PERIODIC HOMEOMORPHISMS OF THE 3-SPHERE[1]

BY
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1. Statement of results

Let $\mathfrak{M}$ be a triangulated 3-sphere, and let f be a periodic simplicial homeomorphism of $\mathfrak{M}$ onto itself. Suppose that f preserves orientation and has a fixed point; let F be the fixed-point set of f; and let n be the period of f. It has been shown by P. A. Smith [S][2] that when n is a prime, F is always a (simple closed) polygon; and we shall show, in the last section of the present paper, that for arbitrary n the same conclusion follows. In the rest of this paper, therefore, we shall assume that F is a polygon. A well-known conjecture due to Smith, discussed by Eilenberg in [E], asserts that F is never knotted.

A partial solution of Smith's problem has been given by Montgomery and Samelson [MS]. They have shown that if f is an involution (i.e., is of period 2), then (1) if F is a simplicial standard torus knot, then F is unknotted, and (2) if F is unknotted, then f is equivalent to a rotation.

In the present paper, we generalize the second of these results, to homeomorphisms of arbitrary period. Thus our main result is:

1.1. THEOREM. *If f: $\mathfrak{M} \to \mathfrak{M}$ is periodic and preserves orientation, and F is unknotted, then f is equivalent to a rotation.*

The proof is based on the following preliminary result:

1.2. THEOREM. *There is a polyhedral disk with handles M_1 such that the boundary of M_1 is F and such that the iterated images*

$$M_i = f^{i-1}(M_1)$$

intersect one another only in F.

Here by a disk with handles we mean, of course, a compact, connected, orientable 2-manifold with boundary, bounded by a 1-sphere.

Theorem 1.2 has been proved, for involutions, by Montgomery and Samelson.

2. 2-spines of 3-dimensional complexes

Let $\mathfrak{K}$ be a complex, and let n be a positive integer. Then $\beta_n \mathfrak{K}$ denotes the set of all points of $\mathfrak{K}$ that do not have open neighborhoods in $\mathfrak{K}$, homeomorphic to Euclidean n-space E^n. The "n-dimensional interior" $\mathfrak{K} - \beta_n \mathfrak{K}$

Received July 27, 1960.

[1] Sponsored by the Office of Ordnance Research, U.S. Army and the Air Force Office of Scientific Research.

[2] Letters in square brackets refer to the bibliography at the end of the paper.

of $\mathfrak{L}$ is denoted by $\mathrm{Int}_n \mathfrak{L}$. Note that if $\mathfrak{L}$ is an n-manifold with boundary, then $\beta_n \mathfrak{L}$ is the "intrinsic boundary" $\partial\mathfrak{L}$, and $\mathrm{Int}_n \mathfrak{L}$ is the interior $\mathrm{Int}\ \mathfrak{L}$.

If v is a vertex of the complex $\mathfrak{L}$, then $\mathrm{St}(v)$ denotes the closed star of v in $\mathfrak{L}$.

Now let $\mathfrak{K}$ be a finite proper subcomplex of a triangulated 3-manifold $\mathfrak{M}$. Let σ_1^3 be a 3-simplex of $\mathfrak{K}$, such that some 2-face σ_1^2 of σ_1^3 lies in $\beta_3 \mathfrak{K}$. Let

$$\mathfrak{K}_1 = \mathfrak{K} - [\mathrm{Int}\ \sigma_1^3 \cup \mathrm{Int}\ \sigma_1^2].$$

(Note that $\mathfrak{K}_1$ is not necessarily the closure of $\mathfrak{K} - \sigma_1^3$.)

We proceed by induction to define $\mathfrak{K}_2, \cdots, \mathfrak{K}_n$, where n is the number of 3-simplices of $\mathfrak{K}$. Given $\mathfrak{K}_i$ $(i < n)$, let σ_{i+1}^3 be a 3-simplex of $\mathfrak{K}_i$, such that $\beta_3 \mathfrak{K}_i$ contains a 2-face σ_{i+1}^2 of σ_{i+1}^3. Let

$$\mathfrak{K}_{i+1} = \mathfrak{K}_i - [\mathrm{Int}\ \sigma_{i+1}^3 \cup \mathrm{Int}\ \sigma_{i+1}^2].$$

Let $\mathfrak{L}$ be $\mathfrak{K}_n$. Then $\mathfrak{L}$ is a 2-dimensional subcomplex of $\mathfrak{K}$. A subcomplex of $\mathfrak{K}$, obtainable by the above process, will be called a 2-*spine* of $\mathfrak{K}$.

Of course, in the case in which $\mathfrak{K}$ is obtained by deleting an open 3-simplex from a triangulated 3-manifold $\mathfrak{M}$, the above process is the process usually employed in describing $\mathfrak{M}$ as a 3-cell with identifications on its boundary, as in [ST, pp. 206–211].

By a trivial induction, we see that each of the complexes $\mathfrak{K}_i$ is a deformation retract of $\mathfrak{K}$. Thus, in particular, we have

2.1. LEMMA. *$\mathfrak{L}$ is a deformation retract of $\mathfrak{K}$.*

A simplicial homeomorphism of $\mathfrak{M}$ onto itself is called *regular* if $f^i(\sigma) = \sigma$ only when $f^i \mid \sigma$ is the identity. (Here f^i denotes the i^{th} iterate of f. Note that this is not really a restriction on f; it is merely a requirement that $\mathfrak{M}$ be sufficiently finely subdivided. If f is simplicial relative to $\mathfrak{M}$, then f is automatically regular relative to the first barycentric subdivision of $\mathfrak{M}$.) If $\mathfrak{K}$ is a subcomplex of $\mathfrak{M}$, and $f(\mathfrak{K}) = \mathfrak{K}$, then $\mathfrak{K}$ is called *f-invariant*.

2.2. LEMMA. *If f: $\mathfrak{M} \to \mathfrak{M}$ is regular, and $\mathfrak{K}$ is an f-invariant subcomplex of $\mathfrak{M}$, then $\mathfrak{K}$ has an f-invariant 2-spine.*

Proof. Let $\sigma_1^2 \subset \sigma_1^3 \in \mathfrak{K}$. Then σ_1^3 is the only 3-simplex of $\mathfrak{K}$ that contains σ_1^2; all of the simplices $\sigma_i^2 = f^{i-1}(\sigma_1^2)$ lie in $\beta_3 \mathfrak{K}$; and $\sigma_i^2 = \sigma_j^2$ only if $f^{i-j} \mid \sigma_1^2$ is the identity. Let $\sigma_i^3 = f^{i-1}(\sigma_1^3)$; and let $n + 1$ be the smallest integer such that $\sigma_{n+1}^3 = \sigma_1^3$. In forming a 2-spine, it is plain that we can delete the sets

$$\mathrm{Int}\ \sigma_i^3 \cup \mathrm{Int}\ \sigma_i^2 \qquad (1 \leqq i \leqq n)$$

before deleting any other sets of the same type. Thus we obtain an f-invariant complex $\mathfrak{K}'$. Repeating this scheme, until all of the 3-simplices of $\mathfrak{K}$ are used up, we obtain an f-invariant 2-spine $\mathfrak{L}$, as desired.

3. Slab-systems and slab-neighborhoods

The slab-neighborhoods to be defined in this section are related to the regular neighborhoods of J. H. C. Whitehead [W]. For reasons of convenience, we set them up *ab initio*, in a special way.

Let $\mathfrak{M}$ be a combinatorial 3-manifold (not necessarily finite), and let $\mathfrak{M}_1$ be a subdivision of $\mathfrak{M}$. In the usual way, let us set up a barycentric coordinate system in $\mathfrak{M}_1$. Then to each point p of each simplex

$$\sigma^i = v_0 v_1 \cdots v_i$$

of $\mathfrak{M}_1$ there corresponds a nonnegative real-valued function f_p, defined over the set of vertices v_j of σ^i, such that

$$\textstyle\sum_j f_p(v_j) = 1.$$

Evidently the domains of definition of the functions f_p can be extended to all of the 0-skeleton $\mathfrak{M}_1^0$, by defining $f_p(v)$ as $= 0$ for every vertex v of $\mathfrak{M}_1$ which is not a vertex of any simplex of $\mathfrak{M}_1$ that contains p.

