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UNIFORMLY SWEEPING OUT DOES NOT IMPLY MIXING

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1. Introduction

Let T be an invertible measure preserving transformation on a measure space that is isomorphic to the unit interval with Lebesgue measure. It was shown in [F1] that if T is mixing, then T is uniformly sweeping out (see §2 for definitions). A sequential counterexample to the converse was given in [F2] where a transformation was constructed that is not mixing on a sequence but is uniformly sweeping out on the sequence.

In [C], Chacon constructed another example of a rank one transformation that is weakly mixing but not mixing that is different from Chacon's transformation [F3, 86–89]. In [FK] the example in [C] was shown to be lightly mixing, not partially mixing, and not lightly 2-mixing which implies not sweeping out of order 2.

Our purpose is to show the transformation T in [C] is uniformly sweeping out. Thus T is rank one, not partially mixing, uniformly sweeping out, but not sweeping out of order 2. This is in contrast to Kalikow's theorem which states that rank one mixing implies 2-mixing [KA].

We also note that it is not difficult to construct a partially mixing transformation that is not uniformly sweeping out.

It was shown in [FT] that $(2k - 1)$-mixing implies uniformly sweeping out of order k, $k \geq 1$. Thus mixing of all orders implies uniformly sweeping out of all orders. Concerning the converse, we do not know if uniform sweeping out of all orders implies mixing.

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2. Preliminaries

Let $(X, \mathscr{B}, \mu)$ be a measure space isomorphic to the unit interval with Lebesgue measure and let T be an invertible measure preserving map of X

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onto X. T is *lightly mixing* if for all sets A and B of positive measure we have

$$\liminf_{n\to\infty} \mu(T^n A \cap B) > 0. \tag{2.1}$$

Lightly mixing was introduced in [BCQ] where it is called sequence mixing. It is easy to show that T is lightly mixing if and only if for every set A of positive measure and increasing sequence of integers (k_i) we have $\mu(\bigcup_{i=1}^{\infty} T^{k_i} A) = 1$. In [F1] this property is referred to as *sweeping out*.

A transformation T is *partially mixing* if there exists $\beta > 0$ such that for all measurable sets A and B we have

$$\liminf_{n\to\infty} \mu(T^n A \cap B) \geq \beta\mu(A)\mu(B). \tag{2.2}$$

A transformation T is α-mixing, $0 < \alpha \leq 1$, if (2.2) holds for $\beta = \alpha$ but does not hold for $\beta > \alpha$. The first example of a lightly mixing transformation T that is not partially mixing was constructed in [BCQ] where T is the infinite direct product of a partially mixing transformation. In [KI1] King proved that a countable Cartesian product of lightly mixing transformations is lightly mixing. The question was asked in [KI1] whether a lightly mixing transformation that is not partially mixing could be constructed directly rather than being obtained as an infinite product. In [FK] it was shown that the rank one example [C] constructed directly by cutting and stacking is lightly mixing, not partially mixing, and not lightly 2-mixing.

A transformation T is *uniformly sweeping out* if for each set A of positive measure and $\varepsilon > 0$ there exists a positive integer $N = N(A, \varepsilon)$ such that $\mu(\bigcup_{i=1}^{n} T^{k_i} A) > 1 - \varepsilon$ for all $k_1 < k_2 < \cdots < k_N$. Mixing implies uniformly sweeping out [F1] and we will show that the transformation in [C] provides a counterexample to the converse. We also note King proved that a countable Cartesian product of uniformly sweeping out transformations is uniformly sweeping out [KI2].

A transformation T is lightly 2-mixing if for all sets A, B, and C of positive measure we have

$$\liminf_{m,n\to\infty} \mu(T^m(T^n A \cap B) \cap C) > 0. \tag{2.3}$$

A transformation T is *sweeping out of order* 2 if for each pair of sets A and B of positive measure and increasing sequences (k_i) and (j_i) we have $\mu(\bigcup_{i=1}^{\infty} T^{k_i}(T^{j_i} A \cap B)) = 1$. It is easy to show that sweeping out of order 2 is equivalent to lightly 2-mixing. A transformation T is *uniformly sweeping out of order* 2 if for each pair of sets A and B of positive measure and $\varepsilon > 0$, there exists $N = N(A, B, \varepsilon)$ such that $\mu(\bigcup_{i=1}^{N} (T^{k_i}(T^{j_i} A \cap B)) > 1 - \varepsilon$ for all $j_i < j_2 < \cdots < j_N$ and $k_1 < k_2 < \cdots k_N$. Clearly uniform sweeping out of order 2 implies sweeping out of order 2. Higher order uniform sweeping out is defined in general in [FT].

