THE ANALOGUE OF THE PISOT-VIJAYARAGHAVAN NUMBERS IN FIELDS OF FORMAL POWER SERIES[1]

BY

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Introduction

A Pisot-Vijayaraghavan number (PV number) is an algebraic integer θ greater than 1 all of whose conjugates, other than θ itself, lie in the open unit disc $|z| < 1$. Vijayaraghavan [9] proved that there exist PV numbers of all degrees, and Pisot [7] proved that in every real algebraic number field there exist PV numbers which generate the field.

The PV numbers have the following basic property, as is easily seen by considering the trace. If θ is a PV number and λ is an algebraic integer in the field generated by θ, then $\| \lambda\theta^n \|$ goes (exponentially) to zero as $n \to +\infty$. Here $\| \lambda\theta^n \|$ denotes the difference, taken positively, between $\lambda\theta^n$ and the nearest integer.

On the other hand, this property of the PV numbers characterizes them to a considerable extent, as the following two results show. Hardy [3] and Vijayaraghavan [9] proved that if θ is an *algebraic* number greater than 1, if λ is a nonzero real number, and if $\| \lambda\theta^n \| \to 0$ as $n \to +\infty$, then θ is a PV number, and λ is in the algebraic number field generated by θ. Secondly, Pisot [6] proved that if θ is a *real* number greater than 1, if λ is a nonzero real number, and if $\sum_{n=1}^{\infty} \| \lambda\theta^n \|^2$ converges, then θ is algebraic and therefore a PV number. An exposition of these two results may be found in [1, Ch. 8]. A comprehensive bibliography is given in [8].

It is reasonable to conjecture that we can suppress the hypothesis that θ is algebraic in the first of these two theorems or, equivalently, that we can replace the hypothesis of the convergence of $\sum_{n=1}^{\infty} \| \lambda\theta^n \|^2$ in the second theorem by the assumption that $\lim_{n\to+\infty} \| \lambda\theta^n \| = 0$. In fact this is the principal unsolved problem of the theory at the present time (cf. [10]).

In this paper we construct an analogous theory in the following parallel situation (cf. [4]). In place of the rational integers we consider the ring $k[x]$ of polynomials in an indeterminate x with coefficients in a given field k. In place of the field of rational numbers we consider the field $k(x)$ of rational functions in x with coefficients in k. In place of the field of real numbers we consider the field $k\{x^{-1}\}$ of pole-like formal Laurent series about ∞ with

Received July 21, 1961.

[1] This work was supported by the Office of Naval Research through a contract with the University of Illinois, and also by the Jet Propulsion Laboratory of the California Institute of Technology under a program sponsored by the National Aeronautics and Space Administration.

coefficients in k, that is, series of the form

$$z = a_h x^h + a_{h-1} x^{h-1} + \cdots = \sum_{j=-h}^{\infty} a_{-j} x^{-j}.$$

The field $k(x)$ may be considered as a subfield of $k\{x^{-1}\}$ by formally expanding each rational function about ∞. In place of the integral part of a real number we consider that part of such a Laurent series involving only non-negative powers of x, that is, we write

$$[z] = a_h x^h + a_{h-1} x^{h-1} + \cdots + a_0 .$$

In place of the fractional part of a real number we consider that part of a Laurent series involving only negative powers of x, that is, we consider

$$((z)) = a_{-1} x^{-1} + a_{-2} x^{-2} + \cdots .$$

In place of the ordinary absolute value of a real number we use the absolute value defined by putting

$$| z | = c^h,$$

provided $a_h \neq 0$, where c is some fixed constant greater than 1. Thus if $z \in k\{x^{-1}\}$, we have

$$z = [z] + ((z)), \qquad [z] \in k[x], \qquad | ((z)) | < 1.$$

Note that the concepts of "integral part" and "nearest integer" coincide in this context. In place of the field of complex numbers we consider the algebraic closure K of $k\{x^{-1}\}$. Our absolute value has a unique extension to K as follows. If $w \in K$ (so that w is algebraic over $k\{x^{-1}\}$), if N denotes the norm function for the extension $k\{x^{-1}\}(w)$ over $k\{x^{-1}\}$, and if n is the degree of $k\{x^{-1}\}(w)$ over $k\{x^{-1}\}$, then we have

$$| w | = | Nw |^{1/n}.$$

The absolute value defined on K in this way is the one we shall use throughout this paper. An element of $k\{x^{-1}\}$ which is not in k will be called a PV element if and only if it is integral over $k[x]$ and all of its conjugates with respect to $k(x)$, other than itself, have absolute value less than one when regarded as elements of K.

All of the theorems on PV numbers quoted above carry over to this new context, although most of them require either a separability assumption or the assumption that k be perfect. Since our absolute value has the property

$$| z_1 + z_2 | \leqq \max (| z_1 |, | z_2 |),$$

it will turn out that the conjecture mentioned in the fourth paragraph becomes trivially true in the present context, provided we have a perfect base field (see Theorem 3.4 below). For here a series converges if and only if its n^{th} term approaches zero.