If $v \in \mathfrak{M}_1^0$, and $0 < \varepsilon < 1$, let

$$\mathfrak{S}(v, \varepsilon)$$

be the set of all points p of $\mathfrak{M}_1$ such that

$$f_p(v) \geqq 1 - \varepsilon.$$

Then $\mathfrak{S}(v, \varepsilon)$ intersects every simplex of $\mathfrak{M}_1$ either in the empty set or in a simplex; in fact, $\mathfrak{S}(v, \varepsilon)$ is the image of $\mathrm{St}(v)$ under a homeomorphism which throws every simplex of $\mathrm{St}(v)$ linearly into itself.

More generally, if $\sigma^i = v_0 v_1 \cdots v_i$ is an i-simplex of $\mathfrak{M}_1$, with $i \leqq 2$, and $0 < \varepsilon < 1$, let

$$\mathfrak{S}(\sigma^i, \varepsilon)$$

be the set of all points p of $\mathfrak{M}_1$ such that

$$\textstyle\sum_j f_p(v_j) \geqq 1 - \varepsilon.$$

Then $\partial\mathfrak{S}(\sigma^i, \varepsilon)$ intersects every 3-simplex of $\mathfrak{M}_1$ in the empty set, a "triangle," or a "plane quadrilateral"; by this we mean that if $\sigma^3 \in \mathfrak{M}_1$ is linearly imbedded in E^3, then the intersections

$$\partial\mathfrak{S}(\sigma^i, \varepsilon) \cap \sigma^3$$

appear as plane sections of σ^3, containing no vertex of σ^3. This is a consequence of the fact that for simplices in Euclidean spaces, the barycentric coordinates and the Cartesian coordinates depend linearly on one another.

Now for each $\sigma^0 \in \mathfrak{M}_1^0$, let

$$\mathfrak{S}(\sigma^0) = \mathfrak{S}(\sigma^0, \tfrac{1}{3});$$

for each $\sigma^1 \in \mathfrak{M}_1^1$, let

$$\mathfrak{S}(\sigma^1) = \mathfrak{S}(\sigma^1, \tfrac{1}{4});$$

and for each $\sigma^2 \epsilon \mathfrak{M}_1^2$, let

$$\mathfrak{S}(\sigma^2) = \mathfrak{S}(\sigma^2, \tfrac{1}{5}).$$

Then the collection consisting of the sets $\mathfrak{S}(\sigma^i)$ $(\sigma^i \epsilon \mathfrak{M}_1^2)$ is the *slab-system for* $\mathfrak{M}_1$. If $\mathfrak{L}$ is a subcomplex of $\mathfrak{M}_1$, then the set $\mathfrak{S}(\mathfrak{L})$ of all elements $\mathfrak{S}(\sigma^i)$ of $\mathfrak{S}$ such that $\sigma^i \epsilon \mathfrak{L}$ is a *slab-system* for $\mathfrak{L}$; and the set

$$\mathfrak{N}(\mathfrak{L}) = \mathfrak{L} \cup \bigcup \mathfrak{S}(\sigma^i) \qquad (\sigma^i \epsilon \mathfrak{L}^2)$$

is a *slab-neighborhood* of $\mathfrak{L}$ in $\mathfrak{M}$. Note that $\mathfrak{S}(\mathfrak{L})$ and $\mathfrak{N}(\mathfrak{L})$ depend not only on $\mathfrak{L}$, but also on $\mathfrak{M}_1$, because the same complex $\mathfrak{L}$ may be a subcomplex of two different subdivisions of $\mathfrak{M}$.

Slab-neighborhoods have the elementary geometric properties that one would expect:

3.1. LEMMA. *If σ is a simplex of $\mathfrak{M}_1$, then $\mathfrak{N}(\sigma)$ is a combinatorial* 3*-cell.*

3.2. LEMMA. *If J is a polygon which forms a subcomplex of $\mathfrak{M}_1$, then $\mathfrak{N}(J)$ is a solid torus.*

(By a solid torus we mean, of course, a set homeomorphic to the Cartesian product of a disk and a circle.)

3.3. LEMMA. *If $\mathfrak{K}$ is a subcomplex of $\mathfrak{M}_1$, and $\mathfrak{K}$ is a* 3*-manifold with boundary, then $\mathfrak{K}$ is combinatorially equivalent to $\mathfrak{N}(\mathfrak{K})$, under a piecewise linear homeomorphim of $\mathfrak{M}$ onto itself.*

3.4. LEMMA. *Let $\mathfrak{K}$ be a subcomplex of $\mathfrak{M}_1$, and let $\mathfrak{L}$ be a* 2*-spine of $\mathfrak{K}$. Then $\mathfrak{N}(\mathfrak{K})$ and $\mathfrak{N}(\mathfrak{L})$ are combinatorially equivalent. And we can choose a piecewise linear homeomorphism h, of $\mathfrak{M}$ onto itself, such that*

(1) $h(\mathfrak{N}(\mathfrak{K})) = \mathfrak{N}(\mathfrak{L})$ *and*

(2) *h is the identity on $\partial\mathfrak{N}(\mathfrak{L}^1) \cap \partial\mathfrak{N}(\mathfrak{K})$.*

Here it should be understood that $\mathfrak{M}_1$ is a subdivision of the triangulated 3-manifold $\mathfrak{M}$, and $\mathfrak{N}(\mathfrak{K})$ and $\mathfrak{N}(\mathfrak{L})$ are the slab-neighborhoods induced by the slab-system for $\mathfrak{M}_1$.

3.5. LEMMA. *Let f be a simplicial homeomorphism of $\mathfrak{M}_1$ onto itself. Then slab-neighborhoods in $\mathfrak{M}_1$ are f-invariant. That is to say, if $\mathfrak{L}$ is a subcomplex of $\mathfrak{M}_1$, then*

$$\mathfrak{N}(f(\mathfrak{L})) = f(\mathfrak{N}(\mathfrak{L})).$$

3.6. LEMMA. *Let $\mathfrak{L}$ be a* 2*-spine of $\mathfrak{K}$, as in Lemma* 3.4, *and let f be a periodic homeomorphism as in Lemma* 3.5. *Suppose that f is regular and $\mathfrak{L}$ is f-invariant. Then the homeomorphism h given by Lemma* 3.4 *can be chosen in such a way that $hf = fh$.*

Here Lemma 3.5 is a trivial consequence of the fact that for every simplex σ of $\mathfrak{M}_1$ we have

$$\mathfrak{S}(f(\sigma)) = f(\mathfrak{S}(\sigma)).$$

Demonstrative proofs of the rest of the lemmas of this section require somewhat tedious geometric arguments. We shall first give the geometric lemmas that are required, and then sketch the arguments.

3.7. LEMMA. *Every triangulated* 3-*manifold is a combinatorial* 3-*manifold.*

That is, every complex $\mathrm{St}(v)$ is a combinatorial 3-cell. This is Theorem 1 of [M_5].

3.8. LEMMA. *Every triangulated* 3-*manifold with boundary is a combinatorial* 3-*manifold with boundary.*

The proof is trivial; it was given parenthetically in the proof of Theorem 9.2 of [M_8].

3.9. LEMMA. *Let C be a polyhedral* 3-*manifold with boundary, bounded by a* 2-*sphere, in a combinatorial* 3-*manifold $\mathfrak{M}$; and suppose that there is a piecewise linear homeomorphism ϕ, of C into E^3. Then C is a combinatorial* 3-*cell.*

Proof of lemma. Evidently $\phi(\partial C)$ is a polyhedral 2-sphere in E^3; and $\phi(\partial C) = \partial\phi(C)$. Therefore, by Theorem 1 of [M_2], $\phi(C)$ is a combinatorial 3-cell. Therefore so also is C.

3.10. LEMMA. *Let C be a combinatorial* 3-*cell which is a subcomplex of a subdivision $\mathfrak{M}'$ of a combinatorial* 3-*manifold $\mathfrak{M}$. Let J be a polygon in ∂C, and let P be a finite polyhedron which is a (closed) neighborhood of $C - J$, such that P lies in the star of a vertex of $\mathfrak{M}$. Let D_1 and D_2 be the closures of the two components of $\partial C - J$. Then there is a piecewise linear homeomorphism f, of $\mathfrak{M}$ onto itself, such that*
(1) $f(D_1) = D_2$ *and*
(2) $f \mid (\mathfrak{M} - P)$ *is the identity.*

We proceed to indicate the proofs of Lemmas 3.1–3.4 and 3.6.
For each simplex σ of $\mathfrak{M}$, let

$$\mathfrak{S}'(\sigma) = \mathrm{Cl}[\mathfrak{S}(\sigma) - \mathfrak{S}(\partial\sigma)].$$

(Here Cl indicates closure.) Every set $\mathfrak{S}'(\sigma)$ is a polyhedral 3-manifold with boundary, lying in the star of some vertex of $\mathfrak{M}$. Since every complex $\mathrm{St}(v)$ can be mapped combinatorially into E^3, it follows by Lemma 3.9 that all sets $\mathfrak{S}'(\sigma)$ are combinatorial 3-cells.

