplier. We conclude that $\lambda(h) = \lambda(fhf^{-1})$. We have shown that λ is an invariant of the conjugacy class of H in Mod (Γ).

4. Cyclic group case

Results in this section are essentially contained in Harvey [14] and Gilman [12]. It seems convenient to include a formulation of these results in the context of this paper.

Let H and H' be subgroups of Mod (Γ). We have seen that if H and H' are conjugate then corresponding elements have the same rotation constants. Clearly the Fuchsian lifts of H and H' have the same signature $(g', v_1, v_2, \ldots, v_l)$.

THEOREM. *Suppose that H, $H' \subseteq$ Mod (Γ) are cyclic subgroups of order k. Assume that the Fuchsian lifts have the same signature*

$$(g', v_1, \ldots, v_l)$$

and that $k/v = 1$ or 2, with $v = \text{lcm}\,(v_1, \ldots, v_l)$. If H and H' have generators h and h' with $\lambda(h^s) = \lambda((h')^s)$ $s = 1, \ldots, k$, then H and H' are conjugate on Mod (Γ).

In the previous section we saw that if $H' = fHf^{-1}$ and Riemann surfaces X and X' correspond to fixed points of H and H' then there is a quasiconformal homeomorphism $X \to X'$ which conjugates the induced automorphism groups. In other words the coverings $X \to X/H$ and $X' \to X'/H'$ are topologically equivalent (we denote with the same symbols H and H' the automorphism groups). It is not hard to verify that the converse holds.

Suppose we have two coverings $\pi\colon X \to X/H$ and $\pi'\colon X' \to X'/H'$. We want to find a homeomorphism (quasiconformal) $f\colon X/H \to X'/H'$ which lifts. It is sufficient (see Fulton [11]) to look at the regular coverings $\pi\colon Y \to Y/H$ and $\pi'\colon Y' \to Y'/H'$. Here Y and Y' are obtained from X and X' respectively by removing the branch points. From elementary covering space theory, a necessary and sufficient condition for the existence of a lift is that

$$f_*(\pi_*(\Pi_1(Y, y_0)) = \pi'_*(\Pi_1(Y', y'_0)) \quad \text{with}\, f(\pi(y_0)) = \pi'(y'_0).$$

Now, assume that the coverings are cyclic of order k. Then

$$\pi_*(\Pi_1(Y, y_0)) \quad \text{and} \quad \pi_*(\Pi_1(Y', y'_0))$$

are given by Ker ϕ and Ker ϕ' for surjective homomorphisms

$$\phi\colon \Pi_1(Y/H, \pi(y_0)) \to \mathbf{Z}_k, \qquad \phi'\colon \Pi_1(Y'/H', \pi'(y'_0)) \to \mathbf{Z}_k.$$

LEMMA. $f_*(\text{Ker}\ \phi) = \text{Ker}\ \phi'$ *if and only if* $\phi' \circ f_* = r\phi$ *with* $(r, k) = 1$.

Proof. If $f_*(\text{Ker}\ \phi) = \text{Ker}\ \phi'$ then $\phi' \circ f_* = \phi$ on Ker ϕ. But since

$$\Pi_1(Y/H, \pi(y_0))/\text{Ker}\ \phi \simeq \mathbf{Z}_k,$$

the values of $\phi' \circ f_*$ and ϕ will be determined by the values on a generator. Therefore $\phi' \circ f_* = r\phi$ with $(r, k) = 1$.

If we can find generators $a_1, \ldots, a_{g'}, b_1, \ldots, b_{g'}, x_1, \ldots, x_l$ for $\Pi_1(Y/H, \pi(y_0))$ and $a'_1, \ldots, a'_{g'}, b'_1, \ldots, b'_{g'}, x'_1, \ldots, x'_l$ for $\Pi_1(Y'/H', \pi'(y'_0))$ such that

$$r\phi(a_i) = \phi'(a'_i), \quad r\phi(b_i) = \phi'(b'_i) \quad \text{and} \quad r\phi(x_j) = \phi'(x'_j)$$

then a homeomorphism f with $f_*(a_i) = a'_i$, $f_*(b_i) = b'_i$ and $f_*(x_j) = x'_j$ will lift.

Proof of the theorem. Harvey [14] has shown that one can always find generators $a_1, \ldots, a_{g'}, b_1, \ldots, b_{g'}, x_1, \ldots, x_l$ with $\phi(a_i) = 0$, $i = 1, \ldots, g'$; $\phi(b_i) = 0$, $i = 2, \ldots, g'$ and $\phi(b_1)$ equal to a generator of $\mathbf{Z}_{k/v}$. Next, one verifies that $\lambda(h^s)$, $s = 1, \ldots, k$, for some generator h, determines the cardinal number of

$$\{x_i | \phi(x_i) = j, j = 1, \ldots, k-1\}.$$

Let $h \in H$ be the generator corresponding to 1 under the isomorphism

$$H \simeq \Pi_1(Y/H, \pi(y_0))/\pi_*(\Pi_1(Y, y_0)) \simeq \mathbf{Z}_k$$

(the last isomorphism is the canonical isomorphism induced by ϕ). Now, consider a generator x_i (recall that $x_i^{v_i} = 1$), x_i corresponds to a fixed point of h^{k/v_i}. Let c be a small loop about the branch point corresponding to x_i. If $\phi(x_i^r) \equiv k/v_i \pmod{k}$, $1 \leq r \leq v_i - 1$, then h^{k/v_i} takes the initial point of a lift of rc to its endpoint. That is h^{-k/v_i} must be locally a rotation $z \to \exp(2\pi i r/v_i)z$. So, if the rotation constant of h^{k/v_i} at a fixed point (corresponding to x_i) is $\exp(2\pi i r/v_i)$ then $\phi(x_i) = sk/v_i$ with $rs \equiv 1 \pmod{v_i}$.

Therefore if there are generator h and h' for the groups H and H' with $\lambda(h^s) = \lambda((h')^s)$ $s = 1, \ldots, k$, we can find generators satisfying the conditions in the paragraph after the lemma. The coverings $X \to X/H$ and $X' \to X'/H'$ will be topologically equivalent.

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