Throughout, x will be an indeterminate over the ground field k, the algebraic closure of $k\{x^{-1}\}$ will be denoted by K, and T will denote the trace function for the extension $k(x)(\theta)$ over $k(x)$, where θ is algebraic over $k(x)$.

1. Existence of PV elements

DEFINITION. *If k is a given field, an element of $k\{x^{-1}\}$ is called a PV element if and only if it is not in k, it is integral over $k[x]$, and all of its conjugates with respect to $k(x)$ have absolute value less than* 1 (*under the absolute value defined earlier on the algebraic closure K of $k\{x^{-1}\}$*).

Note that a PV element is necessarily separable over $k(x)$ and also has absolute value greater than 1. Another immediate consequence of the definition is that a positive integral power of a PV element is a PV element.

THEOREM 1.1. *Suppose k is an arbitrary field. Then the field $k\{x^{-1}\}$ contains PV elements of all degrees over $k(x)$.*

Proof. Suppose r is a given positive integer. We shall prove that the algebraic equation

$$t^r - xt^{r-1} - 1 = 0 \tag{1}$$

has a root t_0 in $k\{x^{-1}\}$ which is a PV element of degree r. Since every element of $k[x]$ which is not in k is a PV element, this is of course trivial when $r = 1$.

We begin by proving that the polynomial $t^r - xt^{r-1} - 1$ is irreducible over $k(x)$. If not, by Gauss's lemma (see [12, pp. 32–33]) it could be factored nontrivially over $k[x]$. But since $t^r - xt^{r-1} - 1$ is linear in x, any such factorization must be of the form $\{f(t)x - g(t)\}h(t)$, where $f(t)$, $g(t)$, and $h(t)$ are polynomials in t with coefficients in k. Hence $f(t)h(t) = -t^{r-1}$ and $g(t)h(t) = 1 - t^r$, so that

$$\{tf(t) - g(t)\}h(t) = -1.$$

Thus $h(t)$ is of degree zero, so that $t^r - xt^{r-1} - 1$ is irreducible over $k(x)$. Hence any root of (1) is of degree n over $k(x)$.

Next we prove that (1) has a root t_0 in $k\{x^{-1}\}$ such that $|t_0| > 1$. In fact it is easy to see that (1) has a root of the form

$$t_0 = x(1 + a_1 x^{-r} + a_2 x^{-2r} + \cdots).$$

For

$$\begin{aligned} t_0^r - xt_0^{r-1} - 1 &= (t_0 - x)t_0^{r-1} - 1 \\ &= \left\{\textstyle\sum_{j=0}^{\infty} a_{j+1} x^{-rj}\right\}\left\{1 + \textstyle\sum_{j=1}^{\infty} a_j x^{-rj}\right\}^{r-1} - 1 \\ &= \{a_1 - 1\} + \{a_2 + f_1(a_1)\}x^{-r} + \cdots \\ &\qquad + \{a_{i+1} + f_i(a_1, \cdots, a_i)\}x^{-ir} + \cdots, \end{aligned}$$

where $f_i(a_1, \cdots, a_i)$ is a polynomial in $a_1, \cdots, a_i$ of degree at most r.

Thus $a_1, a_2, \cdots$ can be determined in turn in such a way that

$$t_0^r - xt_0^{r-1} - 1 = 0.$$

For example $a_1 = 1$ and $a_2 = 1 - r$.

Now the conjugates of t_0 with respect to $k(x)$ are zeros of the polynomial

$$\frac{t^r - xt^{r-1} - 1}{t - t_0} = t^{r-1} + (t_0 - x)t^{r-2} + t_0(t_0 - x)t^{r-3} + \cdots \qquad (2)$$
$$+ t_0^{r-3}(t_0 - x)t + t_0^{r-2}(t_0 - x),$$

which has coefficients in $k\{x^{-1}\}$. But the polynomial (2) is irreducible over $k\{x^{-1}\}$. For the subring R of $k\{x^{-1}\}$ consisting of those Laurent series containing only nonpositive powers of x is a unique factorization domain with the single prime element x^{-1}, and the polynomial (2) has coefficients in R and satisfies the conditions of the Eisenstein irreducibility criterion (see [11, Ch. IV]). In fact

$$t_0^{j-1}(t_0 - x) = x^{j-r} + (j - r)x^{j-2r} + \cdots \qquad (1 \leqq j \leqq r - 1).$$

Finally, since the polynomial (2) is irreducible over $k\{x^{-1}\}$, any of its zeros in K has absolute value equal to

$$| t_0^{r-2}(t_0 - x) |^{1/r} = | x^{-1} - x^{-1-r} + \cdots |^{1/r} = c^{-1/r} < 1,$$

in view of the definition of absolute value in K. Thus t_0 is a PV element of degree r. This completes the proof of Theorem 1.1.