Now let σ be any simplex of $\mathfrak{M}$, with faces $\tau_1, \cdots, \tau_k$. It is a straightforward matter to show that the combinatorial 3-cells $\mathfrak{S}'(\tau_i)$ can be arranged in an order $C_1, C_2, \cdots, C_k$, in such a way that each set C_i intersects the union of its predecessors in a disk. It follows, by induction, that $\bigcup C_i$ is a combinatorial 3-cell. Since $\bigcup C_i = \mathfrak{N}(\sigma)$, this proves Lemma 3.1.

To prove Lemma 3.2, let the edges and vertices of J be

$$v_1, \quad e_1, \quad v_2, \quad e_2, \quad \cdots, \quad v_k, \quad e_k,$$

in the cyclic order of their appearance on J. Then the sets $\mathfrak{S}'(v_i)$, $\mathfrak{S}'(e_i)$ are combinatorial 3-cells, and intersect one another only when they appear consecutively, in which case they intersect in disks. It follows that their union $\mathfrak{N}(J)$ is a solid torus.

To prove Lemma 3.3, we first recall that by definition, $\mathfrak{N}(\mathfrak{K})$ is the union of $\mathfrak{K}$ and the sets $\mathfrak{S}(\sigma)$, where σ runs through the lower-dimensional faces of $\mathfrak{K}$. Evidently it is sufficient to let σ run through the faces of the complex $B = \partial\mathfrak{K}$.

Let σ be any 2-face of B; let

$$C_1 = \mathrm{Cl}[\mathfrak{S}'(\sigma) - \mathfrak{K}];$$

and let

$$C_2 = \mathrm{Cl}[\mathfrak{N}(\mathfrak{K}) - C_1].$$

Then ∂C_1 is the union of two disks D_1, D_2, where

$$D_1 \subset \partial\mathfrak{N}(\mathfrak{K}) \quad \text{and} \quad D_2 = C_1 \cap C_2 .$$

By Lemma 3.10 there is a piecewise linear homeomorphism

$$\begin{aligned} f:\ & \mathfrak{M} \to \mathfrak{M} \\ :\ & D_1 \to D_2 \\ :\ & \mathfrak{N}(\mathfrak{K}) \to C_2 . \end{aligned}$$

By repeated application of this procedure we can show that there is a piecewise linear homeomorphism $\mathfrak{M} \to \mathfrak{M}$, $\mathfrak{N}(\mathfrak{K}) \to \mathfrak{K} \cup \mathfrak{N}(B^1)$, where B^1 is of course the 1-skeleton of B. By iterating the process further, for the edges and vertices of B, we can complete the proof of Lemma 3.3.

To prove Lemma 3.4, we recall that the 2-spine $\mathfrak{L}$ was the last in a sequence

$$\mathfrak{K} = \mathfrak{K}_0 , \quad \mathfrak{K}_1 , \quad \cdots , \quad \mathfrak{K}_n$$

of complexes, where n is the number of 3-simplices in $\mathfrak{K}$ and

$$\mathfrak{K}_{i+1} = \mathfrak{K}_i - [\mathrm{Int}\ \sigma_i^2 \cup \mathrm{Int}\ \sigma_i^3].$$

Let

$$C = \mathrm{Cl}[\mathfrak{N}(\mathfrak{K}_i) - \mathfrak{N}(\mathfrak{K}_{i+1})].$$

It is easy to verify that C is a combinatorial 3-cell, intersecting $\mathfrak{K}_{i+1}$ in a disk which lies in both ∂C and $\partial\mathfrak{K}_{i+1}$. By Lemma 3.9 it follows that if the lemma becomes a true proposition when $\mathfrak{L}$ is replaced by $\mathfrak{K}_i$, then the lemma also becomes a true proposition when $\mathfrak{L}$ is replaced by $\mathfrak{K}_{i+1}$. Since $\mathfrak{L} = \mathfrak{K}_n$, the lemma thus follows by induction.

Note that this induction argument can also be regarded as a construction of the desired piecewise linear homeomorphism L. That is, we may construct homeomorphisms

$$\begin{aligned} h_i :\ & \mathfrak{M}_1 \to \mathfrak{M}_1 \\ :\ & \mathfrak{N}(\mathfrak{K}_i) \to \mathfrak{N}(\mathfrak{K}_{i+1}), \end{aligned}$$

taking each of these so that it differs from the identity only in a small neighborhood of the corresponding combinatorial 3-cell C; we can then let h be the resultant of all these. Thus, if we are forming an f-invariant 2-spine $\mathfrak{L}$ of $\mathfrak{K}$, as in the proof of Lemma 2.2, we can first take

$$h_1 : \ \mathfrak{N}(\mathfrak{K}) \to \mathfrak{N}(\mathfrak{K}_1),$$

such that h_1 differs from the identity only in a small neighborhood of C, and then follow h_1 by the mappings

$$h_i = f^{i-1} h_1 f^{1-i}.$$

These mappings commute with each other, because the sets on which they differ from the identity are disjoint; and it follows that their resultant H_1 commutes with f. Repeating this process, in a fashion analogous to the proof of Lemma 2.2, we obtain a piecewise linear homeomorphism of the sort required in Lemma 3.6.

4. Periodic homeomorphisms of homological 3-spheres

This section will be devoted to the proof of Theorem 1.2. Accordingly, we shall assume that $\mathfrak{M}$ is a triangulated 3-manifold; f is an orientation-preserving simplicial homeomorphism of $\mathfrak{M}$ onto itself, of period n; the fixed-point set of f is a polygon F. Finally, we suppose without loss of generality that f is regular (relative to $\mathfrak{M}$) in the sense of Section 2.

4.1. Lemma. *f has period exactly $= n$ at each point of $\mathfrak{M} - F$.*

Proof of lemma. Suppose that f has period $m < n$ at some point of $\mathfrak{M} - F$. Let $g = f^m$, and let G be the fixed-point set of g. Then G is a (simple closed) polygon, by the result of Section 6. Thus the polygon F is a proper subset of the polygon G, which is obviously impossible.

Let $\mathfrak{S}$ be the slab-system for $\mathfrak{M}$, and let

$$\mathfrak{T} = \mathfrak{N}(F)$$

be the induced slab-neighborhood of F. By Lemma 3.2, $\mathfrak{T}$ is a solid torus. Let e be any edge of F; let v be a vertex of e; and let

$$D = \mathfrak{N}(v) \cap \mathrm{Cl}[\mathfrak{N}(e) - \mathfrak{N}(v)].$$

Then D is a polyhedral disk. Since the slab-system $\mathfrak{S}$ is f-invariant, and $f \mid F$ is the identity, it follows that all of the sets $\mathfrak{T}$, $\mathfrak{N}(v)$, $\mathfrak{N}(e)$ have the same property. Therefore so also do D and ∂D. Let

$$J = \partial D.$$

This polygon J will be fixed, throughout the proof. Let

$$\mathfrak{K} = \mathrm{Cl}(\mathfrak{M} - \mathfrak{T}).$$

4.2. LEMMA. *$H^1(\mathfrak{K})$ is infinite cyclic, and is generated by a 1-cycle Z_J on J.*

This follows from the Alexander Duality Theorem (with integer coefficients) for homological 3-spheres. Knowing of no convenient reference for the latter result, we indicate an elementary proof of the lemma.

We shall use, hereafter, a subdivision $\mathfrak{M}'$ of $\mathfrak{M}$, such that $\mathfrak{T}$ is a subcomplex of $\mathfrak{M}'$. All cycles mentioned will be cycles in the elementary sense, on $\mathfrak{M}'$.

