BORNOLOGICAL STRUCTURES[1]

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Recently, several variations on a bornological theme have appeared (see the examples below). The purpose of our remarks is to suggest in §1 a framework sufficiently general to permit a unified treatment of these (and, it is hoped, future) variations, to consider within this framework in §2 the problem of when the Cartesian product of spaces having a certain bornological property inherits that property and the relation of this problem with Ulam's measure problem, and finally to state in §3 some new results concerning bornological properties of partially ordered locally convex spaces. In particular we shall generalize in §3 the theorem that the topology of a Banach lattice E is the finest locally solid topology on E. All vector spaces are assumed to be over the reals, although our discussion in §1 does not require this.

1. Structured spaces

A *structure* on a vector space E is a filter base $\mathcal{V}$ containing E of convex equilibrated subsets of E such that for all $V \in \mathcal{V}$ and all scalars λ, $\lambda V \in \mathcal{V}$ (in particular, therefore, $\{0\} \in \mathcal{V}$). A *structured space* is a pair (E, $\mathcal{V}$) where $\mathcal{V}$ is a structure on E. If $(E, \mathcal{V})$ and $(F, \mathcal{W})$ are structured spaces, we shall say f is a *structure homomorphism* from $(E, \mathcal{V})$ into $(F, \mathcal{W})$ if f is a linear transformation from E into F, $f(\mathcal{V}) \subseteq \mathcal{W}$, and $f^{-1}(\mathcal{W}) \subseteq \mathcal{V}$. Clearly the composition of two structure homomorphisms is a structure homomorphism. Topology $\mathfrak{J}$ on E is *compatible* with structure $\mathcal{V}$ if $\mathfrak{J}$ is a locally convex topology on E, and if $\mathcal{V}$ contains a fundamental system of neighborhoods of zero for $\mathfrak{J}$; a *structured locally convex space* is a triple $(E, \mathcal{V}, \mathfrak{J})$ where $\mathcal{V}$ is a structure on E, $\mathfrak{J}$ a topology on E compatible with $\mathcal{V}$. When no confusion arises, we shall denote a structured locally convex space $(E, \mathcal{V}, \mathfrak{J})$ simply by E. There is always at least one topology on E compatible with any given structure, namely, the topology whose only open sets are E and $\emptyset$.

Let $(E, \mathcal{V}, \mathfrak{J})$ be a structured locally convex space, $\mathcal{B}$ a class of subsets of E. We shall say V is a *$\mathcal{B}$-bornivore* set if $V \in \mathcal{V}$, V is absorbing, and V absorbs each $B \in \mathcal{B}$. $(E, \mathcal{V}, \mathfrak{J})$ is called a *$\mathcal{B}$-bornological structure* if each $B \in \mathcal{B}$ is bound for $\mathfrak{J}$, and if every $\mathcal{B}$-bornivore subset of E is a neighborhood of zero for $\mathfrak{J}$. Note that if $(E, \mathcal{V}, \mathfrak{J})$ is $\mathcal{B}$-bornological, and if $\mathcal{W}$ is a structure on E weaker than $\mathcal{V}$ (i.e., if $\mathcal{W} \subseteq \mathcal{V}$) but with which $\mathfrak{J}$ is compatible, then $(E, \mathcal{W}, \mathfrak{J})$ is $\mathcal{B}$-bornological. A linear transformation f from E into locally convex space F is *$\mathcal{B}$-bounded* if $f(B)$ is bound in F for all $B \in \mathcal{B}$; f is *$\mathcal{B}$-borno-*

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logical if for every neighborhood W of zero in F, $f^{-1}(W)$ contains a $\mathcal{B}$-bornivore set.

Example 1. The collection of all nonempty convex, equilibrated subsets of vector space E is a structure on E, which we shall call its *linear structure*. Any linear transformation from one vector space into another is a structure homomorphism with respect to the associated linear structures. Any locally convex topology on E is compatible with its linear structure. Let E be a linearly structured locally convex space. (a) If $\mathcal{B}$ is the class of all bound sets, then E is a $\mathcal{B}$-bornological structure if and only if E is a bornological space [2, p. 10; 4, Exercise 12, p. 13]. (b) If $\mathcal{B}$ is the class of all convex, equilibrated, bound, sequentially complete subsets, E is a $\mathcal{B}$-bornological structure if and only if E is an ultrabornological space [4, Exercise 11, p. 34] (a uniform space A is sequentially complete if every Cauchy sequence in A converges to a point of A). (c) If $\mathcal{B} = \emptyset$, E is a $\mathcal{B}$-bornological structure if and only if the topology of E is the finest locally convex topology on E.

Example 2. Let E be an algebra. The collection of all scalar multiples of all nonempty convex equilibrated idempotent subsets is a structure on vector space E, which we shall call the *algebraic structure* on E determined by the given multiplication. Every algebraic homomorphism from E into algebra F is a structure homomorphism for the associated algebraic structures, but not conversely, in general. The topologies on E compatible with its algebraic structure are precisely the locally m-convex topologies on algebra E. Let E be an algebraically structured locally convex space (with respect to some multiplication on E). (a) If $\mathcal{B}$ is the class of all bound idempotent sets, E is a $\mathcal{B}$-bornological structure if and only if E is an i-bornological algebra [13, Proposition 5]. (b) If $\mathcal{B} = \emptyset$, then E is a $\mathcal{B}$-bornological structure if and only if the topology of E is the finest locally m-convex topology on algebra E.