THEOREM 1.2. *Suppose k is an arbitrary field. Then any separable extension of $k(x)$ of finite degree which is contained in $k\{x^{-1}\}$ can be generated by a PV element.*

Proof. Since any separable field extension of finite degree is simple (cf. [12, pp. 84–86]), we may restrict attention to extensions of $k(x)$ of the form $k(x)(\xi)$, where ξ is an element of $k\{x^{-1}\}$ which is algebraic and separable over $k(x)$. Suppose ξ is of degree $s + 1$ over $k(x)$ and its conjugates in K are $\xi_0 = \xi, \xi_1, \cdots, \xi_s$. Let $\omega_0, \cdots, \omega_s$ be a $k[x]$-basis for the integral closure of $k[x]$ in $k(x)(\xi)$ (cf. [12, pp. 264–266]). Let $\omega_0^{(r)}, \cdots, \omega_s^{(r)}$ be the conjugate basis for $k(x)(\xi_r)$ $(r = 1, \cdots, s)$. Any element of $k(x)(\xi)$ which is integral over $k[x]$ then has the form

$$\sum_{i=0}^{s} X_i \omega_i,$$

where $X_0, \cdots, X_s$ are in $k[x]$.

If $f(t)$ is the minimal polynomial for ξ over $k(x)$, we must consider how $f(t)$ factors in $k\{x^{-1}\}$. Suppose

$$f(t) = (t - \xi)g_1(t) \cdots g_v(t),$$

where $g_l(t)$ is an irreducible polynomial of degree s_l with coefficients in $k\{x^{-1}\}$ $(1 \leqq l \leqq v)$ and $s_1 + \cdots + s_v = s$. Suppose the notation is chosen so that

$$\begin{aligned} \xi_1, \cdots, \xi_{s_1} &\quad \text{are zeros of } g_1, \\ \xi_{s_1+1}, \cdots, \xi_{s_1+s_2} &\quad \text{are zeros of } g_2, \\ &\vdots \\ \xi_{s_1+\cdots+s_{v-1}+1}, \cdots, \xi_s &\quad \text{are zeros of } g_v. \end{aligned}$$

For $1 \leqq l \leqq v$ and $s_1 + \cdots + s_{l-1} + 1 \leqq h \leqq s_1 + \cdots + s_l$ let $\lambda_1^{(h)}, \cdots, \lambda_{s_l}^{(h)}$ be a field basis for $k\{x^{-1}\}(\xi_h)$ over $k\{x^{-1}\}$, where we assume that for fixed l the respective bases for $k\{x^{-1}\}(\xi_{s_1+\cdots+s_{l-1}+1}), \cdots, k\{x^{-1}\}(\xi_{s_1+\cdots+s_l})$ are conjugate over $k\{x^{-1}\}$. Thus for $1 \leqq l \leqq v$ and $s_1 + \cdots + s_{l-1} + 1 \leqq h \leqq s_1 + \cdots + s_l$ we have

$$\omega_i^{(h)} = \sum_{j=1}^{s_l} \alpha_{ij}^{(l)} \lambda_j^{(h)} \qquad (0 \leqq i \leqq s),$$

where $\alpha_{ij}^{(l)}$ is in $k\{x^{-1}\}$. Thus

$$\sum_{i=0}^{s} X_i \, \omega_i^{(h)} = \sum_{j=1}^{s_l} \lambda_j^{(h)} \Big(\sum_{i=0}^{s} \alpha_{ij}^{(l)} X_i \Big) \tag{1}$$

for $1 \leqq l \leqq v$ and $s_1 + \cdots + s_{l-1} + 1 \leqq h \leqq s_1 + \cdots + s_l$. Let M and L be integers such that

$$\max_{l,i,j} | \alpha_{ij}^{(l)} | = c^M, \qquad \max_{h,j} | \lambda_j^{(h)} | = c^L.$$

We wish to choose $X_0, \cdots, X_s$ as elements of $k[x]$ not all zero such that

$$\Big| \sum_{i=0}^{s} X_i \, \omega_i^{(h)} \Big| < 1 \qquad (1 \leqq h \leqq s). \tag{2}$$

Suppose we try to do this with $X_0, \cdots, X_s$ of absolute value at most c^m, that is, let us try to satisfy (2) with

$$X_i = u_{i0} + u_{i1} x + \cdots + u_{im} x^m \qquad (0 \leqq i \leqq s),$$

where $u_{i0}, \cdots, u_{im}$ are in k. In view of (1) we need only make

$$\Big| \sum_{i=0}^{s} \alpha_{ij}^{(l)} X_i \Big| < c^{-L} \qquad (1 \leqq l \leqq v, \ \ 1 \leqq j \leqq s_l). \tag{3}$$