Let P be a polygon in $\partial\mathfrak{T}$, such that P carries a generator Z_P of $H^1(\mathfrak{T})$, and such that P crosses J in exactly one point $p = J \cap P$. Let Z_F and Z_J be 1-cycles obtained by assigning orientations to F and J respectively. Since $Z_F \sim 0$ on $\mathfrak{M}$, it follows that Z_F is homologous on $\mathfrak{T}$ to a 1-cycle Z'_F on $\partial\mathfrak{T}$, such that $Z'_F \sim 0$ on $\mathfrak{K}$. But the set $\{Z_J, Z_P\}$ generates $H^1(\partial\mathfrak{T})$. Therefore Z'_F is homologous on $\partial\mathfrak{T}$ to a linear combination $iZ_P + jZ_J$. Since $Z_J \sim 0$ on $\mathfrak{T}$, and both Z_P and Z_F are generators of $H^1(\mathfrak{T})$, it follows that $i = \pm 1$. Therefore Z'_F is homologous on $\partial\mathfrak{T}$ to a 1-cycle Z''_F carried by a polygon which crosses J exactly once. Therefore $\{Z_J, Z''_F\}$ generates $H^1(\partial\mathfrak{T})$.

Now every 1-cycle Z on $\mathfrak{K}$ is homologous to zero on $\mathfrak{M}$, so that such a Z is homologous on $\mathfrak{K}$ to a 1-cycle Z' on $\partial\mathfrak{T}$. Therefore

$$Z \sim iZ_J + jZ''_F \qquad \text{on } \mathfrak{K}.$$

But $Z''_F \sim 0$ on $\mathfrak{K}$. Therefore $Z \sim iZ_J$ on $\mathfrak{K}$, so that Z_J generates $H^1(\mathfrak{K})$. And Z_J is of infinite order in $H^1(\mathfrak{K})$. For otherwise we would have

$$iZ_J \sim 0 \qquad \text{on } \mathfrak{K}$$

and

$$iZ_J \sim 0 \qquad \text{on } \mathfrak{T} = \mathrm{Cl}(\mathfrak{M} - \mathfrak{K}),$$

so that by the Mayer-Vietoris Theorem $\mathfrak{M}$ would carry a nonbounding 2-cycle.

By 2.2, let $\mathfrak{L}$ be an f-invariant 2-spine of $\mathfrak{K}$. Then $J \subset \mathfrak{L}$. And since $\mathfrak{L}$ is a deformation retract of $\mathfrak{K}$ (by Lemma 2.1), we know that $H^1(\mathfrak{L})$ is isomorphic to $H^1(\mathfrak{K})$, and that Z_J generates $H^1(\mathfrak{L})$.

4.3. LEMMA. *There is a connected acyclic linear graph G_1 such that* (1) $G_1 \subset \mathfrak{L}^1$, (2) *the iterated images $G_i = f^{i-1}(G_1)$ are disjoint,* (3) *each set $G_i \cap J$ is connected, and* (4) *$\bigcup G_i$ contains every vertex of $\mathfrak{L}$.*

Proof of lemma. Evidently there is a G_1 which satisfies (1), (2), and (3). (For example, let G_1 be any vertex of J.) Suppose further that G_1 is maximal with respect to this property. We shall show that G_1 also satisfies (4).

Suppose first that there is an edge $e = v_0 v_1$ of J, such that $v_0 \cup v_1$ does not lie in $\bigcup G_i$. Then we may suppose that $v_0 \in \bigcup G_i$, or, in particular, that $v_0 \in G_1$. Then

$$\bigcup f^i(v_1) \cap \bigcup G_i = 0;$$

the iterated images $f^i(v_1)$ $(1 \leqq i \leqq n - 1)$ are all different; and the same is true of the sets $f^i(e)$. Therefore the sets $G'_i = G_i \cup f^{i-1}(e)$ are disjoint;

$G_1' \cap e$ is obviously connected; and $G_1' \cap J$ is connected. This means that G_1 was not maximal.

Suppose, on the other hand, that every edge of J has both its vertices in $\bigcup G_i$, but that some other edge e of $\mathfrak{K}$ fails to have this property. We may suppose, as before, that $e = v_0 v_1$, with $v_0 \in G_1$ and $v_1 \notin \bigcup G_i$. Then $v_1 \notin J$. Let $G_1' = G_1 \cup e$. Then G_1' satisfies (1) and (2), trivially; and (3) is also satisfied, because $G_1' \cap J = G_1 \cap J$. Thus G_1 was not maximal; and this contradiction completes the proof of the lemma.

The sets $G_i \cap J$ are broken lines. If G_i and G_j are consecutive on J, then $G_i \cap J$ and $G_j \cap J$ are joined by an edge of J. Evidently there are exactly n such edges; let them be

$$e_1, \quad e_2, \quad \cdots, \quad e_n,$$

in the cyclic order of their occurrence on J. Then $e_{i+1} = f^j(e_i)$ for some j; f^j has period n, and its iterates are precisely those of f, in some order. Therefore we may assume, as a matter of convenience, that $j = 1$, so that $f(e_i) = e_{i+1}$. We may also assume that e_1, $G_1 \cap J$, and e_2 are consecutive on J.

Let $b_i = G_i \cap J$, and let $c_i = e_i \cup b_i$, with the orientation induced by Z_J. Then any linear combination $\sum \eta_j c_j$, with integer coefficients, is a 1-chain. For $1 \leqq i \leqq n$, let v_i be the "left-hand" vertex of c_i (which is also the left-hand vertex of e_i).

Each vertex of $\mathfrak{K}$ lies in some G_i. If $v = v_i$, let $b_v = 0$. Otherwise, let b_v be the (unique) broken line from v_i to v in G_i, oriented positively from v_i to v, so as to be a 1-chain.

If $C = \sum \alpha_j \sigma_j$ is a chain on $\mathfrak{K}$, then $f(C)$ denotes the chain $\sum \alpha_j f(\sigma_j)$, where $f(\sigma_j)$ has the orientation induced by f; that is, $f(vv') = f(v)f(v')$. It is clear that the function $v \to b_v$ is f-invariant, in the sense that $b_{f(v)} = f(b_v)$. For if $v \in G_i$, then b_v is uniquely determined by the property of being a 1-chain on G_{i+1}, carried by the unique broken line from v_{i+1} to $f(v)$ in G_{i+1}, oriented positively from v_{i+1} to $f(v)$. And $f(b_v)$ is a chain which has this property.

Let $\sigma = vv'$ be an (oriented) edge of $\mathfrak{K}$. If $\sigma \subset \bigcup G_i$, let $C_\sigma = 0$. Otherwise, let

$$C_\sigma = b_v + \sigma - b_{v'}.$$

It is clear that for each such C_σ there is a 1-chain C_σ' on J, with constant coefficient $= 1$, such that $C_\sigma - C_\sigma'$ is a 1-cycle. (To obtain such a C_σ', we need merely assign the appropriate orientation to one of the broken lines in J joining the "end-points" of b_v and $b_{v'}$; if these "end-points" are the same, we take $C_\sigma' = 0$.) Here C_σ' is *not* uniquely determined by σ, except when $C_\sigma' = 0$.

Now $C_\sigma - C_\sigma'$ is homologous on $\mathfrak{K}$ to an integral multiple mZ_J of Z_J. Thus

$$C_\sigma - C_\sigma' \sim mZ_J.$$

Therefore

$$C_\sigma \sim mZ_J + C_\sigma',$$

and the right-hand member is a 1-chain on J. Let

$$C''_\sigma = mZ_J + C'_\sigma .$$

We assert that C''_σ *is* uniquely determined by σ. Obviously C_σ is so determined. And given an alternative $'C'_\sigma$, $'C''_\sigma$, such that

$$C_\sigma \sim m'Z_J + 'C'_\sigma$$

and

$$'C''_\sigma = m'Z_J + 'C'_\sigma ,$$

it follows that

$$'C''_\sigma \sim C''_\sigma ,$$

so that

$$C''_\sigma - 'C''_\sigma \sim 0 \qquad \text{on } \mathfrak{K}.$$

But $C''_\sigma - 'C''_\sigma$ is a 1-cycle on J, and $H^1(\mathfrak{K})$ has a generator on J. It follows that $C''_\sigma - 'C''_\sigma \sim 0$ on J. Therefore, since J is a polygon, we have $C''_\sigma - 'C''_\sigma = 0$, which was to be proved.

