ILLINOIS JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 1989

UNITS IN ABELIAN GROUP RINGS AND MEROMORPHIC FUNCTIONS

BY

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Section 0

Let $\mathbf{Z}\Gamma$ denote the integral group ring of the finite, abelian group Γ. The set of units of $\mathbf{Z}\Gamma$ is written $\mathbf{Z}\Gamma^{\times}$. Higman's theorem (see [4]) gives the structure of $\mathbf{Z}\Gamma^{\times}$ as

$$\mathbf{Z}\Gamma^{\times} \cong T \times \mathbf{Z}^{r_\Gamma}, \tag{1}$$

where r_Γ is a non-negative integer, and

$$T = \{\pm\gamma : \gamma \in \Gamma\}. \tag{2}$$

Let $\| \quad \|$ denote any Euclidean norm on $\mathbf{C}\Gamma$, that is, a function which takes non-negative, real values and satisfies:
(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|\lambda x\| = |\lambda|\,\|x\|$ for $x \in \mathbf{C}\Gamma$, $\lambda \in \mathbf{C}$,
(3) $\|x + y\| \leq \|x\| + \|y\|$.
In a recent paper [2] we studied the series

$$\iota_{\| \ \|}(s) = \sum_{\substack{x \in \mathbf{Z}\Gamma^{\times} \\ \|x\| > 1}} (\log\|x\|)^{-s}, \quad s \in \mathbf{C}. \tag{3}$$

Note. From now on we exclude from any summation those terms which are undefined, e.g., in the above, those x with $\|x\| = 1$. Since $\mathbf{Z}\Gamma^{\times}$ is a discrete subset of $\mathbf{C}\Gamma$ there are only a finite number of these terms and they clearly do not affect the type of results in which we are interested.

THEOREM A [2, p. 35]. *Let r_Γ denote the torsion free rank of $\mathbf{Z}\Gamma^{\times}$. Then*:
(i) $\iota_{\| \ \|}(s)$ *has half-plane of convergence* $\mathrm{Re}(s) > r_\Gamma$.
(ii) $\iota_{\| \ \|}(s)$ *has analytic continuation to* $\mathrm{Re}(s) > r_\Gamma - 1$ *where it is analytic apart from a simple pole at $s = r_\Gamma$. The residue is independent of the choice of norm* $\| \quad \|$.

Received July 15, 1987.

Let $\hat{\Gamma}$ denote the group of characters of Γ, $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbf{C}^{\times})$. These can be used to provide a useful example of a Euclidean norm. For each $\chi \in \hat{\Gamma}$, $x = \sum_{\gamma \in \Gamma} x_\gamma \gamma \in \mathbf{C}\Gamma$, let

$$\iota_\chi(x) = \sum_{\gamma} \chi(\gamma) x_\gamma. \tag{4}$$

Then

$$\|x\| = |x| = \max_{\chi \in \hat{\Gamma}} \{ |\iota_\chi(x)| \} \tag{5}$$

gives a Euclidean norm, the first property following from the fact that the ι_χ are independent linear forms.

THEOREM 1. *The series* $\iota_{|\ |}(s)$ *has analytic continuation to* $\mathrm{Re}(s) > r_\Gamma - 2$. *The only singularities in this half-plane are simple poles at* $s = r_\Gamma$ *and* $s = r_\Gamma - 1$.

In [2] we also introduced a refinement of the series (3). Let $c_{\|\ \|}(1)$ denote the $\|\ \ \|$-unit ball defined by

$$c_{\|\ \|}(1) = \{ x \in \mathbf{C}\Gamma : \|x\| = 1 \}. \tag{6}$$

Suppose W is an open subset of $c_{\|\ \|}(1)$ with characteristic function f_W. Define

$$\iota_{W, \|\ \|}(s) = \sum_{x \in \mathbf{Z}\Gamma^{\times}} f_W(x^1)(\log \|x\|)^{-s}, \quad s \in \mathbf{C} \tag{7}$$

where $x^1 = x\|x\|^{-1}$ denotes the central projection of x onto $c_{\|\ \|}(1)$.