(If $M < -L$, then we can trivially satisfy (3) by taking each X_i equal to 1. Thus we may assume in what follows that $L + M \geqq 0$.) We shall try to choose the u's so that the highest power of x occurring in the Laurent series for

$$\sum_{i=0}^{s} \alpha_{ij}^{(l)} X_i$$

is x^{-L-1} or lower $(1 \leqq l \leqq v, \ 1 \leqq j \leqq s_l)$. Now the highest power of x occurring in any of the α's is x^M, and the highest power of x occurring in the X's is x^m. Thus we need only choose the u's so that the coefficient of x^h in each of the s expressions

$$\sum_{i=0}^{s} \alpha_{ij}^{(l)} X_i \qquad (1 \leqq l \leqq v, \ \ 1 \leqq j \leqq s_l)$$

is zero for

$$-L \leqq h \leqq M + m.$$

This gives $(L + M + m + 1)s$ homogeneous linear equations for the

$(m+1)(s+1)$ unknowns u_{ig} $(0 \leqq i \leqq s,\ 0 \leqq g \leqq m)$. Thus if

$$m \geqq (L+M)s,$$

we have a nontrivial solution of these equations, which provides a nontrivial solution of the set of inequalities (3) and therefore of the inequalities (2).

Put

$$\theta = \sum_{i=0}^{s} X_i\, \omega_i\,,$$

where $X_0, X_1, \cdots, X_m$ is such a nontrivial solution of (2). The conjugates of θ over $k(x)$ are

$$\theta_r = \sum_{i=0}^{s} X_i\, \omega_i^{(r)} \qquad (1 \leqq r \leqq s).$$

Thus θ is an element of $k(x)(\xi)$ such that θ is integral over $k[x]$, $\theta \neq 0$, $\theta_r \neq 0$ $(1 \leqq r \leqq s)$, and $|\theta_r| < 1$ $(1 \leqq r \leqq s)$. Since $|\theta\theta_1 \cdots \theta_r| \geqq 1$, we have $|\theta| > 1$. Hence θ is a PV element. Clearly θ has exact degree s over $k(x)$, for otherwise it would coincide with one of its conjugates. Thus θ generates $k(x)(\xi)$ over $k(x)$. This completes the proof of Theorem 1.2.

Arguments similar to the preceding were used by Mahler in [5]. We remark that we have essentially proved and used a crude analogue of Minkowski's theorem on complex linear forms.

We remark also that a slight modification of the proof of Theorem 1.2 shows that, given some separable extension of $k(x)$ of finite degree which is contained in $k\{x^{-1}\}$, there are actually infinitely many PV elements θ which generate it. To see this we need only replace the right side of (2) by c^{-N}, where N is an arbitrary nonnegative integer, and then replace L by $L+N$ in the subsequent part of the proof.

The following lemma will be needed not only for the next theorem but also for Theorem 3.3. More general results of this type are known, but this lemma is sufficient for our purposes. The proof given below was communicated to us by Irving Reiner.

LEMMA 1.1. *Suppose k is a perfect field. Then any element of $k\{x^{-1}\}$ which is algebraic over $k(x)$ is also separable over $k(x)$.*

Proof. The result is trivial if k has characteristic zero. So let us assume that k has characteristic p, where p is some prime number. Suppose $\alpha(x)$ is in $k\{x^{-1}\}$ and $\alpha(x)$ is a zero of the polynomial $g(t)$ with coefficients in $k(x)$, where the degree of $g(t)$ is as small as possible. Without loss of generality we may assume that $g(t)$ has coefficients in $k[x]$. Suppose that $g(t)$ were inseparable. Then (cf. [12, p. 67]) $g(t)$ has the form

$$(1) \qquad g(t) = A_n(x)t^{np} + A_{n-1}(x)t^{(n-1)p} + \cdots + A_0(x),$$

where $A_i(x)$ is in $k[x]$ for $i = 0, 1, \cdots, n$ and $A_n(x) \neq 0$. We may write

$$(2) \qquad A_i(x) = A_{i0}(x^p) + xA_{i1}(x^p) + \cdots + x^{p-1}A_{i,p-1}(x^p) \qquad (i = 0, 1, \cdots, n),$$

where $A_{ij}(x)$ is in $k[x]$. Since $g(\alpha(x)) = 0$, we have from (1) and (2)

$$(3) \quad \textstyle\sum_{j=0}^{p-1} x^j\{A_{nj}(x^p)\alpha(x)^{pn} + A_{n-1,j}(x^p)\alpha(x)^{p(n-1)} + \cdots + A_{0j}(x^p)\} = 0.$$