3. Example

For reference we will repeat the construction of the rank one transformation T in [C] which is most conveniently defined in terms of the n-blocks B_n for $n = 1, 2, 3, \ldots$. Let $B_1 = (0)$ and let s denote a spacer. By induction, we define $B_{n+1} = B_n B_n s$. If h_n is the length of B_n, then $h_{n+1} = 2h_n + 1$. It follows that $h_n = 2^n - 1$ for $n \geq 1$. We let $H_n = h_n + 1 = 2^n$ for $n \geq 1$.

In terms of cutting and stacking, let C_n denote the single column of height h_n corresponding to B_n. Therefore C_{n+1} is obtained by cutting C_n in half and stacking the right half above the left half with an additional spacer level denoted by S_{n+1} placed on top. We can begin with $C_1 = ([0, 1/2))$ and let $S_{n+1} = [1 - 1/2^n, 1 - 1/2^{n+1})$ for all $n \geq 1$. Thus we obtain $T = \lim_{n \to \infty} T_{C_n}$ defined on $[0, 1)$. In Figure 3.1 we show C_n of height h_n with top level S_n. The arrows show the action of T.

Let $\mu(C_n)$ denote the measure of the union of the levels in C_n; hence

$$\mu(C_n) = h_n(1/2^n) = (2^n - 1)/2^n = 1 - 1/2^n = 1 - 1/H_n.$$

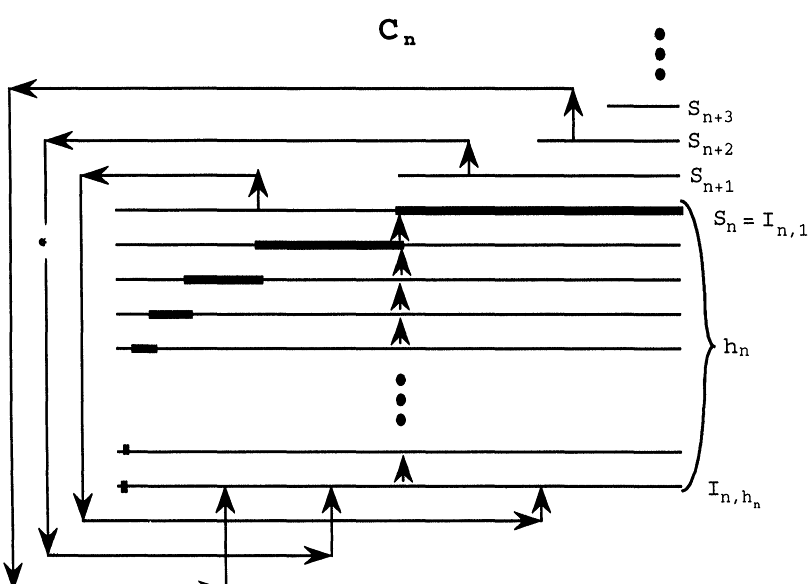

FIG. 1

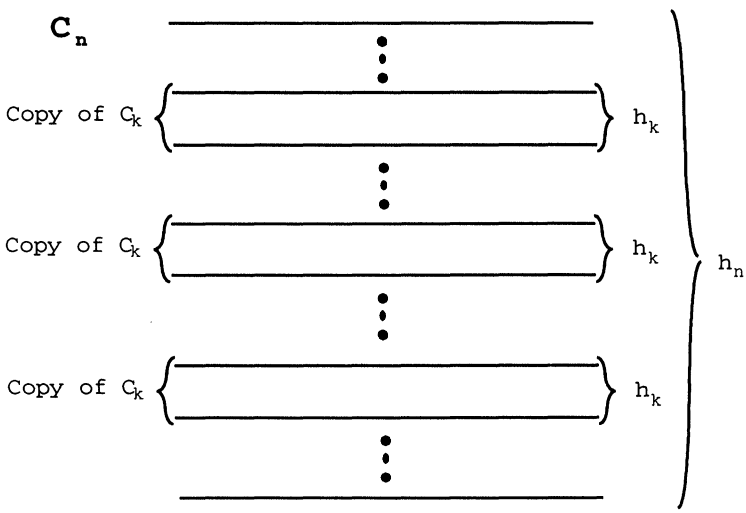

FIG. 2

Let $I_{n,i}$ denote the ith level of C_n starting at the top for $1 \leq i \leq h_n$ as in Figure 3.1. The construction implies that $T^{h_n} S_n$ is the union of the spacer interval S_{n+h_n} and the h_n intervals $T^{h_n} S_n \cap I_{n,i}$ for $1 \leq i \leq h_n$, which are indicated by bold lines in Figure 3.1. The interval lengths decrease by a factor of $1/2$ and we have $\mu(T^{h_n} S_n \cap I_{n,i}) = \mu(S_n)/2^i$ for $1 \leq i \leq h_n$. We will refer to the configuration of these intervals as in Figure 3.1 as a *crescent*.