Example 3. Let E be a partially ordered vector space. If $a, b \in E$, $[a, b]$ denotes the set of all $x \in E$ such that $a \leqq x \leqq b$; subset A of E is *order-bound* if there exist $a, b \in E$ such that $A \subseteq [a, b]$. A is *order convex* if for all $a, b \in A$, $[a, b] \subseteq A$, and A is *o-convex* [7, p. 570] if A is convex and order convex. We shall call a subset of E *positive* if it is contained in the positive cone of E. The collection of all nonempty equilibrated o-convex subsets of E is a structure on E, which we shall call the *order structure* determined by the given partial ordering. Every positive linear transformation from E into a partially ordered vector space F is a structure homomorphism for the associated order structures. A topology on E is compatible with its order structure if and only if it is a *locally o-convex topology* [7, p. 570] (locally full and locally convex in the terminology of [10]). Let E be an order structured locally convex space (with respect to some partial ordering on E) in (a)–(c) below. (a) If $\mathcal{B}$ is the class of all bound subsets of E, E is a $\mathcal{B}$-bornological structure if and only if E is *o-bornological* [7, Proposition 6.3]. (b) If $\mathcal{B}$ is the class of all positive bound sets, E is a $\mathcal{B}$-bornological structure if and only if E is

P-o-bornological [7, Proposition 7.1]. (c) If $\mathfrak{B} = \emptyset$, E is a $\mathfrak{B}$-bornological structure if and only if its topology is the finest locally o-convex topology on E. (d) Let E be a partially ordered linearly structured locally convex space (Example 1). If $\mathfrak{B}$ is the class of all order-bound subsets, then E is $\mathfrak{B}$-bornological if and only if its topology is the *order-bound topology* $\mathfrak{I}_b$ [10, p. 20].

The above examples exhaust the bornological concepts hitherto introduced. It is possible to give many others at once. For example, analogous to ultra-bornologicity, one could in Examples 2 and 3 consider a variety of classes of sequentially complete bound sets. If E is a partially ordered algebra (that is, an algebra with a partial ordering whose set of positive elements is a convex idempotent cone), the collection of all scalar multiples of all nonempty o-convex, equilibrated, idempotent sets is a structure on E, and one can consider classes of bound sets for topologies compatible with this structure. A subset of a vector lattice E is *solid* [10, p. 37] if $x \in S$ and $|y| \leqq |x|$ imply $y \in S$. The class of all nonempty convex solid sets is a structure on E, and we shall say a topology is *locally solid* if it is compatible with this structure (our terminology differs slightly from that of [10, p. 40] in that we require a locally solid topology to be locally convex). A locally solid topology is always locally o-convex [10, Theorem 8.1].

PROPOSITION 1. *Let E and F be structured locally convex spaces, f a linear transformation from E into F, $\mathfrak{B}$ a class of bound subsets of E. If f is continuous, f is $\mathfrak{B}$-bornological. If f is $\mathfrak{B}$-bornological, f is $\mathfrak{B}$-bounded. If f is a structure homomorphism, then f is $\mathfrak{B}$-bornological if and only if f is $\mathfrak{B}$-bounded.*

The proof is similar to an argument given in [13, p. 203].

PROPOSITION 2. *Let $(E, \mathfrak{V}, \mathfrak{I})$ be a structured locally convex space, $\mathfrak{B}$ a class of bound subsets of E. The class of all $\mathfrak{B}$-bornivore subsets of E is a fundamental system of neighborhoods of zero for a topology $\mathfrak{I}^\star$ on E compatible with $\mathfrak{V}$, finer than $\mathfrak{I}$, and for which each $B \in \mathfrak{B}$ is bound. $(E, \mathfrak{V}, \mathfrak{I})$ is $\mathfrak{B}$-bornological if and only if $\mathfrak{I} = \mathfrak{I}^\star$. Further, if f is a structure homomorphism from E into any structured locally convex space F, then f is $\mathfrak{B}$-bounded from $(E, \mathfrak{I})$ into F if and only if f is continuous from $(E, \mathfrak{I}^\star)$ into F.*

Proof. The first two assertions are evident, and the third follows from Proposition 1.

PROPOSITION 3. *The topology generated by a family of topologies compatible with a structure $\mathfrak{V}$ on E is again compatible with $\mathfrak{V}$.*

Proof. The assertion follows at once from the fact $\mathfrak{V}$ is a filter base.

PROPOSITION 4. *Let E' be a total subspace of the algebraic dual of E, and let $\mathfrak{V}$ be a structure on E. If there exists a topology on E compatible with $\mathfrak{V}$ and the duality between E and E', there exists a finest such topology; in this* *case the collection of all members of* $\mathfrak{V}$ *which are neighborhoods of zero for the Mackey topology* $\tau(E, E')$ *is a fundamental system of neighborhoods of zero for that finest topology.*

Proof. The collection of all members of $\mathfrak{V}$ which are neighborhoods of zero for the Mackey topology is clearly a fundamental system of neighborhoods of zero for a topology $\mathfrak{T}$ on E compatible with $\mathfrak{V}$. $\mathfrak{T}$ is finer than the given topology, and so its dual contains E'; $\mathfrak{T}$ is weaker than $\tau(E, E')$, and so its dual is contained in E'. Clearly every topology on E compatible with $\mathfrak{V}$ and the duality between E and E' is weaker than $\mathfrak{T}$.

The class of all absorbing members of structure $\mathfrak{V}$ is clearly a fundamental system of neighborhoods of zero for a topology compatible with $\mathfrak{V}$ and finer than all other compatible topologies; it is therefore the finest topology on E compatible with $\mathfrak{V}$.