By a familiar property of fields of characteristic p we have

$$\alpha(x)^{pi} = \{\alpha(x)^p\}^i = \{\beta(x^p)\}^i,$$

where $\beta(x)$ is in $k\{x^{-1}\}$ and is obtained from $\alpha(x)$ by raising each coefficient to the p^{th} power. Hence the j^{th} term in the sum in (3) contains only powers of x with exponents congruent to j modulo p. Thus each term in the sum must be separately zero, that is,

$$(4) \quad A_{nj}(x^p)\alpha(x)^{pn} + A_{n-1,j}(x^p)\alpha(x)^{p(n-1)} + \cdots + A_{0j}(x^p) = 0 \qquad (j = 0, 1, \cdots, p-1).$$

Since k is perfect, every coefficient of $A_{ij}(x)$ is a p^{th} power, and so there exists $B_{ij}(x)$ in $k[x]$ such that

$$A_{ij}(x^p) = \{B_{ij}(x)\}^p \qquad (i = 0, 1, \cdots, n; j = 0, 1, \cdots, p-1).$$

Thus (4) may be written

$$(5) \quad \{B_{nj}(x)\alpha(x)^n + B_{n-1,j}(x)\alpha(x)^{n-1} + \cdots + B_{0j}(x)\}^p = 0 \qquad (j = 0, 1, \cdots, p-1).$$

Since $A_n(x) \neq 0$, there exists a j such that $A_{nj}(x) \neq 0$ and so $B_{nj}(x) \neq 0$. Then by (5), $\alpha(x)$ is a zero of the polynomial

$$B_{nj}(x)t^n + B_{n-1,j}(x)t^{n-1} + \cdots + B_{0j}(x),$$

which has coefficients in $k[x]$ but lower degree than $g(t)$. Thus the assumption that $g(t)$ is inseparable leads to a contradiction. This completes the proof of Lemma 1.1.

THEOREM 1.3. *Suppose k is a perfect field. Then any extension of $k(x)$ of finite degree which is contained in $k\{x^{-1}\}$ can be generated by a PV element.*

Proof. By Lemma 1.1 any extension of $k(x)$ of finite degree which is contained in $k\{x^{-1}\}$ is separable. Thus Theorem 1.2 gives the desired result.

2. Basic Property

THEOREM 2.1. *Suppose k is an arbitrary field and θ is a PV element of $k\{x^{-1}\}$ of degree r over $k(x)$. Suppose λ is an element of $k(x)(\theta)$ such that*

$$T(\lambda\theta^N), \quad T(\lambda\theta^{N+1}), \quad \cdots, \quad T(\lambda\theta^{N+r-1})$$

are all in $k[x]$ for some integer N. Then

$$\lim_{n\to+\infty}((\lambda\theta^n)) = 0.$$

Remark. The condition on λ is certainly satisfied by any element λ of

$k(x)(\theta)$ such that $\lambda\theta^N$ is integral over $k[x]$ for some integer N. However, in general it is also satisfied by other elements as well (cf. [12, pp. 298–312]). For example, if θ is a PV element of degree 2 over $k(x)$, we could have $\lambda = (\theta - \theta')^{-1}$, where θ' is the conjugate of θ, for then the above condition is satisfied with $N = 0$.

Proof. Suppose the conjugates of θ over $k(x)$ are $\theta_1 = \theta, \theta_2, \cdots, \theta_r$, and suppose the minimal polynomial of θ over $k(x)$ is

$$t^r + A_{r-1}t^{r-1} + \cdots + A_0,$$

where $A_0, \cdots, A_r$ are in $k[x]$. Then

$$T(\lambda\theta^n) + A_{r-1}T(\lambda\theta^{n-1}) + \cdots + A_0T(\lambda\theta^{n-r}) = T\{\lambda\theta^{n-r}(\theta^r + A_{r-1}\theta^{r-1} + \cdots + A_0)\} = 0$$

for any integer n. Hence by induction $T(\lambda\theta^n)$ is in $k[x]$ for all $n \geqq N$. Let λ_i be the image of λ under the isomorphism of $k(x)(\theta)$ onto $k(x)(\theta_i)$ which takes θ into θ_i and leaves $k(x)$ fixed ($i = 2, \cdots, r$). Then

$$\lambda\theta^n = T(\lambda\theta^n) - \lambda_2\theta_2^n - \cdots - \lambda_r\theta_r^n,$$

and so for $n \geqq N$ we have

$$|((\lambda\theta^n))| \leqq \max_{i=2,\cdots,r}|\lambda_i\theta_i^n|.$$

The assertion of the theorem follows since $|\theta_i| < 1$ ($i = 2, \cdots, r$).

3. Characterization

Our discussion in this section is somewhat parallel to the exposition in [1, Ch. 8] of the two theorems characterizing PV numbers which are quoted in the Introduction.