To see the intuitive significance of C''_σ, we should suppose that the linear graphs G_i are shrunk to points. Each σ then becomes a broken line (or polygon) σ' with both of its end-points in the image of J. C''_σ then measures the "number of times that the image of σ winds around $\mathfrak{K}$." If the end-points of σ' are the same (which can happen, if both end-points of σ lie in the same G_i), then C''_σ may be either $= 0$ or a positive or negative multiple of Z_J. If the end-points of σ' are different, then C''_σ may be either a chain joining the end-points of σ' or the sum of such a chain with a cycle mZ_J.

4.4. LEMMA. *The function* $\sigma \to C''_\sigma$ *is f-invariant*; *that is to say*,

$$C''_{f(\sigma)} = f(C''_\sigma).$$

Proof of lemma. It has already been observed that $b_{f(v)} = f(b_v)$. Since

$$C_\sigma = b_v + \sigma - b'_v \qquad (\sigma = vv'),$$

it follows that

$$C_{f(\sigma)} = f(b_v) + f(\sigma) - f(b_{v'}) = f(C_\sigma).$$

Since

$$C_\sigma \sim C''_\sigma ,$$

we have[3]

$$C_\sigma - C''_\sigma = \partial \sum \beta_j \sigma_j^2 .$$

Therefore

$$f(C_\sigma) - f(C''_\sigma) = f[\partial \sum \beta_j \sigma_j^2] = \partial f[\sum \beta_j \sigma_j^2].$$

Thus $f(C''_\sigma)$ is a 1-chain on J, homologous to $f(C_\sigma) = C_{f(\sigma)}$ on $\mathfrak{K}$. But $C''_{f(\sigma)}$ is uniquely determined by these conditions. Therefore $f(C''_\sigma) = C''_{f(\sigma)}$.

[3] Here, and occasionally hereafter, we use the symbol ∂ in a second sense, as the algebraic boundary operator applied to chains. It should be plain, in each context, which meaning is intended.

For each $\sigma = vv' \in \mathfrak{K}^1$, such that σ does not lie in $\cup G_i$, we shall define a locally one-to-one mapping

$$\phi_\sigma : \quad \sigma \to J \quad \text{(into)}$$

in the following way. Let

$$C''_\sigma = mZ_J + C'_\sigma = \sum \eta_i c_i .$$

Since all coefficients in

$$C'_\sigma = \sum \delta_i c_i$$

are the same, and $= \pm 1$, it follows that η_i takes on only two values, differing by a constant ± 1. Therefore C''_σ can be represented by a path which starts at the initial point of b_v, proceeds to the initial point of $b_{v'}$ (in one of the two possible ways on J), and then goes around J a certain number of times, preserving its initial direction. Thus this path can be chosen so as to be a locally one-to-one mapping. Taking the linear interval σ as its pre-image, we obtain ϕ_σ. It is readily verified that ϕ_σ is independent of the orientation assigned to σ.

Such a ϕ_σ can be defined for each σ in $\mathfrak{K}^1$. Moreover, this can be done in such a way that the mappings ϕ_σ are f-invariant, in the sense that

$$\phi_{f^i(\sigma)} = f^i \phi_\sigma f^{-i}.$$

For suppose that a particular ϕ_σ has been defined, representing the chain C''_σ. By 4.4, $C''_{f(\sigma)} = f(C''_\sigma)$. Therefore $f\phi_\sigma f^{-1}$ has all the properties desired for $\phi_{f(\sigma)}$, and can be used as a definition of the latter. Similarly we can define $\phi_{f^i(\sigma)}$ as $f^i \phi_\sigma f^{-i}$. We proceed in this fashion in every orbit $\{f^i(\sigma)\}$.

Note that if $\sigma = e_i \subset J$, then ϕ_σ maps e_i homeomorphically onto c_i, leaving the left-hand end-point of e_i fixed.

If $\sigma = vv' \subset \cup G_i$, let ϕ_σ be the mapping which throws σ onto the initial point of b_v (which is also the initial point of $b_{v'}$).

Consider now an oriented 2-face s of $\mathfrak{K}$. Let

$$\partial s = \sum \sigma_j \qquad (1 \leqq j \leqq 3).$$

Evidently $\sum \sigma_j = \sum C_{\sigma_j}$, because each b_v, $b_{v'}$ appears twice in the latter sum, with opposite signs. Therefore

$$\sum C_{\sigma_j} \sim 0 \quad \text{on } \mathfrak{K}.$$

But

$$C_{\sigma_j} \sim C''_{\sigma_j},$$

so that

$$\sum C''_{\sigma_j} \sim 0 \quad \text{on } \mathfrak{K}.$$

Let S_s be a polygon, obtained by collapsing into points the edges of s that lie in $\cup G_i$. (S_s is a polygon, rather than a single point, because the G_i's are acyclic.) Let g be the mapping $\partial s \to S_s$. Let

$$\phi_s : \quad S_s \to J$$

be defined by the condition

$$\phi_s(x) = \phi_{\sigma_j}[g^{-1}(x)] \qquad (x \in g(\sigma_j)).$$

Then ϕ_s is a piecewise linear mapping of S_s into J, such that each set ϕ_s^{-1} is finite. The components β_k of the sets $\phi_s^{-1}(c_i)$ are broken lines, and each mapping $\phi_s \mid \beta_k$ is a homeomorphism. Let us number the β_k's in the order of their appearance on S_s, in the positive direction on ∂s, starting from a base-point x_0. For each k, let $\tau_k = \phi_s(\beta_k)$. Then each τ_k is $= c_i$ for some i. Thus the mapping ϕ_s can be represented by a sequence

$$\tau_1^{\alpha_1}\tau_2^{\alpha_2} \cdots \tau_k^{\alpha_k}\tau_{k+1}^{\alpha_{k+1}} \cdots \tau_m^{\alpha_m},$$

where α_k is 1 or -1, according as $\phi_s \mid \beta_k$ preserves or reverses orientation.

Now J is a (simple closed) polygon. Therefore the injection of the fundamental group (or the edge-path group) of J into $H^1(J)$ is an isomorphism onto. But the injection of ϕ_s is the 1-chain $\sum C''_{\sigma_j}$, which is $= 0$. Therefore ϕ_s, considered as a mapping, is contractible; and so ϕ_s cannot be everywhere locally one-to-one. This means that there is a k such that

$$\tau_k = \tau_{k+1}$$

and

$$\alpha_k = -\alpha_{k+1}.$$

(Here the subscripts are taken modulo m.)

For $1 \leqq i \leqq n$, let p_i be an interior point of e_i. Let

$$Q = \cup \phi_s^{-1}(p_i) \qquad (\sigma \in \mathfrak{K}^1, \quad 1 \leqq i \leqq n).$$

Since the mappings ϕ_σ are f-invariant, it follows that Q is f-invariant. If k is as in the two equations immediately above, then some two points x, y of some set $\phi_s^{-1}(p_i)$ are successive in Q on ∂s; that is to say, $x \cup y$ does not separate any other two points of Q from one another in ∂s. Therefore there is a broken line B_k, from x to y, lying except for its end-points in Int s. For convenience, we suppose that $k = m - 1$.

Let us delete the last two terms from the sequence representing ϕ_s, obtaining a mapping ϕ'_s. B_k decomposes s into two disks, one of which intersects Q only in $x \cup y$. Let s' be the other of these two disks.

Now ϕ_s is contractible. Therefore, as before, there is a $k' < m - 2$, such that $\tau_{k'} = \tau_{k'+1}$ and $\alpha_{k'} = -\alpha_{k'+1}$, with subscripts taken modulo $m - 2$. Therefore there is a broken line $B_{k'}$, joining two points x', y' of $\partial s' \cap Q$, such that $x' \cup y'$ does not separate any two points of Q from one another in ∂s.

Thus, in a finite number of such steps, we obtain a sequence $B_1, B_2, \cdots, B_{m/2}$ of disjoint broken lines, such that (1) each B_k lies, except for its end-points, in Int s, (2) for each k, the end-points of B_k lie in a single set $\phi_s^{-1}(p_i) \cap \partial s$, (3) $Q \cap \partial s \subset \cup B_k$, and (4) $\cup B_k$ contains no vertex of s.