PROPOSITION 5. *Let* $(E, \mathfrak{V}, \mathfrak{T})$ *be a structured locally convex space,* E' *the dual of* $(E, \mathfrak{T})$, $\mathfrak{B}$ *a class of bound subsets of* E. *The following conditions are equivalent:* (1) E *is a* $\mathfrak{B}$*-bornological structure.* (2) *Every* $\mathfrak{B}$*-bornological linear transformation from* E *into any locally convex space is continuous.* (3) *Every* $\mathfrak{B}$*-bounded structure homomorphism from* E *into any structured locally convex space is continuous.* (4) *If* $\mathfrak{S}$ *is any topology on* E *compatible with* $\mathfrak{V}$ *such that the identity map from* $(E, \mathfrak{T})$ *into* $(E, \mathfrak{S})$ *is* $\mathfrak{B}$*-bounded, then it is continuous.* (5) $\mathfrak{T}$ *is the finest of those topologies on* E *compatible with* $\mathfrak{V}$ *and for which each* $B \in \mathfrak{B}$ *is bound. If in addition* $\mathfrak{T}$ *is separated, the following condition is equivalent to the preceding ones:* (6) $\mathfrak{T}$ *is the finest of those topologies on* E *compatible with* $\mathfrak{V}$ *and the duality between* E *and* E', *and every* $\mathfrak{B}$*-bornological linear form on* E *is continuous.*

Proof. (1) insures that every $\mathfrak{B}$-bornivore set is a neighborhood of zero, and so implies (2). (2) implies (3) by Proposition 1, and clearly (3) implies (4). (4) implies (5), for if $\mathfrak{S}$ is a topology on E compatible with $\mathfrak{V}$ and for which each $B \in \mathfrak{B}$ is bound, the identity map from $(E, \mathfrak{T})$ into $(E, \mathfrak{S})$ is $\mathfrak{B}$-bounded and thus continuous by hypothesis; hence $\mathfrak{T}$ is finer than $\mathfrak{S}$. (5) implies (1), for by Proposition 2, $\mathfrak{T}^\star$ is compatible with $\mathfrak{V}$, and each $B \in \mathfrak{B}$ is bounded for $\mathfrak{T}^\star$; hence as $\mathfrak{T} \subseteq \mathfrak{T}^\star$, by hypothesis $\mathfrak{T} = \mathfrak{T}^\star$, and so E is $\mathfrak{B}$-bornological. Henceforth, assume $\mathfrak{T}$ is separated. (2) and (5) imply (6): If $\mathfrak{S}$ is a topology on E compatible with $\mathfrak{V}$ and with the duality between E and E', every $B \in \mathfrak{B}$ is bound for $\mathfrak{S}$ by a theorem of Mackey [4, Theorem 3, p. 70], and so by (5) $\mathfrak{T} \supseteq \mathfrak{S}$; on the other hand (2) implies every $\mathfrak{B}$-bornological linear form on E is continuous. (6) implies (1): $\mathfrak{T}^\star$ is compatible with $\mathfrak{V}$ and is finer than $\mathfrak{T}$. By the second part of (6) and Proposition 1, $\mathfrak{T}^\star$ is compatible with the duality between E and E'. By the first part of (6), $\mathfrak{T} = \mathfrak{T}^\star$, and therefore $(E, \mathfrak{V}, \mathfrak{T})$ is $\mathfrak{B}$-bornological by Proposition 2.

PROPOSITION 6. *Let* E *be a locally m-convex algebra* [*locally o-convex partially ordered space*], $\mathfrak{B}$ *a class of bound subsets of* E. *Then* E, *with its asso-*

ciated algebraic [order] structure, is a $\mathcal{B}$-bornological structure if and only if every $\mathcal{B}$-bounded algebraic homomorphism [positive linear transformation] from E into any locally m-convex algebra [locally o-convex space] is continuous.

Proof. As every algebraic homomorphism [positive linear transformation] is a structure homomorphism for the associated algebraic [order] structures, the condition implies (4) and is implied by (3) of Proposition 5.

Let $(E_\alpha)_{\alpha \in A}$ be a family of structured locally convex spaces, $(E, \mathcal{V})$ a structured space, and for each $\alpha \in A$ let g_α be a structure homomorphism from E_α into E. The finest topology on E compatible with $\mathcal{V}$ for which each g_α is continuous is called the *structure inductive limit topology* defined by the structured locally convex spaces (E_α) and structure homomorphisms (g_α).

If each structure considered in the preceding definition is the linear [respectively, algebraic, order] structure, the structure inductive limit topology is simply the (linear) inductive limit topology [3, pp. 60–62] [respectively, the algebraic inductive limit topology [13, p. 193], the o-inductive limit topology [7, p. 573]].

PROPOSITION 7. *Let $\mathfrak{I}$ be the inductive limit topology on structured space $(E, \mathcal{V})$ with respect to structured locally convex spaces (E_α) and structure homomorphisms (g_α). Then $V \in \mathcal{V}$ is a neighborhood of zero for $\mathfrak{I}$ if and only if V is absorbing and $g_\alpha^{-1}(V)$ is a neighborhood of zero in E_α for all $\alpha \in A$. If f is any structure homomorphism from E into any structured locally convex space, then f is continuous if and only if $f \circ g_\alpha$ is continuous for all $\alpha \in A$.*

The proof is similar to that of [3, Proposition 1 and Corollary, p. 60].

PROPOSITION 8. *Let E be the structure inductive limit of structured locally convex spaces (E_α) with respect to structure homomorphisms (g_α). If E_α is $\mathcal{B}_\alpha$-bornological for all α and if $\mathcal{B}$ is a family of bound subsets of E containing $\bigcup_\alpha g_\alpha(\mathcal{B}_\alpha)$, then E is $\mathcal{B}$-bornological.*

The proof is similar to that of [13, Proposition 6]. The proposition clearly yields as a special case the fact that the linear inductive limit [algebraic inductive limit, o-inductive limit] of bornological spaces [i-bornological algebras, o-bornological or P-o-bornological spaces] is again bornological [i-bornological, o-bornological or P-o-bornological].