LEMMA 3.1. *Suppose that k is an arbitrary field. Suppose the polynomial $f(t) = A_0 + A_1t + \cdots + A_rt^r$ has coefficients in $k[x]$, is irreducible over $k[x]$, and has r distinct zeros $\theta_1, \theta_2, \cdots, \theta_r$ in K. If $\xi_1, \cdots, \xi_n$ are given elements of K, the system of linear equations*

$$\textstyle\sum_{j=1}^r \theta_j^i Y_j = \xi_i \qquad (0 \leqq i < r)$$

has the unique solution for $Y_1, \cdots, Y_r$ given by

$$\beta_j Y_j = \textstyle\sum_{i=0}^{r-1} \alpha_{ji}\xi_i \qquad (1 \leqq j \leqq r),$$

where

$$\beta_j = f'(\theta_j) \neq 0, \qquad \alpha_{ji} = f_i(\theta_j) \neq 0,$$

with

$$f_i(t) = A_{i+1} + A_{i+2}t + \cdots + A_rt^{r-i-1}.$$

Proof. Since the determinant of the system is the Vandermonde determinant formed from $\theta_1, \theta_2, \cdots, \theta_r$, uniqueness is immediate. The actual calculation of the solution is given in [1, p. 135]. Note that the separability of $f(t)$ is essential.

LEMMA 3.2. *Suppose that k is an arbitrary field. If θ is an element of the algebraic closure K of $k\{x^{-1}\}$ and μ is a nonzero element of K such that μ, $\mu\theta$, $\mu\theta^2, \cdots$ are all equal to polynomials in θ of degree $r-1$ with coefficients in $k[x]$, then θ is integral over $k[x]$ of degree at most r.*

Proof. Let Γ be the $k[x]$-module generated by μ, $\mu\theta$, $\mu\theta^2, \cdots$. By assumption Γ is a submodule of the $k[x]$-module generated by $1, \theta, \cdots, \theta^{r-1}$. Since $k[x]$ is a Euclidean domain, it follows (cf. [2, pp. 56–57]) that Γ has a finite set of generators over $k[x]$, say $\gamma_1, \cdots, \gamma_s$, where we may assume $s \leqq r$. Since $\theta\gamma_i$ is in Γ for $i = 1, \cdots, s$, there exist elements D_{ij} of $k[x]$ such that

$$\theta\gamma_i = \sum_{j=1}^{s} D_{ij}\gamma_j \qquad (i = 1, \cdots, s).$$

Since $\mu \neq 0$, not all of the γ_i are zero, and so

$$\det(\theta I - D) = 0,$$

where I is the s by s identity matrix and D is the s by s matrix formed by the D_{ij}. Thus θ satisfies a monic polynomial equation having coefficients in $k[x]$ and degree s, where $s \leqq r$.

THEOREM 3.1. *Suppose k is an arbitrary field. Suppose θ is an element of $k\{x^{-1}\}$ which is algebraic and separable over $k(x)$ and satisfies $|\theta| > 1$. Suppose there is a nonzero element λ of $k\{x^{-1}\}$ such that $\lim_{n\to+\infty}((\lambda\theta^n)) = 0$. Then θ is a PV element, and λ is an element of $k(x)(\theta)$ such that for some N we have*

$$T(\theta^{i+N}\lambda) \in k[x] \qquad (i = 0, 1, \cdots, r-1).$$

Proof. Suppose that $f(\theta) = 0$, where

$$f(t) = A_r t^r + \cdots + A_0$$

is a separable polynomial with coefficients in $k[x]$ which is irreducible over $k[x]$. We assume $A_r \neq 0$. Suppose the zeros of f in K are $\theta = \theta_1, \theta_2, \cdots, \theta_r$. Let

$$\lambda\theta^n = B_n + \varepsilon_n, \tag{1}$$

where $B_n = [\lambda\theta^n]$ and $\varepsilon_n = ((\lambda\theta^n))$. Since

$$A_0\theta^n + A_1\theta^{n+1} + \cdots + A_r\theta^{n+r} = 0, \tag{2}$$

we have

$$A_0B_n + A_1B_{n+1} + \cdots + A_rB_{n+r} = -A_0\varepsilon_n - \cdots - A_r\varepsilon_{n+r}. \tag{3}$$

Since $|\varepsilon_n| < 1/\max(|A_0|, |A_1|, \cdots, |A_n|)$ for all sufficiently large n, there is an N such that

$$|A_0\varepsilon_n + A_1\varepsilon_{n+1} + \cdots + A_r\varepsilon_{n+r}| < 1 \tag{4}$$

for $n \geqq N$. But $A_0 B_n + \cdots + A_r B_{n+r}$ is in $k[x]$, and so by (3) and (4)

$$A_0 B_n + A_1 B_{n+1} + \cdots + A_r B_{n+r} = 0 \qquad (n \geqq N). \tag{5}$$

By Lemma 3.1 there are elements $\lambda_1, \lambda_2, \cdots, \lambda_r$ of K such that