So far, we have been considering a fixed s. Given the sequence $B_1, B_2, \cdots, B_{m/2}$ for s, the sequences

$$f^{i-1}(B_1), \quad f^{i-1}(B_2), \quad \cdots, \quad f^{i-1}(B_{m/2})$$

have the same properties relative to the corresponding 2-simplices $f^{i-1}(s)$. (We recall that Q is f-invariant.) This means that B-sequences can be defined for all 2-simplices s of $\mathfrak{L}$, in such a way that their union $\mathfrak{B}$ is f-invariant.

Each broken line B_k intersects only one set $\bigcup_s \phi_s^{-1}(p_i)$. Let $\mathfrak{B}_1$ be the union of all broken lines B_k that intersect $\bigcup_s \phi_s^{-1}(p_1)$. We have now proved the following lemma:

4.5. Lemma. *There is a polyhedral linear graph $\mathfrak{B}_1 \subset \mathfrak{L}$, such that*
(1) $\mathfrak{B}_1 \cap \mathfrak{L}^1 = \bigcup_s \phi_s^{-1}(p_1)$,
(2) $\mathfrak{B}_1 \cap \mathfrak{L}^0 = 0$,
(3) *the iterated images $\mathfrak{B}_i = f^{i-1}(\mathfrak{B}_1)$ are disjoint,*
(4) *if s is a 2-simplex of $\mathfrak{L}$, then no point of $\mathfrak{B}_1 \cap \partial s$ is isolated in $\mathfrak{B}_1 \cap s$, and*
(5) $\mathfrak{B}_1 \cap J$ *is a single point* q_1.

(Here (4) is a consequence of the fact that each point of $\mathfrak{B}_1 \cap s$ is an endpoint of some $B_k \cap s$.)

This lemma states all that we shall need of the discussion beginning with Lemma 4.3.

We recall that F is a subcomplex of $\mathfrak{M}$, $\mathfrak{S}$ is the slab-system for $\mathfrak{M}$, $\mathfrak{T}$ is the induced slab-neighborhood of F, $\mathfrak{K} = \mathrm{Cl}(\mathfrak{M} - \mathfrak{T})$, and $\mathfrak{M}'$ is a subdivision of $\mathfrak{M}$, such that $\mathfrak{T}$ and $\mathfrak{K}$ are subcomplexes of $\mathfrak{M}'$. $\mathfrak{L}$ is a 2-spine of $\mathfrak{K}$, defined relative to $\mathfrak{M}'$. J is a polygon in $\partial\mathfrak{T} = \partial\mathfrak{K}$, such that J is latitudinal in $\mathfrak{T}$ and $f(J) = J$.

Let $\mathfrak{S}'$ be the slab-system for $\mathfrak{M}'$, and let $\mathfrak{N}'(\mathfrak{K})$ and $\mathfrak{N}'(J)$ be the induced slab-neighborhoods of $\mathfrak{K}$ and J respectively. Let the vertices and edges of J be

$$v_1, \quad e_1, \quad v_2, \quad \cdots, \quad v_m, \quad e_m,$$

in the cyclic order of their appearance on J. Then the sets

$$\mathfrak{N}'(v_i), \qquad \mathrm{Cl}[\mathfrak{N}'(e_i) - \mathfrak{N}'(\mathfrak{M}'^0)]$$

form a sequence

$$g_1, \quad g_2, \quad \cdots, \quad g_{2m}$$

of combinatorial 3-cells, in which consecutive cells intersect one another in disks. By an obvious geometric construction, we obtain a polygon J', lying in

$$\partial\mathfrak{N}'(J) \cap \partial\mathfrak{N}'(\mathfrak{K}),$$

such that J and J' are *parallel in* $\mathfrak{N}'(J)$. By this we mean that (1) J' intersects each set g_i in a broken line, (2) J' intersects each set $g_i \cap g_{i+1}$ in a point, and (3) there is polyhedron A, homeomorphic to the closed annulus between two concentric circles, such that

$$\partial A = J \cup J',$$

and such that each set $A \cap g_i$ is a disk.

We know J carries a generator Z_J of $\mathfrak{K}$. From this it follows that J' carries a generator $Z_{J'}$ of $\mathfrak{N}(\mathfrak{K})$; the verification is omitted.

By Lemma 3.4, $\mathfrak{N}'(\mathfrak{K})$ is the image of $\mathfrak{N}'(\mathfrak{L})$ under a piecewise linear homeomorphism h, such that $h \mid J'$ is the identity.

Consider now the set $\mathfrak{B}_1$, given by Lemma 4.5. We know that $\mathfrak{B}_1 \cap \mathfrak{L}^0 = 0$; and we may also assume that $\mathfrak{B}_1 \cap \mathfrak{N}'(\mathfrak{L}^0) = 0$, as the latter situation can be obtained by moving the points of $\mathfrak{B}_1$ slightly farther away from $\mathfrak{L}^0$. We may assume further that the components B_k of the sets $\mathfrak{B}_1 \cap s$ are "straight relative to the slab-system $\mathfrak{S}'$," in the sense that (1) each intersection $B_k \cap \partial\mathfrak{N}'(e)$ is a point, (2) each intersection $B_k \cap \mathfrak{N}'(e)$ is a broken line, and (3) each intersection

$$B_k \cap \mathrm{Cl}[\mathfrak{N}'(s) - \mathfrak{N}'(\mathfrak{L}^0)]$$

is a broken line.

Starting with $\mathfrak{B}_1$ we shall construct a 2-manifold $M_{1,1}$ with boundary, lying in $\mathfrak{N}'(\mathfrak{L})$, such that

(1) $\partial M_{1,1} \subset \partial\mathfrak{N}'(\mathfrak{L})$,

(2) the images $M_{i,1} = f^{i-1}(M_{1,1})$ are disjoint,

(3) $\mathfrak{B}_1 \subset M_{1,1}$, and

(4) $\partial M_{1,1} \cap J'$ is a single point, which is a crossing point of $\partial M_{1,1}$ with J'.

The construction is as follows.

Let b be a component of a set $\mathfrak{B}_1 \cap \mathfrak{N}'(e)$, where e is an edge of $\mathfrak{L}$. Then b consists of a finite number of linear intervals $b \cap s$ with a common end-point $w \in e$. Therefore there is a disk D_b, containing b, and lying in $\mathfrak{N}'(e)$, such that ∂D_b lies in $\partial[\mathfrak{N}'(e) - \mathfrak{N}'(\mathfrak{L}^0)]$ and intersects each set $\mathrm{Cl}[\mathfrak{N}'(s) - \mathfrak{N}'(\mathfrak{L}^0)]$ in a broken line. The disks D_b can be chosen so as to be disjoint; and their images $f^{i-1}(D_b)$ will then also be disjoint. For the case $w = q_1$, D_b can be constructed so that $\partial D_b \cap J'$ is a single crossing point of ∂D_b with J'. If $w \neq q_1$, then ∂D_b and J' will automatically be disjoint.

Now let $\mathfrak{C}$ be a set $\mathrm{Cl}[\mathfrak{N}'(s) - \mathfrak{N}'(\mathfrak{L}^1)]$, where s is a 2-face of $\mathfrak{L}$; and let c be a component of

$$\mathfrak{C} \cap [\mathfrak{B}_1 \cup \bigcup D_b].$$

Then $\mathfrak{C}$ is the union of two combinatorial 3-cells $\mathfrak{C}_1$, $\mathfrak{C}_2$, such that

$$\mathfrak{C}_1 \cap \mathfrak{C}_2 = s \cap \mathfrak{C};$$

and c lies in the union of two polygons c_1 and c_2, such that

$$c_1 \subset \partial\mathfrak{C}_1$$

and

$$c_2 \subset \partial\mathfrak{C}_2 .$$

It follows that there is a disk D_c, containing c and lying in $\mathfrak{C}$, such that $\partial D_b = (c_1 \cup c_2) \cap \partial\mathfrak{C}$. The disks D_c can be chosen so as to be disjoint; and their images $f^{i-1}(D_c)$ will then also be disjoint. Let

$$M_{1,1} = \bigcup D_b \cup \bigcup D_c .$$

Then $M_{1,1}$ has the properties desired.

Starting with $M_{1,1}$ we shall construct a surface $M_{1,2}$, having all the stated properties of $M_{1,1}$, such that $\partial M_{1,2}$ has only one component $F_{1,2}$.