Fix k and let $n > k$. The column C_k appears in C_n as 2^{n-k} disjoint groups of h_k consecutive levels of C_n. Each of these groups of h_k consecutive levels will be called a *copy of* C_k. Thus C_k appears in C_n as 2^{n-k} disjoint copies of C_k, as indicated in Figure 3.2.

For example, consider $k = 2$ and $n = 3$, as in Figure 3.3. The two copies of C_2 in C_3 are denoted by $C_{2,i}$ for $i = 1, 2,$. Let I be the top level in C_2; hence I consists of the top levels of the two copies of C_2 in C_3. The right half of the top level in $C_{2,i}$ is denoted by I_i for $i = 1, 2$.

Let $I_i^* = T^{H_3} I_i \cap C_{2,i}$ and $I^* = \bigcup_{i=1}^{2} I_i^*$. We will also refer to I_i^* as a crescent which is indicated by bold lines in Figure 3.3. It is convenient to work with these crescents rather than all of $T^{H_3} I \cap C_2$. Note that if L is one of the bottom six levels in C_3, then $L \subset C_2$ and

$$\mu(I^* \cap L) \geq \mu(L)/16 = \mu(L)/2^{H_2}.$$

The union of these six levels is C_2 and $\mu(C_2) = 1 - 1/H_2$. Furthermore, if

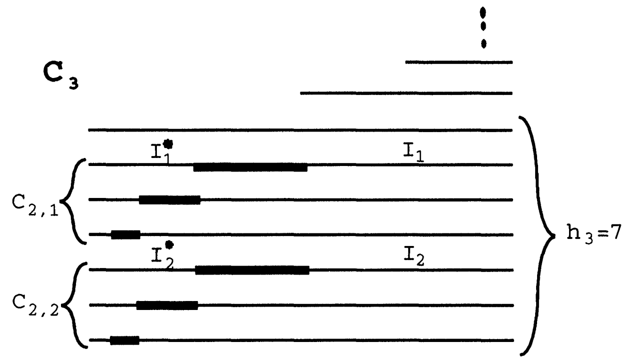

FIG. 3

$0 \le t \le H_3 = 8$, then

$$\mu(T^t I^* \cap L) \ge \mu(L)/2^{H_2}$$

for six levels L in C_3 whose union also has measure $\mu(C_2) = 1 - 1/H_2$.

In general, let $n > k$ and let $C_{k,i}$ be the ith copy of C_k in C_n for $1 \le i \le 2^{n-k}$. Let I be the top level in C_k and let I_i be the right half of the top level in $C_{k,i}$ for $1 \le i \le 2^{n-k}$. Let

$$I_i^* = T^{H_n} I_i \cap C_{k,i} \quad \text{for } 1 \le i \le 2^{n-k}.$$

We refer to I_i^* as a crescent, which is indicated by bold lines in Figure 3.4. If L is a level in $C_{k,i}$, then $\mu(T^{H_n} I_i \cap L) \ge \mu(L)/2^{H_k}$.