$$\textstyle\sum_{j=1}^{r} \theta_j^{i+N} \lambda_j = B_{i+N} \tag{6}$$

for $i = 0, 1, \cdots, r - 1$. But by (2) and (5) it follows that (6) then holds for all nonnegative integers i. The elements $\lambda_1, \lambda_2, \cdots, \lambda_r$ are given by the formulas

$$\beta_j \theta_j^m \lambda_j = \textstyle\sum_{i=0}^{r-1} \alpha_{ji} B_{i+m} \qquad (1 \leqq j \leqq r), \tag{7}$$

where we may take m as any integer not less than N. Here β_j and the α_{ji} are the elements of $k(x)[\theta_j]$ given in Lemma 3.1. Thus λ_j is the image of λ_1 under the isomorphism of $k(x)(\theta_1)$ onto $k(x)(\theta_j)$ which takes θ_1 into θ_j and leaves $k(x)$ fixed. Clearly

$$\lambda_j \neq 0 \qquad (1 \leqq j \leqq r). \tag{8}$$

For if any λ_j were zero, all of them would be zero, and then $B_n = 0$ for $n \geqq N$, which is impossible by (1).

If we take $j = 1$ in (7), the right-hand side of (7) is a polynomial in $\theta = \theta_1$ of degree $r - 1$ with coefficients in $k[x]$. Since m may be taken as any integer not less than N, the hypothesis of Lemma 3.2 is satisfied by $\mu = \beta_1 \lambda_1 \theta^N$. Therefore θ is integral over $k[x]$.

By equation (1) and the validity of (6) for all nonnegative i we have

$$\theta^{i+N}(\lambda - \lambda_1) - \textstyle\sum_{j=2}^{r} \theta_j^{i+N} \lambda_j = \varepsilon_{i+N} \tag{9}$$

for any nonnegative integer i. Thus by Lemma 3.1

$$\beta_1 \theta^m (\lambda - \lambda_1) = \textstyle\sum_{i=0}^{r-1} \alpha_{1i} \varepsilon_{i+m}, \tag{10}$$

and

$$\beta_j \theta_j^m (-\lambda_j) = \textstyle\sum_{i=0}^{r-1} \alpha_{ji} \varepsilon_{i+m} \qquad (2 \leqq j \leqq r), \tag{11}$$

provided $m \geqq N$. Since $\beta_1 \neq 0$ and the right-hand side of (10) is a bounded function of m, we must have $\lambda = \lambda_1$. Thus $T(\theta^{i+N}\lambda) \in k[x]$ for $i = 0, 1, \cdots, r - 1$ by (6). Since $\beta_j \lambda_j \neq 0$ and the right-hand side of (11) tends to zero as $m \to +\infty$, we must have $|\theta_j| < 1$ for $2 \leqq j \leqq r$, which completes the proof.

THEOREM 3.2. *Suppose k is an arbitrary field. Suppose θ is an element of $k\{x^{-1}\}$ such that $|\theta| > 1$ and $f(\theta) = 0$, where $f(t) = A_r t^r + \cdots + A_0$ is a separable polynomial with coefficients in $k[x]$ which is irreducible over $k[x]$. Suppose λ is a nonzero element of $k\{x^{-1}\}$ such that*

$$|((\lambda\theta^n))| < 1/\max(|A_0|, |A_1|, \cdots, |A_r|)$$

for all sufficiently large integers n. Then θ is integral over $k[x]$, and all of its conjugates with respect to $k(x)$ have absolute value at most 1. *Moreover λ is an element of $k(x)(\theta)$ such that for some N we have*

$$T(\theta^{i+N}\lambda) \in k[x] \qquad (i = 0, 1, \cdots, r-1).$$

Proof. The proof is the same as that of Theorem 3.1, except for the last sentence. Under the present hypotheses we can say only that the right side of (11) is bounded, and thus we can infer only that $|\theta_j| \leqq 1$ for $2 \leqq j \leqq r$.

LEMMA 3.3. *Suppose k is an arbitrary field. Let $\xi_0, \xi_1, \cdots$ be a sequence of elements of $k\{x^{-1}\}$ satisfying a recurrence relation*

$$\xi_{n+r} + \alpha_{r-1}\xi_{n+r-1} + \cdots + \alpha_0\xi_n = 0 \qquad (n \geqq N),$$

where $\alpha_0, \cdots, \alpha_{r-1}$ are fixed elements of $k\{x^{-1}\}$. Suppose $B_0, B_1, \cdots$ is a sequence of elements of $k[x]$ such that

$$|B_n - \xi_n| < 1/\max(|\alpha_0|, \cdots, |\alpha_{r-1}|, 1)^2 \qquad (n \geqq N).$$

Then the sequence $B_0, B_1, \cdots$ also satisfies a recurrence relation.