Let $F_{1,2}$ be the component of $\partial M_{1,1}$ that intersects J'. Let

$$U = \partial\mathfrak{N}'(\mathfrak{L}) - (J' \cup F_{1,2}).$$

Then U is homeomorphic to the interior of a disk, so that every polygon $P \subset U$ is the boundary of a unique polyhedral disk D_P in U. Let the components of $\partial M_{1,1}$ be $F_{1,2}, P_1, \cdots, P_m$; and for each j let D_j be the disk in U, bounded by P_j. Let j be such that Int D_j contains no polygon P_k. Evidently the images $f^{i-1}(D_j)$ are disjoint, for otherwise one of them would lie in the interior of another, and f would not be periodic. We may therefore add D_j to $M_{1,1}$, and then retract the resulting surface very slightly into Int $\mathfrak{N}'(\mathfrak{L})$ in the neighborhood of D_j, by a piecewise linear homeomorphism. If the set on which this homeomorphism differs from the identity is a sufficiently small neighborhood of D_j, then the image-surface will satisfy the hypothesis for $M_{1,1}$; and its boundary will have only $m - 1$ components. It follows that the desired $M_{1,2}$ can be obtained by m iterations of this process.

We recall from Lemma 3.6 that the homeomorphism h of Lemma 3.4 can be chosen so that $hf = fh$. Let $M_{1,3} = h^{-1}(M_{1,2})$. Then

(1) $M_{1,3} \subset \mathfrak{N}'(\mathfrak{K})$,

(2) $\partial M_{1,3} = M_{1,3} \cap \partial\mathfrak{N}'(\mathfrak{K})$,

(3) the images $M_{i,3} = f^{i-1}(M_{1,3})$ are disjoint, and

(4) $\partial M_{1,3}$ crosses J' at exactly one point $J' \cap M_{1,3}$.

Consider the set

$$\mathfrak{T}' = \mathrm{Cl}[\mathfrak{M} - \mathfrak{N}'(\mathfrak{K})].$$

By Lemma 3.3, $\mathfrak{T}'$ is combinatorially equivalent to $\mathfrak{T} = \mathfrak{N}(F)$. Let the vertices and edges of F be

$$w_1, \quad a_1, \quad w_2, \quad a_2, \quad \cdots, \quad w_m, \quad a_m,$$

in the cyclic order of their occurrence on F. Then the sets

$$g_{2i} = \mathrm{Cl}[\mathfrak{N}(a_i) - \mathfrak{N}(\mathfrak{M}^0)]$$

and

$$g_{2i-1} = \mathfrak{N}(w_i)$$

form a decomposition of $\mathfrak{T}$ into 3-cells g_i; and different sets g_i, g_j intersect only if i and j are consecutive modulo $2m$, in which case $g_i \cap g_j$ is a disk lying in the boundary of each of them. For each i, let

$$g_i' = g_i \cap \mathfrak{T}' = \mathrm{Cl}[g_i - \mathfrak{N}'(\mathfrak{K})].$$

It is then a straightforward matter to show that the g_i''s form a decomposition of $\mathfrak{T}'$, having the properties just stated for the g_i's. (Here we are appealing directly to the geometric definition of a slab-system.)

To complete the construction of the M_1 of Theorem 1.2, it would suffice to construct a polyhedron A_1, homeomorphic to a plane annulus, such that

$$A_1 = F \cup \partial M_{1,3},$$

$$A_1 \subset \mathfrak{T}',$$

$$A_1 \cap \partial\mathfrak{T}' = \partial M_{1,3},$$

and the images $A_i = f^{i-1}(A_1)$ intersect one another only in F. We could then define M_1 as $M_{1,3} \cup A_1$. For the sake of convenience, however, we shall first define a new surface $M_{1,4}$ for which the corresponding A_1 is more readily constructed.

From condition (4) for $M_{1,3}$, it follows that $H^1(\partial\mathfrak{T}')$ is generated by a pair $\{Z_{J'}, Z_{1,3}\}$, where $Z_{1,3}$ is a cycle on $F_{1,3} = \partial M_{1,3}$. There is no loss of generality in supposing also that $F_{1,3}$ is in general position relative to the g_i''s, in the sense that every intersection

$$F_{1,3} \cap g_i' \cap g_{i+1}'$$

consists of a finite number of "true crossing points" of $F_{1,3}$ with the polygon

$$P_i = g_i' \cap g_{i+1}' \cap \partial\mathfrak{T}'.$$

If intersection numbers ± 1 are assigned to these intersections in the usual way, using orientations of $F_{1,3}$ and P_i, then the sum of these intersection numbers, for fixed i, must be ± 1. We shall show that there is an $M_{1,4}$, having all the stated properties of $M_{1,3}$, such that the intersections of $F_{1,4}$ with the polygons P_i are single points.

Let $F_{i,3} = f^{i-1}(F_{1,3})$. Let B be a component of a set

$$F_{i,3} \cap g_j' = F_{i,3} \cap \partial g_j'.$$

Then B is a broken line. If every such B joins a point of g_{j-1}' to a point of g_{j+1}', then $M_{1,3}$ has the property desired for $M_{1,4}$. Otherwise, there is a broken line B', lying in P_j (or P_{j-1}) such that $B \cup B'$ is the boundary of a disk D_B, lying in $\partial g_j' \cap \partial\mathfrak{T}'$. The image-sets $f^k(B)$, $f^k(B')$ have the same properties, so that we may assume that $i = 1$. The iterated images $f^i(D_B)$ must be disjoint, because otherwise one would lie in the interior of another, and f would not be periodic. First we add D_B to $M_{1,3}$, obtaining $M_{1,3}'$. Let D' be a polyhedral disk in $\partial g_{2m}' \cap \partial\mathfrak{T}'$, such that $\partial D' \cap \partial M_{1,3}'$ is a broken line containing B' in its interior. We take D' in a sufficiently small neighborhood of B' so that the image-sets $f^i(D')$ are disjoint. We then retract $M_{1,3}' \cup D'$ slightly away from D_B into $\mathfrak{M} - \mathfrak{T}'$, in the neighborhood of D_B, by a piecewise linear homeomorphism which is the identity except in a small neighborhood U of D_B. (See Lemma 7 of [M_3].) If U is taken as a sufficiently small neighborhood of D_B, then the resulting surface $M_{1,3}''$ will have all the stated properties of $M_{1,3}$. And the number of points in

$$\partial M_{1,3}'' \cap \bigcup P_j$$

is less than the number of points in $F_{1,3} \cap \bigcup P_j$. It follows by induction that $M_{1,4}$ can be obtained in a finite number of steps of the sort just described.

We can now define A_1.

(I). Consider a set g'_{2i}. $F \cap g'_{2i}$ is a linear interval, with end-points x_0, x_2 and mid-point x_1. Evidently g'_{2i} lies in the union of all 3-simplices of $\mathfrak{M}$ that contain $F \cap g'_{2i}$, so that the points of the annulus $\partial g'_{2i} \cap \partial \mathfrak{T}'$ are joined to the points x_0, x_1, x_2 by unique straight lines; if $p \in P_{2i-1}$, then the interval $x_0 p$ lies in $g'_{2i} \cap g'_{2i-1}$; and similarly, if $q \in P_{2i}$, then $x_2 q \subset g'_{2i} \cap g'_{2i+1}$. (Here, and also in step (II) below, we are appealing directly to the geometric definition of a slab-system.) Let the end-points of $B = F_{1,4} \cap g'_{2i}$ be $p \in P_{2i}$, and $q \in P_{2i}$. Then g'_{2i} contains unique 2-simplices $x_0 p x_1$, pqx_1, and $x_1 x_2 q$, spanned by the indicated points. And the join $J(B, x_1)$ of B with x_1 is a polyhedral disk. Thus

$$A_{1,2i} = x_0 p x_1 \cup pqx_1 \cup x_1 x_2 q \cup J(B, x_1)$$

is a polyhedral disk, bounded by

$$B \cup x_0 p \cup x_2 q \cup (F_{1,4} \cap g'_{2i}).$$

The iterated images $A_{j,2i} = f^{j-1}(A_{1,2i})$ intersect only in F, because the iterated images of B are disjoint, and f is simplicial.

(II). Consider a set g'_{2i-1}. Then $F \cap g'_{2i-1}$ is the union of two linear intervals $x_0 v$ and vx_1, where $v \in \mathfrak{M}^0$. And the points of $\partial g'_{2i-1}$ are joined to v by unique linear intervals, each of which lies in a single simplex of $\mathfrak{M}$. Let $A_{1,2i-1}$ be the join of v with

$$g'_{2i-1} \cap (A_{1,2i-2} \cup A_{1,2i} \cup F_{1,4}).$$

Then the iterated images of $A_{1,2i-1}$ intersect only in F, because the set which we joined with v has this property.