2. Cartesian products of structured spaces

Let $((E_\alpha, \mathcal{V}_\alpha, \mathfrak{I}_\alpha))_{\alpha \in A}$ be a family of structured locally convex spaces, E_α' the dual of $(E_\alpha, \mathfrak{I}_\alpha)$ for all $\alpha \in A$. Let $E = \prod_\alpha E_\alpha$, let $\mathfrak{I}$ be the Cartesian product topology of E, and let E' be the dual of $(E, \mathfrak{I})$. For each $\alpha \in A$ let i_α be the canonical injection mapping from E_α into E. Clearly $[V \subseteq E\colon V$ is convex and equilibrated, and $i_\alpha^{-1}(V) \in \mathcal{V}_\alpha$ for all $\alpha]$ is a structure on E, which we shall call the *Cartesian product structure* on E determined by the structured spaces $((E_\alpha, \mathcal{V}_\alpha))_{\alpha \in A}$.

The linear structure on E, for example, is the Cartesian product structure of the linear structures on the E_α. If each $\mathcal{V}_\alpha$ is the algebraic [order] structure on E_α defined by a given multiplication [partial ordering] on E_α, these multiplications [partial orderings] induce a multiplication [partial ordering] on E whose associated algebraic [order] structure is weaker than the Cartesian product structure determined by the $\mathcal{V}_\alpha$.

PROPOSITION 9. *For all $\alpha \in A$, i_α is a structure homomorphism from E_α into E. The Cartesian product topology $\mathfrak{I}$ on E is compatible with its Cartesian product structure $\mathcal{V}$.*

Proof. By definition of $\mathcal{V}$, $\overset{-1}{i_\alpha}(\mathcal{V}) \subseteq \mathcal{V}_\alpha$ for all α. For any $V \in \mathcal{V}_\alpha$, $\overset{-1}{i_\beta}(i_\alpha(V)) = \{0\} \in \mathcal{V}_\beta$ if $\beta \neq \alpha$, and $\overset{-1}{i_\alpha}(i_\alpha(V)) = V \in \mathcal{V}_\alpha$; hence $i_\alpha(V) \in \mathcal{V}$. Thus $i_\alpha(\mathcal{V}_\alpha) \subseteq \mathcal{V}$, so i_α is a structure homomorphism. If $V = \prod_\alpha V_\alpha$ is a neighborhood of zero for $\mathfrak{I}$ where each $V_\alpha \in \mathcal{V}_\alpha$, then $\overset{-1}{i_\alpha}(V) = V_\alpha \in \mathcal{V}_\alpha$ for all α, so $V \in \mathcal{V}$. Hence $\mathfrak{I}$ is compatible with $\mathcal{V}$.

Using the fact that if $\mathfrak{I}_\alpha$ is the Mackey topology $\tau(E_\alpha, E_\alpha')$ for all α then $\mathfrak{I}$ is the Mackey topology $\tau(E, E')$ [5, §2, Corollary of Theorem 2; 4, Exercise 6a), p. 80], we shall extend it to Cartesian products of structured locally convex spaces.

PROPOSITION 10. *If for all $\alpha \in A$, $\mathfrak{I}_\alpha$ is separated and is the finest topology on E_α compatible with $\mathcal{V}_\alpha$ and the duality between E_α and E_α', then $\mathfrak{I}$ is the finest topology on E compatible with $\mathcal{V}$ and the duality between E and E'.*

Proof. It suffices by Proposition 4 to show that if W is a neighborhood of zero for $\tau(E, E')$, and if $W \in \mathcal{V}$, then W is a neighborhood of zero for $\mathfrak{I}$. Since $\tau(E, E')$ is the Cartesian product topology on E determined by the topologies $\tau(E_\alpha, E_\alpha')$, there exists a finite subset B of A such that for all $(x_\alpha) \in E$, if $x_\alpha = 0$ for all $\alpha \in B$, then $(x_\alpha) \in W$. Let n be the number of elements in B. Let $V_\alpha = \overset{-1}{i_\alpha}((n + 1)^{-1}W)$ for all $\alpha \in B$, let $V_\alpha = E_\alpha$ for all $\alpha \in A - B$, and let $V = \prod_\alpha V_\alpha$. For each $\alpha \in A$, $V_\alpha \in \mathcal{V}_\alpha$, and V_α is a neighborhood of zero for $\tau(E_\alpha, E_\alpha')$, whence V_α is a neighborhood of zero for $\mathfrak{I}_\alpha$. Thus V is a neighborhood of zero for $\mathfrak{I}$. It remains to show $V \subseteq W$: Let $(x_\alpha) \in V$. Let $y_\alpha = 0$ if $\alpha \in B$, $y_\alpha = x_\alpha$ if $\alpha \in A - B$. Then $(x_\alpha) = (y_\alpha) + \sum_{\beta \in B} i_\beta(x_\beta)$. Since $((n + 1)y_\alpha) \in W$, we have $(y_\alpha) \in (n + 1)^{-1}W$. For $\beta \in B$, $i_\beta(x_\beta) \in (n + 1)^{-1}W$. Hence as W is convex, and as (x_α) is the sum of $n + 1$ members of $(n + 1)^{-1}W$, $(x_\alpha) \in W$. Thus $V \subseteq W$, and the proof is complete.

COROLLARY. *Under the hypotheses of Proposition 10, if $\mathcal{W}$ is any structure on E weaker than the Cartesian product structure $\mathcal{V}$ but with which the Cartesian product topology is compatible, then $\mathfrak{I}$ is the finest topology on E compatible with $\mathcal{W}$ and with the duality between E and E'.*

The proof follows at once from the observation that if $\mathcal{V}$ and $\mathcal{W}$ are two structures on a vector space such that $\mathcal{W} \subseteq \mathcal{V}$, any topology compatible with $\mathcal{W}$ is also compatible with $\mathcal{V}$. The corollary is applicable of course, to the case where the structures considered are all algebraic structures or are all order structures.