In [2, p. 36] we proved the following result.

THEOREM B. *There are finitely many points* $P_1, \dots, P_n$ *on* $c_{\|\ \|}(1)$ *such that*:

(i) *If* W *contains all of the* P_i *then* $\iota_{W, \|\ \|}(s) - \iota_{\|\ \|}(s)$ *is analytic in* $\mathrm{Re}(s) > r_\Gamma - 1$.

(ii) *If* W *contains none of the* P_i *then* $\iota_{W, \|\ \|}(s)$ *is analytic in* $\mathrm{Re}(s) > r_\Gamma - 1$.

Now we will define a set of line segments upon the surface of $c_{|\ |}(1)$ which we will use to generalise Theorem B. For $\chi \in \hat{\Gamma}$, let e_χ denote the idempotent

$$|\Gamma|^{-1} \sum_{\gamma \in \Gamma} \bar{\chi}(\gamma)\gamma. \tag{8}$$

It follows from the orthogonality relations that for $x \in \mathbf{C}\Gamma$,

$$x = \sum_{\chi \in \hat{\Gamma}} e_\chi \iota_\chi(x). \tag{9}$$

Say that a character $\chi \in \hat{\Gamma}$ is *non-degenerate* if $\mathbf{Q}(\chi)$—the extension of $\mathbf{Q}$ generated by the values of χ—is not $\mathbf{Q}$ or an imaginary quadratic extension of $\mathbf{Q}$. Let

$$\rho_\chi = e_\chi + e_{\bar{\chi}} \tag{10}$$

and write E for the set of all ρ_χ together with their translates under T (as in (2)),

$$E = \left\{ \delta\rho_\chi : \chi \in \hat{\Gamma} \text{ non-degenerate}, \delta \in \Gamma \right\}. \tag{11}$$

Given any $P_i, P_j \in E$, it is clear that the line segment

$$\iota_{ij} = P_i + tP_j, \quad 0 \leq t \leq 1, \tag{12}$$

lies in $c_{|\ |}(1)$. Let L denote the set of all the ι_{ij} for $P_i \neq P_j$ in E.

THEOREM 2. *Suppose W is an open subset of $c_{|\ |}(1)$.*
(i) *If $L \subset W$ then $\iota_{W, |\ |}(s) - \iota_{|\ |}(s)$ is analytic in* $\mathrm{Re}(s) > r_\Gamma - 2$.
(ii) *If $W \cap L = \varnothing$ and $\|\ \ \|$ denotes any norm then the function $\iota_{W, \|\ \|}(s)$ is analytic in* $\mathrm{Re}(s) > r_\Gamma - 2$.

Note. Suppose K is a totally real extension of $\mathbf{Q}$ with $n + 1 = [K : \mathbf{Q}]$ and O_K denotes the ring of algebraic integers of K. Let

$$H(x) = \max_{\sigma : k \to \mathbf{R}} \{|\sigma(x)|\}.$$

Our methods apply to the series

$$U(s) = \sum_{x \in O_K^\times} \log H(x)^{-s}, \quad s \in \mathbf{C}. \tag{13}$$

The regulator of K appears in some rather interesting ways in the residues for the poles of this function at $s = n$ and $s = n - 1$. The trick of using the "first approximation" (see the proof of Proposition 1) works well on the Riemann zeta function and can be used to "rediscover" the local theory of that function. It might be interesting to see whether the same is true of the series (15).

The "clustering" phenomenon has some interesting consequences for Galois-module theory. In [3] a relationship was established with the divisibility properties of normal integral bases in tame, abelian number fields. These finer results also have application in this theory and will be published shortly.

In [1], Bushnell initiated the study of questions such as these although he used the series $\sum_{x \in \mathbf{Z}\Gamma^\times} \|x\|^{-s}$. This is difficult to work with because, when

$r_\Gamma > 1$, the singularities are squashed together at $s = 0$ and it seems difficult to obtain an analytic continuation.