LEMMA 3.1. *Let $n > k$ and let G be a union of some of the top levels of the copies of C_k in C_n. Let $G^* = \bigcup_{I_i \subset G} I_i^*$. If $0 \le t \le H_n$, then $\mu(T^t G^* \cap L) \ge \mu(L)/2^{H_k}$ for a class of levels L in C_n whose union has measure $(\mu(G)/\mu(I))(1 - 1/H_k)$, where I is the top level in C_k.*

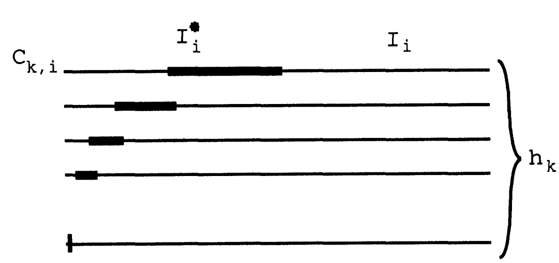

FIG. 4

Proof. A crescent $I_i^* = T^{H_n} I_i \cap C_{k,i}$ as in Figure 3.4 starts in the left half of C_n. As t increases it moves upward under T^t until it passes through the top left of C_n and then moves into the lower right half of C_n. For $0 \le t \le H_n$, $T^t I_i^*$ intersects h_k levels L of C_n in measure at least $\mu(L)/2^{H_k}$. Let r be the number of top levels in G. Therefore G^* consists of r crescents. Hence if $0 \le t \le H_n$, then $\mu(T^t G^* \cap L) \ge \mu(L)/2^{H_k}$ for rh_k levels L in C_n. Now

$$rh_k \mu(L) = r\mu(C_k)/2^{n-k} = \frac{r\mu(L)}{2^{n-k}\mu(L)}\mu(C_k)$$

$$= \frac{\mu(G)}{\mu(I)}(1 - 1/H_k).$$

LEMMA 3.2. *Let $n > k$ and let G^* be as in Lemma 3.1. If $0 \le t \le H_n$, then $T^t G^*$ and $C_n - T^t G^*$ are unions of levels in C_m for $m \ge n + H_k$.*

Proof. If L is a level in C_n such that $\mu(T^t G^* \cap L) > 0$, then $T^t G^* \cap L$ is an interval whose length is a multiple of $\mu(L)/2^{H_k}$. This interval will appear as a union of levels in C_m for $m \ge n + H_k$. Moreover, $L - T^t G^*$ will consist of two intervals with lengths that are multiples of $\mu(L)/2^{H_k}$. These intervals will also appear as unions of levels in C_m for $m \ge n + H_k$.

The inverse transformation T^{-1} acts on levels of a column C_n in a similar way that T does. In this case we let I be the bottom level in C_k and let I_i be the left half of the bottom level of a copy $C_{k,i}$ of C_k in C_n. The corresponding crescent $I_i^* = (T^{-H_n} I_i) \cap C_{k,i}$ is shown in Figure 3.5.

Lemmas 3.1 and 3.2 with T and top replaced by T^{-1} and bottom, respectively, are proved in exactly the same way. We remark that it is not difficult to show that T and T^{-1} are isomorphic.

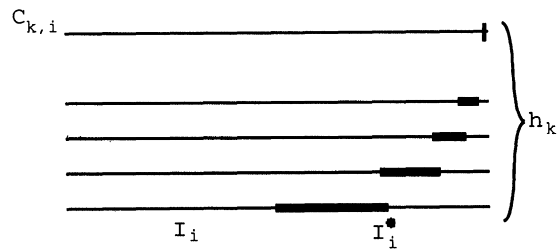

FIG. 5

4. Subset selection

In this section we will prove that given a sufficiently large set of integers, we can select a certain subset with certain growth properties.

LEMMA 4.1. *Let k be a positive integer and let P be a set of 2^{2k} positive integers. There exists a subset $S = \{s_1 < s_2 < \cdots < s_k\} \subset P$ such that either* (a) *or* (b) *holds*:

(a) $s_k - s_i \geq 2(s_k - s_{i+1})$ *for* $i = 1, 2, \ldots, k-1$.

(b) $s_{i+1} - s_1 \geq 2(s_i - s_1)$ *for* $i = 1, 2, \ldots, k-1$.