PROPOSITION 11. *Let A be finite. For all $\alpha \in A$ let $\mathcal{B}_\alpha$ be a class of bound subsets of E_α, and let $\mathcal{B}$ be a class of bound subsets of E containing $\bigcup_\alpha i_\alpha(\mathcal{B}_\alpha)$. If E_α is $\mathcal{B}_\alpha$-bornological for all $\alpha \in A$, then E is $\mathcal{B}$-bornological.*

Proof. Let f be a $\mathcal{B}$-bounded structure homomorphism from E into a structured locally convex space F. As $\mathcal{B} \supseteq i_\alpha(\mathcal{B}_\alpha)$, $f_\alpha = f \circ i_\alpha$ is a $\mathcal{B}_\alpha$-bounded structure homomorphism from E_α into F and hence is continuous. But then as $f((x_\alpha)) = \sum_{\alpha \in A} f_\alpha(x_\alpha)$, f is clearly continuous. Hence by Proposition 5, E is $\mathcal{B}$-bornological.

Henceforth, therefore, we may assume A is infinite. Each $x = (x_\alpha) \in E$ defines a continuous linear mapping $x^\wedge$: $(\lambda_\alpha) \to (\lambda_\alpha x_\alpha)$ from $\mathbf{R}^A$, the Cartesian product of a family of real lines indexed by A, into E; we shall show that the question of when E inherits bornological properties from the E_α can be reduced to a set-theoretical question about A and a question concerning the maps $x^\wedge$. Recall that the collection of all convex equilibrated sets absorbing every order-bound set is a fundamental system of neighborhoods of zero for the order-bound topology on a partially ordered vector space (Example 3(d)). A theorem of Mackey [8] implies that the Cartesian product topology of $\mathbf{R}^A$ is the order-bound topology of $\mathbf{R}^A$ (regarded as a lattice) if and only if A admits no Ulam measure: By (6) of Proposition 5, Proposition 10, and Proposition 1, the Cartesian product topology of $\mathbf{R}^A$ is the order-bound topology if and only if every linear form on $\mathbf{R}^A$ which is bounded on the order-bound subsets of $\mathbf{R}^A$ is continuous, or equivalently [4, Proposition 10, p. 75] is in the linear subspace of the algebraic dual of $\mathbf{R}^A$ spanned by the projection mappings; Mackey's theorem asserts that the latter condition is equivalent to the assertion that A admits no Ulam measure. (This also follows from the more general result of Nachbin (contained in the proof of [9, Theorem 2]) that the partially ordered space $\mathcal{C}(A)$ of all continuous real-valued functions on a completely regular space A, equipped with the topology of compact convergence, has the order-bound topology if and only if A is a Q-space (this theorem is due also to Shirota [12]). Applying this result to the case where A is discrete and therefore $\mathcal{C}(A) = \mathbf{R}^A$, we see that the Cartesian product topology of $\mathbf{R}^A$ is the order-bound topology if and only if discrete space A is a Q-space, which in turn is equivalent to the assertion that there exist no Ulam measures on A (see the discussion in [13, pp. 206–208]).)

PROPOSITION 12. *Let A be a set admitting no Ulam measure. For each $\alpha \in A$ let $\mathcal{B}_\alpha$ be a family of bound subsets of E_α, and let $\mathcal{B}$ be a family of bound subsets of E containing $\bigcup_\alpha i_\alpha(\mathcal{B}_\alpha)$. If E_α is a separated $\mathcal{B}_\alpha$-bornological structure for all $\alpha \in A$, and if for all $x \in E$ and all positive $v \in \mathbf{R}^A$ every $\mathcal{B}$-bornivore subset of E absorbs $x^\wedge([-v, v])$ (equivalently, the inverse image under $x^\wedge$ of every*

$\mathfrak{B}$-bornivore subset of E absorbs every set of the form $[-v, v]$), then $E = \prod_\alpha E_\alpha$ is $\mathfrak{B}$-bornological.

Proof. By (6) of Proposition 5 and Proposition 10 it suffices to show every $\mathfrak{B}$-bornological linear form on E is continuous. Slight modifications in the argument of Lemmas 2–4 of [13] furnish the desired proof. (Use is made of the fact that the Cartesian product topology of $\mathbf{R}^A$ is the order-bound topology in modifying Lemma 2 and Lemma 4, and use is made of Proposition 11 in modifying Lemma 3.)

Let us apply Proposition 12 to the examples of §1; for each example we shall assume that both $\mathfrak{B}$ and the collections $\mathfrak{B}_\alpha$ are of the type described. Then in all examples, $\bigcup_\alpha i_\alpha(\mathfrak{B}_\alpha) \subseteq \mathfrak{B}$. For Example 2 [Example 3], if $\mathcal{W}$ is the algebraic [order] structure of E, to show $(E, \mathcal{W}, \mathfrak{J})$ is $\mathfrak{B}$-bornological, it suffices to show $(E, \mathcal{V}, \mathfrak{J})$ is $\mathfrak{B}$-bornological by remarks following the definition of $\mathfrak{B}$-bornologicity and the definition of the Cartesian product structure. Now let $x \in E$. For any $v \geqq 0$ in $\mathbf{R}^A$, $[-v, v]$ is compact, convex, and equilibrated, so $x^\wedge([-v, v])$ has the same properties; hence in Examples 1(a), 1(b), and 3(a), the condition concerning $x^\wedge$ is satisfied. As shown in [13, Lemma 1], $x^\wedge$ has the desired property in Example 2(a) if each E_α has an identity. Next, suppose each E_α is a partially ordered space whose cone P_α of positive elements generates E_α. Then the positive cone of E is generating, so there exist $y \geqq 0$, $z \geqq 0$ in E such that $x = y - z$. For any $v \geqq 0$ in $\mathbf{R}^A$, $[-v, v] \subseteq [0, v] + (-[0, v])$, and $y^\wedge([0, v])$ and $z^\wedge([0, v])$ are positive order-bound subsets of E; hence as

$$x^\wedge([-v, v]) \subseteq y^\wedge([0, v]) + (-z^\wedge([0, v])) + (-y^\wedge([0, v])) + z^\wedge([0, v]),$$