Section 1

Important for these finer results is the following definition. Suppose $\chi \in \hat{\Gamma}$ and $x \in \mathbf{C}\Gamma$ has

$$|x| = |\iota_\chi(x)|.$$

Let

$$|x|^* = \max_{\psi \neq \chi, \bar{\chi}} \{|\iota_\psi(x)|\}.$$

We will work mostly inside a subgroup of $\mathbf{Z}\Gamma^\times$ of finite (generalised) index, namely, those $x \in \mathbf{Z}\Gamma^\times$ for which

$$\iota_\chi(x) \in \mathbf{R}^+ \quad \text{for } \chi \in \hat{\Gamma}.$$

Write $R^\times$ for this subgroup.

In §2 we will show how to lift the results back to $\mathbf{Z}\Gamma^\times$.

Suppose $1 > \varepsilon > 0$ and $N_\varepsilon(L)$ denotes the open cylinder of radius ε about L (the metric is that induced by $|\quad|$). Given $\rho_{\chi_1}, \rho_{\chi_2} \in E$ with $\chi_1\chi_2 \neq 1$ define

$$R_{12} = \{x \in \mathbf{C}\Gamma : |x| = |\iota_{\chi_1}(x)|, |x|^* = |\iota_{\chi_2}(x)|\}.$$

Then, for $x \in R^\times$ we have

$$x^1 = x|x|^{-1} \in N_\varepsilon(L) \cap R_{12}$$

if and only if

$$\left|\frac{\iota_\psi(x)}{\iota_{\chi_1}(x)}\right| < \varepsilon \quad \text{for } \psi \neq \chi_i, \bar{\chi}_i, \quad i = 1, 2, \tag{14}$$

and

$$\left|\frac{\iota_{\chi_2}(x)}{\iota_{\chi_1}(x)}\right| \leq 1.$$

Suppose $n_1, \dots, n_t$ are positive integers. For each n_i suppose $L_1^{(i)}, \dots, L_{n_i}^{(i)}$ are n_i real, linearly independent linear forms on $\mathbf{R}^{n_i}$. Define

$$L_{n_i+1}^{(i)}(x) = -\sum_{j=1}^{n_i} L_j^{(i)}(x), \quad x \in \mathbf{R}^{n_i}. \tag{15}$$

Let $N = \sum_{i=1}^{t} n_i$, $V = \oplus \mathbf{R}^{n_i}$, and let $V_{\mathbf{Z}}$ denote the integer points of V. Also, extend the $L_j^{(i)}$ to V in the obvious way and define

$$H(x) = \max_{i,j} \left\{ L_j^{(i)}(x) \right\}. \tag{16}$$

Define

$$F(s) = \sum_{x \in V_{\mathbf{Z}}} H(x)^{-s}. \tag{17}$$

Recall from [2] the following facts:

(1) $F(s)$ is absolutely convergent on $\mathrm{Re}(s) > N$.

(2) $F(s)$ has analytic continuation to $\mathrm{Re}(s) > N - 1$ where it is analytic apart from a simple pole at $s = N$.

(3) Write S_{N+t} for the symmetric group on $N + t$ letters. Fix an ordering of the $N + t$ symbols

$$(i, j), \quad 1 \le i \le t, 1 \le j \le n_i + 1.$$

Then S_{N+t} acts on the set of forms $\{L_j^i\}$. Write σL_j^i for the effect of $\sigma \in S_{N+t}$ upon L_j^i. Also, let

$$c_\sigma = \left\{ x \in \mathbf{R}^N : \sigma L_1^1(x) \ge \cdots \ge \sigma L_{n_1+1}^1(x) \ge \cdots \ge \sigma L_{n_t+1}^t(x) \right\}. \tag{18}$$

Clearly

$$\mathbf{R}^N = \bigcup_{\sigma \in S_{N+T}} c_\sigma.$$

Write B for the set of all boundaries of the c_σ i.e. the set of all $x \in \mathbf{R}^N$ where at least one of the inequalities in (18) is an equality. Let $O_M(s)$ denote any function of $s \in \mathbf{C}$ which is analytic on the half-plane $\mathrm{Re}(s) > M$. Then

$$\sum_{\substack{x \in V_{\mathbf{Z}}, \\ c_x \cap B \ne \varnothing}} H(x)^{-s} = O_{N-1}(s), \tag{19}$$

where c_x denotes the unit cube with centre x.