Proof. We will obtain disjoint subsets A_{2k} and B_{2k} of P such that $A_{2k} \cup B_{2k}$ will have $2k+1$ numbers. Therefore, either A_{2k} or B_{2k} will have at least k numbers.

Given an interval $I = [m, M]$, the left and right halves will be denoted by

$$L(I) = \left[m, \frac{m+M}{2}\right] \quad \text{and} \quad R(I) = \left[\frac{m+M}{2}, M\right],$$

respectively.

Let $a_1 = \min P$ and $b_1 = \max P$. Let $A_1 = \{a_1\}$ and $B_1 = \{b_1\}$. Let $I_1 = [a_1, b_1]$; hence $I_1 \cap P = P$. Thus, I_1 has 2^{2k} numbers.

We now proceed by induction. Let $i < 2k$. After the ith step, we have

$$A_i = \{a_1 < a_2 < \cdots < a_{u_i}\} \quad \text{and} \quad B_i = \{b_{v_i} < \cdots < b_2 < b_1\},$$

where $u_i + v_i = i + 1$. Also, $a_{u_i} < b_{v_i}$ and if $I_i = [a_{u_i}, b_{v_i}]$, then $I_i \cap P$ has at least 2^{2k-i+1} numbers.

If $L(I_i) \cap P$ has at least 2^{2k-i} numbers, then define $b_{v_i+1} = \max L(I_i) \cap P$, $v_{i+1} = v_i + 1$, and $u_{i+1} = u_i$. Otherwise, $R(I_i) \cap P$ has at least 2^{2k-i} numbers and we define $a_{u_i+1} = \min R(I_i) \cap P$, $u_{i+1} = u_i + 1$, and $v_{i+1} = v_i$. In either case, $I_{i+1} = [a_{u_{i+1}}, b_{v_{i+1}}]$ will have at least 2^{2k-i} numbers.

Proceeding inductively, we arrive at

$$A_{2k} = \{a_1 < a_2 < \cdots < a_{u_{2k}}\} \quad \text{and} \quad B_{2k} = \{b_{v_{2k}} < \cdots < b_2 < b_1\},$$

where $u_{2k} + v_{2k} = 2k + 1$. Consider $u_{i+1} = u_i + 1$ for $i < 2k$; hence

$$a_{u_i+1} \geq \frac{a_{u_i} + b_{v_i}}{2} \geq \frac{a_{u_i} + a_{u_{2k}}}{2}. \tag{1}$$

From (1) we obtain

$$a_{u_{2k}} - a_{u_i} \geq 2(a_{u_{2k}} - a_{u_i+1}) \tag{2}$$

Next consider $v_{i+1} = v_i + 1$ for $i < 2k$; hence

$$(3) \qquad b_{v_{i+1}} \leq \frac{b_{v_i} + a_{u_i}}{2} \leq \frac{b_{v_i} + b_{v_{2k}}}{2}.$$

From (3) we obtain

$$(4) \qquad 2\left(b_{v_{i+1}} - b_{v_{2k}}\right) \leq b_{v_i} - b_{v_{2k}}.$$

If A_{2k} has at least k numbers, let S be the largest k numbers in A_{2k}; hence $s_k = a_{u_{2k}}$. It follows from (2) that S satisfies (a). Otherwise B_{2k} has at least k numbers. In this case let S be the smallest k numbers in B_{2k}; hence $s_1 = b_{v_{2k}}$. It follows from (4) that S satisfies (b).

LEMMA 4.2. *Let M and H be positive integers. Let P be a set of $2^{2(M+1)H}$ positive integers. There exists $S = \{s_1 < s_2 < \cdots < s_{M+1}\}$ such that either (a) or (b) holds:*

(a) $s_{M+1} - s_i \geq 2^H(s_{M+1} - s_{i+1})$ *for* $i = 1, 2, \ldots, M$.

(b) $s_{i+1} - s_1 \geq 2^H(s_i - s_1)$ *for* $i = 1, 2, \ldots, M$.

Proof. Apply Lemma 4.1 to get a subset of $(M+1)H$ numbers satisfying either (a) or (b) of Lemma 4.1. Extract every H-th number to obtain a subset of $M + 1$ numbers satisfying either (a) or (b) above.