$x^\wedge$ has the desired properties in Examples 3(b), 3(c), and 3(d). In summary, we have

PROPOSITION 13. *Let A be a set admitting no Ulam measure. For each of the seven following types, if each member of a family of separated spaces indexed by A is of that type, the Cartesian product of that family is also.* (1) *Bornological space.* (2) *Ultrabornological space.* (3) *i-bornological algebra with identity.* (4) *o-bornological partially ordered space.* (5) *P-o-bornological partially ordered space whose positive cone is generating.* (6) *Partially ordered locally o-convex space whose positive cone is generating and whose topology is the finest locally o-convex topology.* (7) *Partially ordered locally convex space whose positive cone is generating and whose topology is the order-bound topology.*

(1) is a theorem of Donoghue and Smith [5, §2, Theorem 7 and Corollary of Theorem 2], (2) generalizes [4, Exercise 11 e), p. 35], and (3) is [13, Theorem 6]. The hypothesis concerning the positive cone in (5), (6), and (7) cannot be removed without other restrictions. For if we choose equality as the partial ordering of $\mathbf{R}$, the positive cone of $\mathbf{R}^A$ is $\{0\}$, and thus $\mathbf{R}^A$ is *P-o*-bornological or has the finest locally *o*-convex topology or has the order-

bound topology if and only if $\mathbf{R}^A$ has the finest locally convex topology. But if A is infinite, the algebraic dual of $\mathbf{R}^A$ strictly contains the topological dual of $\mathbf{R}^A$ for the Cartesian product topology, and hence that topology is not the finest locally convex topology.

3. Partially ordered locally convex spaces

A partially ordered vector space E has the *decomposition property* [10, p. 27] if for all $x, y \geqq 0$, $[0, x] + [0, y] = [0, x + y]$. Every vector lattice has the decomposition property [10, Lemma 7.2], though not every partially ordered space with the decomposition property is a lattice. For our next results, we need the following algebraic lemma:

LEMMA. *Let E be a partially ordered vector space having the decomposition property, let P be its positive cone, and let $E_0 = P + (-P)$. For any convex equilibrated subset W of E, there exists an equilibrated o-convex subset U of E such that $U \subseteq 3W \cap E_0$ and $U \cap P = [x \geqq 0: [0, x] \subseteq W]$. If in addition E is a lattice, then for each $u \in U$, $u^+ \in W$.*

Proof. Let $V = [x \geqq 0: [0, x] \subseteq W] \subseteq E_0$. If $x, y \in V$ and if $0 \leqq \alpha \leqq 1$, by the decomposition property

$$[0, \alpha x + (1 - \alpha)y] = [0, \alpha x] + [0, (1 - \alpha)y]$$
$$= \alpha[0, x] + (1 - \alpha) [0, y] \subseteq \alpha W + (1 - \alpha)W = W,$$

so $\alpha x + (1 - \alpha)y \in V$. Thus V is convex. Clearly $V \cup (-V)$ is equilibrated, so the convex envelope Z of $V \cup (-V)$ is also equilibrated and contained in E_0. As V is convex, $Z = [\lambda x - (1 - \lambda)y: x, y \in V \text{ and } 0 \leqq \lambda \leqq 1]$ by [3, Proposition 8, p. 45]. Clearly $Z \cap P \supseteq V$; on the other hand, if $z \in Z$ and if $z \geqq 0$, there exist $x, y \in V$ and $\lambda \in [0, 1]$ such that $0 \leqq z = \lambda x - (1 - \lambda)y \leqq \lambda x \leqq x$, so $[0, z] \subseteq [0, x] \subseteq W$, and therefore $z \in V$. Hence $Z \cap P = V$. Let U be the order-convex envelope of Z, that is, the union of all sets of the form $[v, w]$ where $v, w \in Z$ and $v \leqq w$. As Z is convex and equilibrated, U is o-convex and equilibrated. Suppose $u \in U$. Then there exist $x, y, z, w \in V$ and $\alpha, \beta \in [0, 1]$ such that $\alpha x - (1 - \alpha)y \leqq u \leqq \beta z - (1 - \beta)w$. Let $v = u - \alpha x + (1 - \alpha)y$. Then

$$0 \leqq v \leqq \beta z + (1 - \alpha)y - (1 - \beta)w - \alpha x$$
$$\leqq \beta z + (1 - \alpha)y \leqq z + y \in V + V = 2V.$$

Therefore $v \in 2W \cap E_0$. Since $x, y \in V \subseteq W \cap E_0$, we have

$$u = v + \alpha x - (1 - \alpha)y \in [2W + \alpha W + (1 - \alpha)W] \cap E_0 = 3W \cap E_0 .$$

Thus $U \subseteq 3W \cap E_0$. Clearly $U \cap P \supseteq Z \cap P = V$. Suppose $u \in U$ and $u \geqq 0$. Then there exists $b \in Z$ such that $0 \leqq u \leqq b$, whence $b \in Z \cap P = V$, and thus $[0, u] \subseteq [0, b] \subseteq W$, that is, $u \in V$. Therefore $U \cap P = V$. Suppose further that E is a lattice, and let $u \in U$. Then there exist $x, y \in V$ and

$\lambda \in [0, 1]$ such that $u \leqq \lambda x - (1 - \lambda)y \leqq \lambda x \leqq x$. Consequently as $x \geqq 0$, $0 \leqq u^+ \leqq x$, so by the definition of V, $u^+ \in W$.