(4) Define

$$I(s) = \int_{\mathbf{R}^N - O} H(y)^{-s}\,dy, \quad s \in \mathbf{C}, \tag{20}$$

where O is some open ball which contains the origin. The function $I(s)$ has analytic continuation to the whole plane where it is analytic apart from a set of simple poles at the points $s = 1, 2, \ldots, N$.

PROPOSITION 1. *The function* $F(s) - I(s)$ *has analytic continuation to the half-plane* $\mathrm{Re}(s) > N - 2$ *with a simple pole at* $s = N - 1$.

Proof. We can ignore the finite number of x for which $c_x \cap 0 \neq \varnothing$. Write

$$F(s) - I(s) = \sum_{x \in V_{\mathbf{Z}}} \left(H(x)^{-s} - \int_{c_x} H(y)^{-s}\,dy \right). \tag{21}$$

Now we will use a first approximation to the mean value theorem to write the integral in the form

$$H(x + t_x)^{-s} \quad \text{for some } t_x \text{ with } |t_x| \leq 1.$$

Case (i). $c_x \cap B \neq \varnothing$. The bound on $|t_x|$ implies

$$H(x + t_x) = H(x) + O(1).$$

Thus, the sum in (21) over all such x is

$$\begin{aligned} &\sum_{\substack{x \in V_{\mathbf{Z}},\\ c_x \cap B \neq \varnothing}} H(x)^{-s}\left\{1 - \left(1 + O\left(H(x)^{-1}\right)\right)^{-s}\right\} \\ &\quad = \sum_{\substack{x \in V_{\mathbf{Z}},\\ c_x \cap B \neq \varnothing}} H(x)^{-1-s} + O_{N-2}(s) \\ &\quad = O_{N-2}(s) \end{aligned}$$

by remark (3).

Case (ii). $c_x \cap B = \varnothing$. For these x we have

$$H(y) = L_j^{(i)}(y) \quad y \in c_x \text{ for some } (i, j).$$

We claim that in this case

$$H(t_x) = c_{ij} + O\left(H(x)^{-1}\right) \tag{22}$$

where c_{ij} is constant. That is

$$|H(t_x) - c_{ij}| \leq \frac{|f(s)|}{|H(x)|}$$

where $f(s)$ is analytic function which is independent of x. We will prove this later.

Now to proceed, write (using case (i))

$$\begin{aligned} F(s) - I(s) &= \sum_{\substack{x \in V_{\mathbf{Z}}, \\ c_x \cap B = \varnothing}} H(x)^{-s}\left(1 - (1 + H(t_x)/H(x))^{-s}\right) + O_{N-2}(s) \\ &= s \sum_{\substack{x \in V_{\mathbf{Z}} \\ c_x \cap B = \varnothing}} H(t_x) H(x)^{-1-s} + O_{N-2}(s) \\ &= s \sum_{i,j} c_{ij} \sum_{\substack{x \in V_{\mathbf{Z}}, \\ c_x \cap B = \varnothing}} H(x)^{-1-s} + O_{N-2}(s) \end{aligned}$$

By remarks (2) and (3) the inner sum has analytic continuation to $\mathrm{Re}(s) > N - 2$ with only a simple pole at $s = N - 1$.

Notice that the coefficient in the first term is non-zero so the singular behaviour of $F(s)$ at $s = N - 1$ is determined by that of $I(s)$ at $N - 1$ and by that of $F(s)$ at $s = N$. Also observe that the constant $\sum c_{ij}$ is a combinational constant multiplied by the product of the inverse determinants of the sets of the forms $L_j^{(i)}(x)$.