LEMMA 4.3. *Let $\delta > 0$ and let H be a positive integer. Suppose r_n, $n = 1, 2, 3, \ldots$, is a sequence of real numbers such that $r_1 = 1$ and*

$$r_n \leq r_{n-1} - \frac{1}{2^H}(r_{n-1} - \delta) \quad \text{for } n > 1.$$

Then

$$r_n \leq \left(1 - \frac{1}{2^H}\right)^{n-1} + \delta \quad \text{for } n = 1, 2, 3, \ldots.$$

Proof. We have

$$r_1 = 1 < 1 + \delta = \left(1 - \frac{1}{2^H}\right)^0 + \delta.$$

Assume

$$(1) \qquad r_{n-1} \leq \left(1 - \frac{1}{2^H}\right)^{n-2} + \delta.$$

Hence

$$(2)\qquad \begin{aligned} r_n &\le r_{n-1} - \frac{1}{2^H}(r_{n-1} - \delta) \\ &= r_{n-1}\left(1 - \frac{1}{2^H}\right) + \delta/2^H \\ &\le \left(1 - \frac{1}{2^H}\right)^{n-1} + \delta\left(1 - \frac{1}{2^H}\right) + \delta/2^H \\ &= \left(1 - \frac{1}{2^H}\right)^{n-1} + \delta. \end{aligned}$$

Thus the lemma follows by induction.

5. Main result

Let T be the transformation constructed in Section 3.

THEOREM 5.1. *The transformation T is uniformly sweeping out.*

Proof. Let A be a set of positive measure and $\varepsilon > 0$. Choose k sufficiently large so that $1/H_k < \varepsilon/100$ and there exists a level I in C_k such that $\mu(A \cap I) \ge (1 - \varepsilon/100)\mu(I)$. We can assume I is the top level in C_k and $A = A \cap I$. Choose M so that

$$(1)\qquad (1 - 1/2^{H_k})^M < \varepsilon/100.$$

There exists $n > k$ so large that there exists a union G of top levels of copies of C_k in C_n such that

$$(2)\qquad \mu(G \Delta A) < (\varepsilon/100M)\mu(A).$$

It follows that $\mu(G \cap I) \ge (1 - \varepsilon/50)\mu(I)$. Choose N as

$$(3)\qquad N = 2^{2(M+1)H_n}.$$

Let P be a set of positive integers with N numbers. By Lemma 4.2 with $H = H_n$, there exists a subset S with $M + 1$ numbers satisfying either (a) or (b). First assume (b) is satisfied. Let $s'_1 = 0$, $s'_2 = s_3 - s_1$, $s'_3 = s_4 - s_1, \ldots, s'_M = s_{M+1} - s_1$. Note that $s'_2 = s_3 - s_1 \ge 2^{H_n}(s_2 - s_1) \ge 2^{H_n} > H_n$.

We can write $s'_u = H_{n_u} + t_u$ where $0 \le t_u < H_{n_u}$ for $2 \le u \le M$. We have

$$\begin{aligned} s'_{u+1} &\ge 2^{H_n} s'_u = 2^{H_n}(H_{n_u} + t_u) \\ &= H_{n_u + H_n} + 2^{H_n} t_u > H_{n_u + H_k}. \end{aligned}$$

Thus we have

$$s'_{u+1} > H_{n_u + H_k}, \qquad 2 \le u < M. \tag{4}$$

Let G_u^* correspond to G^* in Lemma 3.1 for $n = n_u$, $u = 2, 3, \ldots, M$. Let $R_1 = X$, $R_2 = G^c$, and

$$R_u = \left(G \cup \bigcup_{i=2}^{u-1} T^{t_i} G_i^* \right)^c \tag{5}$$

for $u = 3, 4, \ldots, M + 1$. Intuitively, R_u is the remainder at stage u. From (5) we have

$$R_{u+1} = R_u \cap \left(T^{t_u} G_u^*\right)^c. \tag{6}$$

It follows from (4) and Lemma 3.2 that $R_u \cap C_{n_u}$ is a union of levels in C_{n_u}. Let $r_u = \mu(R_u)$. We will apply Lemma 4.3 to estimate r_u.