Let us call a partially ordered locally convex space *p-bornological* if every convex, equilibrated, absorbing set absorbing all positive bound subsets of E is a neighborhood of zero. (E is thus p-bornological if and only if the associated linearly structured locally convex space is a $\mathfrak{B}$-bornological structure, where $\mathfrak{B}$ is the class of all positive bound sets. As for Examples 3(b), 3(c), and 3(d), Proposition 12 implies that if $(E_\alpha)_{\alpha \in A}$ is a family of p-bornological separated spaces each with a generating positive cone, and if A admits no Ulam measure, then $\prod_\alpha E_\alpha$ is a p-bornological space with generating positive cone.)

If $\mathfrak{J}$ is a locally solid topology on lattice E, and if B is a bound subset for $\mathfrak{J}$, then $B^+ = [x^+ : x \in B]$ is clearly bound as zero has a fundamental system of solid neighborhoods; hence $B \subseteq B^+ - (-B)^+$ and so is contained in the difference of two positive bound sets. Consequently, $\mathfrak{J}$ is a bornological topology if and only if it is a p-bornological topology, and $\mathfrak{J}$ is an o-bornological topology if and only if it is a P-o-bornological topology.

In general, a p-bornological space is clearly bornological, and the order-bound topology $\mathfrak{J}_b$ on any partially ordered vector space converts it into a p-bornological space. Since $\mathfrak{J}_b$ is not always locally o-convex [10, p. 21], a p-bornological space need not be locally o-convex. However, it is clear that a p-bornological locally o-convex space is P-o-bornological, and the converse holds in the presence of the decomposition property:

PROPOSITION 14. *Let E be a partially ordered locally convex space with the decomposition property. The following are equivalent*: (1) *E is P-o-bornological*; (2) *E is p-bornological and locally o-convex. If in addition E is a lattice, then the following condition is equivalent to the preceding ones*: (3) *E is bornological and locally solid.*

Proof. As remarked above, (2) implies (1), and if E is a lattice, (3) implies (2). We shall show (1) implies (2) and also their conjunction implies (3) if E is a lattice. Let W be any convex equilibrated absorbing set absorbing all positive bound subsets of E, and let $E_0 = P + (-P)$ where P is the positive cone of E. By the lemma, there exists an o-convex equilibrated set U such that $U \subseteq 3W \cap E_0$ and $U \cap P = [x \geqq 0 : [0, x] \subseteq W]$. U absorbs all positive bound sets: If not, there exists a positive bound set B such that $B \not\subseteq n^2 U$ for all $n \geqq 1$. Let $x_n \in B$ be such that $x_n \notin n^2 U$. Then

$$n^{-2} x_n \notin U \supseteq U \cap P,$$

so there exists y_n such that $0 \leqq y_n \leqq n^{-2} x_n$ and $y_n \notin W$. Therefore

$$0 \leqq n y_n \leqq n^{-1} x_n ,$$

and as $\{x_n\}_{n \geqq 1} \subseteq B$, $n^{-1} x_n \to 0$. Hence as the topology of E is locally o-convex, $n y_n \to 0$, so $\{n y_n\}_{n \geqq 1}$ is a positive bound set. But as $n y_n \notin nW$, W does not absorb all positive bound sets, a contradiction. Hence U absorbs all positive bound sets, and in particular, therefore, all points of E_0. Let $V = U + (W \cap E_1) \subseteq 4W$ where E_1 is an algebraic supplement of E_0 in E; then V is absorbing and also convex and equilibrated. V is o-convex: Suppose $u_1 + w_1 \leqq u_2 + w_2$ where u_1, $u_2 \in U$ and w_1, $w_2 \in W \cap E_1$. If

$$z = (w_2 - w_1) + (u_2 - u_1),$$

then $z \geqq 0$, so $z \in E_0$, and hence $w_2 - w_1 = z - (u_2 - u_1) \in E_0$. Therefore $w_2 - w_1 = 0$, and so

$$[u_1 + w_1, u_2 + w_2] = w_1 + [u_1, u_2] \subseteq (W \cap E_1) + U = V.$$

By hypothesis, therefore, V is a neighborhood of zero. Thus as W contains $4^{-1}V$, W itself is a neighborhood of zero, and we have shown (1) implies (2). Now suppose E is a lattice satisfying (1) and hence (2), and let W be a convex equilibrated neighborhood of zero. Then $E_0 = E$ and $V = U$, a neighborhood of zero by the preceding. By the lemma, $u \in U$ implies $u^+ \in W$, so by (ii) of [10, Theorem 8.1], E is locally solid. Consequently as E is p-bornological, E is bornological, and thus (3) holds.

The author is indebted to H. Schaefer for pointing out that a hypothesis in the author's original version of Propositions 14 and 15 was unnecessary.

PROPOSITION 15. *Let E be a partially ordered vector space having the decomposition property. Then the order-bound topology $\mathfrak{J}_b$ of E is locally o-convex and hence is the finest locally o-convex topology on E. If E is a lattice, $\mathfrak{J}_b$ is locally solid and hence is the finest locally solid topology on E.*

Proof. Again, let $E_0 = P + (-P)$ where P is the positive cone of E. Let W be any convex equilibrated set absorbing all order-bound subsets of E. By the lemma, there exists an o-convex equilibrated set U such that $U \subseteq 3W \cap E_0$ and $U \cap P = [x \geqq 0 : [0, x] \subseteq W]$. For any $x \geqq 0$, U absorbs x: If not, there exists $x \in P$ such that $x \notin nU$, and therefore $n^{-1}x \notin U \supseteq U \cap P$ for all $n \geqq 1$. Hence there exists y_n such that $0 \leqq y_n \leqq n^{-1}x$ and $y_n \notin W$. Then $ny_n \in [0, x]$ for all $n \geqq 1$, but $ny_n \notin nW$, so W does not absorb $[0, x]$, a contradiction. Consequently U absorbs all points of E_0. Let

$$V = U + (W \cap E_1) \subseteq 4W,$$

where E_1 is an algebraic supplement to E_0. As in the proof of Proposition 14, V is an absorbing o-convex equilibrated set and therefore absorbs all order-bound subsets of E. Thus $4^{-1}V$ is an o-convex neighborhood of zero for $\mathfrak{J}_b$ contained in W; hence $\mathfrak{J}_b$ is locally o-convex. But then if E is a lattice, $\mathfrak{J}_b$ is also locally solid by Proposition 14.