Finally, we prove the claim (22). Fix a pair (i, j). By the integral mean value theorem write

$$I_x = \int_{c_x} \left(L_j^{(i)}(x) + L_j^{(i)}(y)\right)^{-s} dy$$

where y runs through c_x. This is

$$L_j^{(i)}(x)^{-s} \int_{c_x} \left(1 + L_j^{(i)}(y) L_j^{(i)}(x)^{-1}\right)^{-s} dy$$

Expand this to order $O(L_j^{(i)}(x)^{-1})$ to obtain

$$\begin{aligned} I_x &= L_j^{(i)}(x)^{-s} \int_{c_x} \left(1 - s L_j^{(i)}(y) L_j^{(i)}(x)^{-1}\right) dy + O\left(L_j^{(i)}(x)^{-2}\right) \\ &= L_j^{(i)}(x)^{-s}\left(1 - s L_j^{(i)}(x) I_{ij}\right) + O\left(L_j^{(i)}(x)\right)^{-2} \end{aligned}$$

where $I_{ij} = \int_{c_x} L_j^{(i)}(y)\,dy$). Now compare with equation (22) and extract the $-1/s$ root to obtain

$$L_j^{(i)}(t_x) = I_{ij} + O\left(L_j^{(i)}(x)\right)^{-1}.$$

Regrouping terms, we have proved that

$$F(s) - I(s) = \sum_{i,\,j} c_{ij} \sum_{\substack{x \in V_{\mathbf{Z}},\\ c_x \cap B = \varnothing}} H(x)^{-s} + O_{N-2}(s).$$

Next we will identify the contribution to the singular behaviour at $s = N$ and $s = N - 1$.

Suppose $x \in c_\sigma$. Then

$$H(x) = \sigma L_1^1(x), \quad H(x)^* = \sigma L_2^1(x).$$

Suppose α_{ij}, β_{ij} are non-negative constants with $\alpha_{11} = \alpha_{12} = 0$. Let Y denote the set of all $x \in \mathbf{R}^N$ with

$$H(x) \geq \sigma L_j^{(i)}(x) + \alpha_{ij}, \quad i = i, \dots, t \quad j = 1, \dots, n_i$$

or

$$H(x)^* \geq \sigma L_j^{(i)}(x) + \alpha_{ij}, \quad i = 1, \dots, t \quad j = 1, \dots, n_i.$$

PROPOSITION 2.

$$\sum_{\substack{x \in V_{\mathbf{Z}},\\ x \notin Y}} H(x)^{-s} = O_{N-2}(s).$$

Proof. Given i, j, k, l let T_{ijkl} denote the set

$$T_{ijkl} = \left\{ x \in \mathbf{R}^N : H(x) < L_j^{(i)}(x) + \alpha_{ij},\ H(x)^* < L_l^{(k)}(x) + \alpha_{kl} \right\}.$$

Let T denote the union of all possible T_{ijkl}. Choose the c_x so that

(i) $T \subset \bigcup_x c_x$,

(ii) $C_x \cap T \neq \varnothing$,

(iii) $C_{x_1} \cap C_{x_2} = \varnothing$ if $x_1 \neq x_2$.

Clearly, given any $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 0$ we have

$$\left| \sum_{\substack{x \in V_{\mathbf{Z}},\\ x \in T}} H(x)^{-s} \right| \leq c_1 \int_{c-0} H(y)^{-r}\,dy$$

where $r = \mathrm{Re}(s) > 0$, and c denotes $\bigcup_{x \in V_Z \cap T} c_x$ and, as before O denotes some open ball containing the origin. It is sufficient to estimate the integral.