Let $A_1 = \bigcup A_{1,j}$, and let $M_1 = M_{1,4} \cup A_1$.

M_1 may not be connected, but it can be made so, simply by deleting all components of M_1 that do not intersect F. To complete the proof of Theorem 1.2, it remains only to show that M_1 is orientable. Evidently $M_1 \cup M_2$ is a 2-manifold. Therefore $M_1 \cup M_2$ carries a nonbounding 2-cycle Z^2, with integers modulo 2 as coefficients. Therefore, by the Alexander Duality Theorem, $M_1 \cup M_2$ separates $\mathfrak{M}$ into exactly two connected open sets U and V. (See [L].) Let $\mathfrak{M}'$ be a subdivision of $\mathfrak{M}$, such that $\bar{U}$ forms a subcomplex of $\mathfrak{M}'$. Then $\bar{U} \cup \bar{V} = \mathfrak{M}'$; $\bar{U} \cap \bar{V} = M_1 \cup M_2$; and $\mathfrak{M}'$ carries a nonbounding 3-cycle with integer coefficients, but neither of the sets U and V has this property. By the Mayer-Vietoris Theorem [L, p. 267], $M_1 \cup M_2$ carries a nonbounding 2-cycle, with integer coefficients, which bounds both on U and on V. Thus $M_1 \cup M_2$ is orientable, and so also is M_1.

5. Simplicial homeomorphisms of the 3-sphere

It remains only to deduce Theorem 1.1 from Theorem 1.2.

Let n be the period of f; let the sets M_i be as in Theorem 1.2; and let D

be a polyhedral disk in $\mathfrak{M}$, such that

$$\partial D = F.$$

We assume also that M_1 is chosen so as to minimize the 1-dimensional Betti number $p_1(M_1)$, with integers modulo 2 as coefficients. Clearly we may not suppose that D is in general position relative to the M_i's, because $D \cap M_i$ contains F for each i. But we may suppose that D is in "almost general position" relative to the M_i's, in the sense that each intersection $D \cap M_i$ is a finite union of polygons, intersecting one another only in F.

As in an analogous situation in the proof of Theorem 1.2, we may suppose that the M_i's appear around F in the stated cyclic order; that is, one of the two components of $\mathfrak{M} - (M_1 \cup M_2)$ is disjoint from $\bigcup M_i$. Thus there are 3-manifolds $\mathfrak{D}_1, \mathfrak{D}_2, \cdots, \mathfrak{D}_n$ with boundary, such that

$$\partial\mathfrak{D}_i = M_i \cup M_{i+1},$$

and

$$\mathfrak{D}_i \cap \bigcup_{j \neq i} \mathfrak{D}_j = \partial\mathfrak{D}_i .$$

Thus Int $\mathfrak{D}_1$ may be thought of as a fundamental domain of $\mathfrak{M}$ relative to f.

Let J be a polygon different from F, lying in a set $D \cap M_i$; let D_J be the (unique) polyhedral disk lying in D and bounded by J; and let Z_J be the nonzero 1-cycle carried by J. (Here, and hereafter, we use coefficients modulo 2.) We may suppose that D_J contains no similar disk $D_{J'}$ $(J' \subset D \cap M_j,\ J' \neq F)$. It follows that

$$\text{Int } D_J \cap \bigcup M_i = 0,$$

so that Int D_J lies in Int $\mathfrak{D}_i$ (or Int $\mathfrak{D}_{i-1}$). We suppose also that $i = 1$.

We assert that $Z_J \sim 0$ on M_1. For suppose not. By moving J slightly into Int M_1 (that is, slightly away from F), we obtain a simple closed polygon J', such that $Z_{J'} \nsim 0$ on M_1, and such that J' bounds a disk

$$D_{J'} \subset \text{Int } \mathfrak{D}_1 \cup J'.$$

J' separates M_1, locally, into two connected open sets. Let us add $D_{J'}$ to M_1, so that $D_{J'}$ has a neighborhood in $M_1 \cup D_{J'}$ which is the union of two disks E_1, E_2, intersecting in $D_{J'}$. E_1 and E_2 can then be "pulled apart at $D_{J'}$," so as to give a surface M_1' which has all the stated properties of M_1, but has a lower 1-dimensional Betti number. (To justify the "pulling apart" operation, see Lemma 7 of [M3].)

Since $Z_J \sim 0$ on M_1, it follows that there is a component of $M_1 - J'$, with closure D_J', such that $\partial D_J' = J$. We assert that D_J' is a disk. For otherwise D_J' would be a disk with one or more handles, and

$$M_1' = (M - D_J') \cup D_J$$

would have all the stated properties of M_1, with a lower 1-dimensional Betti number.

Let $D' = (D - D_J) \cup D_J'$. Then D' is a disk. And $D' - F$ can be retracted slightly into Int $\mathfrak{D}_2$, in the neighborhood of $D'_J - F$, so as to give a disk D'', bounded by F, such that

$$D'' \cap \bigcup M_i \subset D \cap \bigcup M_i$$

and

$$J \cap D'' \subset F.$$

Thus, in replacing D by D'', we have reduced by one the 1-dimensional Betti number of $D \cap \bigcup M_i$. Therefore, in a finite number of such steps, we can obtain a disk D_1, bounded by F, such that

$$D \cap \bigcup M_i = F.$$

This means that Int D_1 lies in some set Int $\mathfrak{D}_i$, so that the disks $D_i = f^{i-1}(D_1)$ intersect one another only in F.

For each i, let C_i be the 3-manifold with boundary bounded by $D_i \cup D_{i+1}$, such that Int C_i is disjoint from $\bigcup D_i$. Then C_1 has a topological image in E^3, because $\mathfrak{M}$ is a 3-sphere. By Theorem 1 of [M_4], or by the Hauptvermutung, Theorem 4 of [M_5], this means that C_1 has a combinatorial image in E^3; and so, by Lemma 3.9, C_1 is a combinatorial 3-cell. Thus $\mathfrak{M}$ is the union of the 3-cells C_i; and $C_i \cap C_j = F$ unless i and j are consecutive modulo n, in which case $C_i \cap C_j$ is a disk. From this it is easily verified that f is homeomorphic to a rotation, which was to be proved.

6. Proof that F is always a polygon

We shall show that if $\mathfrak{M}$, f, F, and n are as in Section 1, with n arbitrary, then F is a polygon. The proof depends on the following:

LEMMA. *If F is 1-dimensional, then F is a 1-manifold.*

It will turn out, of course, that F is always 1-dimensional.

Proof of lemma. We shall assume, as in the main body of the paper, that $\mathfrak{M}$ is sufficiently finely subdivided so that f is regular, in the sense that $f(\sigma) = \sigma$ ($\sigma \in \mathfrak{M}$) only if $f \mid \sigma$ is the identity. It follows that F lies in the 1-skeleton $\mathfrak{M}^1$, and that F is locally Euclidean except possibly at vertices v of $\mathfrak{M}$.

Let v be a vertex of $\mathfrak{M}$, lying in F, let $\mathrm{St}(v)$ be the closed star of v, and let $\beta = \partial \mathrm{St}(v)$. Then $f \mid \beta$ is an orientation-preserving periodic homeomorphism of the 2-sphere β onto itself. It follows from well-known results of B. Kerékjártó [K] that $f \mid \beta$ has exactly two fixed points x and y. Since $f \mid \mathrm{St}(v)$ is a simplicial homeomorphism of $\mathrm{St}(v)$ onto itself, it follows that $F \cap \mathrm{St}(v)$ is the union of the edges vx, vy of $\mathfrak{M}$. Therefore F is locally Euclidean at v, which completes the proof of the lemma.

We shall now show that F is a polygon. Let p be a prime factor of n, and let $q = n/p$. Then f^q has period p. Let F' be the fixed-point set of f^q. By the cited result of Smith, F' is a polygon. Evidently $F \subset F'$; and since F is locally Euclidean, it follows that $F' = F$, so that F is a polygon.

Here we have been merely following, in a straightforward fashion, the suggestions made in the middle paragraph of p. 162 of [S], in which Smith unaccountably denied that he was sketching a proof.

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