Proof. For $n \geqq N + r$ let us put

$$\varepsilon_n = B_n + \alpha_{r-1}B_{n-1} + \cdots + \alpha_0 B_{n-r}.$$

In view of the hypothesis we have

$$\begin{aligned} |\varepsilon_n| &= |(B_n - \xi_n) + \alpha_{r-1}(B_{n-1} - \xi_{n-1}) + \cdots + \alpha_0(B_{n-r} - \xi_{n-r})| \\ &< 1/\max(|\alpha_0|, \cdots, |\alpha_{r-1}|, 1) \qquad (n \geqq N + r). \end{aligned}$$

For $n \geqq N + 2r$ let us put

$$\eta_n = \varepsilon_n + \alpha_{r-1}\varepsilon_{n-1} + \cdots + \alpha_0\varepsilon_{n-r},$$

so that

$$|\eta_n| < 1 \qquad (n \geqq N + 2r).$$

Thus $|\varepsilon_n| \leqq c^{-1}$ if $n \geqq N + r$, and $|\eta_n| \leqq c^{-1}$ if $n \geqq N + 2r$. Let Δ_n be the $n + 1$ by $n + 1$ determinant in which the element in the $(i + 1)^{\text{th}}$ row and $(j + 1)^{\text{th}}$ column is B_{i+j} $(0 \leqq i, j \leqq n)$. By a sequence of row operations the element B_{i+j} may be replaced by ε_{i+j} for all $i \geqq N + r$ without changing the value of the determinant. By a sequence of column operations the element B_{i+j} may be replaced by δ_{i+j} without changing the value of the determinant, where

$$\begin{aligned} \delta_{i+j} &= B_{i+j} \quad \text{if} \quad i < N + r, \quad j < N + r, \\ \delta_{i+j} &= \eta_{i+j} \quad \text{if} \quad i \geqq N + r, \quad j \geqq N + r, \\ \delta_{i+j} &= \varepsilon_{i+j} \quad \text{otherwise}. \end{aligned}$$

Thus if

$$M = \max(|B_0|, |B_1|, \cdots, |B_{2N+2r-2}|, 1),$$

we have

$$|\Delta_n| \leqq M^{N+r}(c^{-1})^{n-N-r+1}.$$

Thus $|\Delta_n| < 1$ if n is sufficiently large. Since Δ_n is in $k[x]$, we have $\Delta_n = 0$ for all sufficiently large n. Therefore, by a theorem of Kronecker (valid for any field), the sequence $B_0, B_1, \cdots$ satisfies a recurrence relation.

THEOREM 3.3. *Suppose k is an arbitrary field. Suppose θ is an element of $k\{x^{-1}\}$ such that $|\theta| > 1$. Suppose there is a nonzero element λ of $k\{x^{-1}\}$ such that*

$$|((\lambda\theta^n))| < 1/|\theta|^2$$

for all sufficiently large n. Then θ is algebraic over $k(x)$.

Proof. Put $\lambda\theta^n = B_n + \varepsilon_n$, where $B_n = [\lambda\theta^n]$ is in $k[x]$ and $|\varepsilon_n| = |((\lambda\theta^n))| < 1$. Applying the preceding lemma with $\xi_n = \lambda\theta^n$ and noting the recurrence relation $\xi_{n+1} = \theta\xi_n$, we see that the sequence $B_0, B_1, \cdots$ satisfies a recurrence relation, say

$$B_{n+r} + A_{r-1}B_{n+r-1} + \cdots + A_0 B_n = 0 \qquad (n \geqq N). \tag{1}$$

Without loss of generality we may assume that $A_0, A_1, \cdots, A_{r-1}$ are in $k(x)$ (cf. [1, pp. 137–138]). Then (1) gives

$$\lambda\theta^n(\theta^r + A_{r-1}\theta^{r-1} + \cdots + A_0) = \varepsilon_{n+r} + A_{r-1}\varepsilon_{n+r-1} + \cdots + A_0\varepsilon_n. \tag{2}$$

Since the right side of (2) is a bounded function of n, we must have

$$\theta^r + A_{r-1}\theta^{r-1} + \cdots + A_0 = 0.$$

Thus Theorem 3.3 is proved.

The following theorem combines Theorems 3.1 and 3.3. We must assume that k is perfect in order to infer the separability of θ from Lemma 1.1.

THEOREM 3.4. *Suppose k is a perfect field. Suppose θ is an element of $k\{x^{-1}\}$ such that $|\theta| > 1$. Suppose there is a nonzero element λ of $k\{x^{-1}\}$ such that $\lim_{n\to+\infty}((\lambda\theta^n)) = 0$. Then θ is a PV element, and λ is an element of $k(x)(\theta)$ such that for some N we have*

$$T(\theta^{i+N}\lambda) \in k[x] \qquad (i = 0, 1, \cdots, r-1).$$

Proof. By Theorem 3.3, θ is algebraic over $k(x)$. By Lemma 1.1, θ is therefore separable over $k(x)$. Thus we may apply Theorem 3.1 to obtain the above conclusions.

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