Change the variables and this becomes a finite sum of integrals of the form

$$\int L_1^{1-r}\, dL$$

where

$$\begin{aligned}
&L_1^i, \ldots, L_{n_i}^i \geq -L_1^i - \cdots - L_{n_i}^i,\\
&L_q^1 \geq L_j^i + c_2,\\
&L_b^a \geq L_l^k + c_3, \quad (i, j) \neq (a, b), (k, l) \neq (1, 1),\\
&L_1^1 < L_d^c + c_4,\\
&L_b^a < L_f^e + c_5.
\end{aligned}$$

Make the transformation

$$\theta_j^i = L_j^i + \sum_{k=1}^{n_i} L_k^i;$$

then our integral is majorised by one of the form

$$\int \left(\Sigma\, \sigma_j\right)^{-r} d\theta_i$$

where the θ_j are N variables which satisfy inequalities of the form

$$\begin{gathered}
\theta_1, \theta_2 \geq \theta_i > c_6 > 0,\\
\theta_1 < \theta_3 + c_7,\\
\theta_2 < \theta_4 + c_8, \quad c_7, c_8 > 0.
\end{gathered}$$

Do the $\theta_N, \ldots, \theta_5$ integrals first to obtain the finite sum of integrals of the form

$$\frac{1}{1-r} \cdots \frac{1}{N-4-r} \int d\theta_1 \ldots d\theta_4 \left(A_1\theta_1 + A_2\theta_2 + A_3\theta_3 + A_4\theta_4 + A_5\right)^{N-4-r}$$

over the region

$$\begin{gathered}
\theta_1 < \theta_3 + c_7,\\
\theta_2 < \theta_4 + c_8,\\
\theta_1, \theta_2 \geq \theta_3, \quad \theta_4 \geq c_6 > 0,
\end{gathered}$$

where $A_1, \ldots, A_5$ are positive constants.

Translating r it is sufficient to prove that the integral

$$\int d\theta_1 \ldots d\theta_4 (A_1\theta_1 + A_2\theta_2 + A_3\theta_3 + A_4\theta_4 + A_5)^{-r+4}$$

is defined for $r > 2$. Do the θ_1 integral for $\theta_3 < \theta_1 < \theta_3 + c_7$ and the θ_2 integral for $\theta_4 < \theta_2 < \theta_4 + c_8$. We obtain

$$\begin{aligned}\frac{1}{2-r}\cdot\frac{1}{1-r}\int d\theta_3\, d\theta_4 \big\{ & (A_6\theta_3 + A_7\theta_4 + c_9 + c_{10} + c_{11})^{2-r} \\ & + (A_6\theta_3 + A_7\theta_4 + c_{11})^{2-r}\big\} \\ - \big\{(A_6\theta_3 + A_7 + \theta_4 + c_9 + c_{11})^{2-r} & - (A_6\theta_3 + A_7\theta_4 + c_{10} + c_{11})^{2-r}\big\}.\end{aligned}$$

Finally, do the θ_3, θ_4 integrals and we obtain

$$\begin{aligned}\frac{1}{4-r}\cdot\frac{1}{3-r}\cdot\frac{1}{2-r}\cdot\frac{1}{1-r}\big\{ & (c_9 + c_{10} + c_{11})^{4-r} + c_{11}^{4-r} \\ & - (c_9 + c_{11})^{4-r} - (c_{10} + c_{11})^{4-r}\big\}\end{aligned}$$

However, it is clear that the expression inside the brackets vanishes when $r = 3$ or $r = 4$ and these zeros cancel any potential singularities.

Section 2

To show how these results apply, suppose $\overline{\mathbf{Q}}$ is an algebraic closure of $\mathbf{Q}$ with $\Omega = \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$. We assume that the values of $\chi \in \hat{\Gamma}$ lie in $\overline{\mathbf{Q}}$ so that Ω acts upon $\hat{\Gamma}$ in the obvious natural way. If ω is any orbit then the field $\mathbf{Q}_\omega = \mathbf{Q}(\chi)$ is independent of $\chi \in \omega$ because $\mathrm{Gal}(\mathbf{Q}(\chi)|\mathbf{Q})$ arises naturally as a quotient of Ω. Choose any $\chi \in \omega$ then the map