Define δ as

$$\delta = 1 - \frac{\mu(G)}{\mu(I)}\left(1 - \frac{1}{H_k}\right) < 1 - \left(1 - \frac{\varepsilon}{50}\right)\left(1 - \frac{\varepsilon}{100}\right) < \frac{\varepsilon}{25}. \tag{7}$$

We have $r_1 = \mu(X) = 1$ and $r_2 = 1 - \mu(G)$. Now

$$\begin{aligned} r_1 - \frac{1}{2^{H_k}}(r_1 - \delta) &= 1 - \frac{1}{2^{H_k}}(1 - \delta) \\ &> 1 - \frac{1}{2^{H_k}} > 1 - \frac{1}{2H_k} > 1 - \mu(G) = r_2. \end{aligned} \tag{8}$$

Let D_u denote the union of levels L in C_{n_u} such that $\mu(T^{t_u} G_u^* \cap L) \ge \mu(L)/2^{H_k}$. By Lemma 3.1 we have $\mu(D_u) = 1 - \delta$. In particular, let L be a level in C_{n_u} such that $L \subset R_u$. Therefore the measure of the union of levels $L \subset R_u$ such that $\mu(T^{t_u} G_u^* \cap L) \ge \mu(L)/2^{H_k}$ is at least $\mu(R_u) - \delta$. Thus

$$\mu\left(R_u \cap T^{t_u} G_u^*\right) \ge \frac{1}{2^{H_k}}\left(\mu(R_u) - \delta\right) \tag{9}$$

From (6) and (9) we obtain

$$\mu(R_{u+1}) \leq \mu(R_u) - \frac{1}{2^{H_k}}(\mu(R_u) - \delta) \tag{10}$$

Thus (8), (10), and Lemma 4.3 with $H = H_k$ imply

$$\begin{aligned} \mu(R_{M+1}) &< (1 - 1/2^{H_k})^M + \delta \\ &< \frac{\varepsilon}{100} + \frac{\varepsilon}{25} < \varepsilon/2. \end{aligned} \tag{11}$$

Hence, (11) implies

$$\mu\left(\left(\bigcup_{i=1}^{M} T^{s'_i}G\right)^c\right) \leq \mu(R_{M+1}) < \varepsilon/2.$$

Therefore,

$$\begin{aligned} \mu\left(\left(\bigcup_{p \in P} T^p A\right)^c\right) &\leq \mu\left(\left(\bigcup_{i=1}^{M} T^{s'_i}A\right)^c\right) \\ &\leq \mu\left(\left(\bigcup_{i=1}^{M} T^{s'_i}G\right)^c\right) + M\mu(G \Delta A) \\ &< \frac{\varepsilon}{2} + M \cdot \frac{\varepsilon}{100M} \\ &< \varepsilon. \end{aligned}$$

Next we assume there is a subset S satisfying (a) of Lemma 4.2. Let $s'_1 = 0$, $s'_2 = s_{M+1} - s_{M-1}$, $s'_3 = s_{M+1} - s_{M-2}, \ldots, s'_M = s_{M+1} - s_1$. Note that

$$s'_2 = s_{M+1} - s_{M-1} \geq 2^{H_n}(s_{M+1} - s_M) > H_n.$$

So, for $2 \leq u \leq M$ we can write $s'_u = H_{n_u} + t_u$ where $0 < t_u < H_{n_u}$. Then property (a) in Lemma 4.2 says we have the relation

$$s'_{u+1} \geq H_{n_u + H_k}, \qquad 2 \leq u < M.$$

Now, we can use Lemmas 3.1 and 3.2 for T^{-1} to get a similar argument showing

$$\mu\left(\left(\bigcup_{i=1}^{M} T^{-s'_i}G\right)^c\right) < \varepsilon/2.$$

Hence, we get

$$\mu\left(\left(\bigcup_{i=1}^{M} T^{-s_i'}A\right)^c\right) < \varepsilon. \tag{12}$$

But, $-s_i' = -(s_{M+1} - s_{M-i+1}) = s_{M-i+1} - s_{M+1}$. Therefore, (12) gives

$$\mu\left(\left(\bigcup_{p\in P} T^{p}A\right)^c\right) < \varepsilon.$$

Thus T is uniformly sweeping out.

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