$$x \mapsto \left(\iota_\chi(x)\right)_\omega \tag{23}$$

yields an isomorphism

$$\mathbf{Q}\Gamma \cong \prod_\omega \mathbf{Q}_\omega. \tag{24}$$

Let M denote the (unique) maximal order of $\mathbf{Q}\Gamma$. The map (23) gives rise to two further isomorphisms

$$M \cong \prod_\omega O_\omega, \quad M^\times \cong \prod_\omega O_\omega^\times \tag{25}$$

where $O_\omega^\times$ denotes the ring of algebraic integers of $\mathbf{Q}_\omega$. Let $R^\times$ denote the group on the right hand side of (25) which consists of totally real, positive units:

$$R^{\times} = \prod_{\omega} u_{\omega}.$$

This group corresponds to the subgroup M^1 of $M^{\times}$, say. For each ω there is a logarithmic map

$$L_{\omega} : u_{\omega} \to \mathbf{R}^{t_{\omega}+1}, \quad u \mapsto (\log u^{\sigma})_{\sigma \in \omega}$$

where σ is viewed as an element of ω. The components extend to $t_{\omega} + 1$ real linear forms on $\mathbf{R}^{t_{\omega}}$. These forms obviously sum to zero on $\mathbf{R}^{t_{\omega}}$ and any t_{ω} of the forms are linearly independent. Do this for each ω then we are in the set-up of §1. The results there imply that the series,

$$\sum_{x \in M^1} (\log |x|)^{-s}, \quad s \in \mathbf{C},$$

has analytic continuation to $\mathrm{Re}(s) > r_{\Gamma} - 2$ with simple poles at $s = r_{\Gamma}$ and $s = r_{\Gamma} = 1$.

Also, with L as in §0 if $L \cap W = \varnothing$,

$$\sum_{x \in M^1} f_W(x^1)(\log |x|)^{-s} = O_{r_{\Gamma}-2}(s).$$

These results follow because each of these sums is itself a finite sum of expressions dealt with in §1.

First we show how to lift these results to $M^{\times}$. Choose a system of coset representatives for M^1 in $M^{\times}$, $\alpha_1, \dots, \alpha_m$. Choose $0 < \varepsilon < 1$ and let R_i denote those $x \in M^1$ with

$$\frac{|\iota_{\psi}(\alpha_i x)|}{|\alpha_i x|} < \varepsilon$$

for all $\psi \in \hat{\Gamma}$ with $|\iota_{\psi}(\alpha_i x)| \neq |\alpha_i x|$. Notice that for $x \in R_i$, with ε sufficiently small,

$$\log |\alpha_i x| = c_i + \log |x|, \tag{26}$$

while for the others we have at least

$$\log |\alpha_i x| = \log |x| + O(1). \tag{27}$$

Break up the sum defining $\iota_{|\ |}(s)$ and expand using (26) and (27),

$$\begin{aligned}\iota_{|\ |}(s) &= \sum_{i=1}^{m} \sum_{x \in \cup R_i} (\log|x|)^{-s} - s\sum_{i=1}^{m} c_i \sum_{x \in \cup R_i} (\log|x|)^{-1-s} + O_{r_\Gamma - 2}(s)\\ &\quad + \sum_{i=1}^{m} \sum_{x \notin \cup R_i} (\log|x|)^{-s} + s\sum_{i=1}^{m} \sum_{x \notin \cup R_i} O(1)(\log|x|)^{-1-s}\\ &= m \sum_{x \in M^1} (\log|x|)^{-s} - s\sum_{i=1}^{m} c_i \sum_{x \in \cup R_i} (\log|x|)^{-1-s} + O_{r_\Gamma - 2}(s)\end{aligned}$$

where we have absorbed the last sum into the $O_{r_\Gamma - 2}(s)$ by the results in [2]. The analytic continuation for the first sum comes from the above theorem while that for the second comes again from [2].

In the same way we can obtain the analytic continuation of the series where we sum over $\mathbf{Z}\Gamma^{\times}$ rather than the larger group $M^{\times}$. This follows by choosing a finite number of coset representatives and grouping the terms appropriately.

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