The first assertion of Proposition 15 was essentially proved by a different method by Schaefer [14, Theorem (4.9)]; his argument may be modified to eliminate the hypothesis made in his theorem that E admit a separated locally o-convex topology. The second assertion of Proposition 15 was proved by Namioka [10, Theorem 8.5].

In view of Proposition 14 it is natural to ask if a bornological, or more specifically a normed locally o-convex lattice E is necessarily locally solid. The answer is no, as the following example shows.

Let L be the real Hilbert space of all square-summable sequences indexed by the positive integers, and let E be the subspace of all sequences $\alpha = (\alpha_k)_{k \geqq 1}$ in L such that $\alpha_k = 0$ for all but a finite number of indices k, equipped with the induced norm $\| \alpha \| = (\sum \alpha_k^2)^{1/2}$. Let P be the set of all sequences $(\alpha_k) \in E$ such that $\alpha_k \geqq k\alpha_{k+1} \geqq 0$ for all $k \geqq 1$. It is easy to verify that P is an antisymmetric cone and is further closed in prehilbert space E since the projections $(\alpha_k)_{k \geqq 1} \to \alpha_m$ are all continuous. E with the associated partial ordering is a lattice: It suffices to show that for any $\alpha = (\alpha_k)$ in E, α^+ exists [10, Lemma 1.2]. Let p be the largest of those indices j such that $\alpha_j \neq 0$; let $\beta_k = 0$ if $k > p$, and by induction let

$$\beta_{p-r} = \max\{(p - r)\beta_{p-r+1}, \alpha_{p-r} + (p - r)(\beta_{p-r+1} - \alpha_{p-r+1})\}, 0 \leqq r < p.$$

One may then verify that $(\beta_k) = \alpha^+$, so E is a lattice. If

$$0 \leqq \alpha = (\alpha_k) \leqq \beta = (\beta_k)$$

where β is in the closed ball of radius ε, then $0 \leqq \alpha_k \leqq \beta_k$ for all $k \geqq 1$ and hence $\| \alpha \| \leqq \| \beta \| \leqq \varepsilon$; consequently [10, Theorem 4.8], E is locally o-convex. But E is not locally solid: Suppose W is a solid neighborhood of zero contained in the closed ball of radius 1. Then for some $\varepsilon > 0$, W contains the closed ball of radius ε. Let $p > 1$ be such that $(p - 1)!\, \varepsilon > 1$, and let $\alpha = (\alpha_k)$ where $\alpha_p = \varepsilon$ and $\alpha_k = 0$ for $k \neq p$. Then $\| \alpha \| = \varepsilon$, so $\alpha \in W$. But if $\alpha^+ = (\beta_k)$, by the preceding $\beta_{p-1} \geqq (p - 1)\varepsilon$, and by induction $\beta_{p-r} \geqq (p - r)(p - r + 1) \cdots (p - 1)\varepsilon$ for $0 \leqq r < p$. In particular, $\beta_1 \geqq (p - 1)!\, \varepsilon > 1$, so $\| \alpha^+ \| > 1$, and thus $\alpha^+ \notin W$, a contradiction.

By Proposition 14, therefore, the preceding is an example of a bornological (indeed, prehilbert) locally o-convex lattice which is neither p-bornological nor P-o-bornological. In contrast, any complete metrizable locally o-convex lattice is locally solid [10, Theorem 8.2] and thus both p-bornological and P-o-bornological. Note also that the closure P^- of P in L is an antisymmetric cone but does not convert L into a lattice. Indeed, $\alpha = (k^{-1})_{k \geqq 1}$ is not majorized by any element of P^-, for if $\beta = (\beta_k) \in P^-$ and if $\beta \geqq \alpha$, then $\beta_1 \geqq (p - 1)!\, p^{-1}$ for all $p \geqq 1$.

Let us call a partially ordered uniform space A *monotonically sequentially complete* if every increasing Cauchy sequence in A converges to a point of A. Modifying a proof of Goffman [6, Theorem 2], Kist [7, Proposition 7.2] has shown that if E is a separated sequentially complete P-o-bornological space whose positive cone is closed, the topology of E is the finest locally o-convex topology on E. A trivial modification of Kist's argument yields the following more general result, but for completeness we shall give the entire proof.

PROPOSITION 16. *Let E be a separated partially ordered locally convex space, V a convex equilibrated set absorbing all order-bound subsets of E. Then V also* *absorbs all convex, positive, monotonically sequentially complete, bound subsets B of E containing zero. If further E is monotonically sequentially complete, and if the positive cone of E is closed, then V absorbs all positive bound subsets of E.*

Proof. If not, there exists a sequence $(x_n)_{n\geqq 1}$ in B such that $x_n \notin 2^{2n}V$; thus if $y_n = \sum_{1\leqq k\leqq n} 2^{-k}x_k$, $(y_n)_{n\geqq 1}$ is an increasing Cauchy sequence contained in B and hence converges to a point $y \in B$. But also if $m \geqq n$, $y_m - y_n \in B$, and thus $(y_m - y_n)_{m\geqq n}$ is an increasing Cauchy sequence contained in B; hence $y - y_n = \lim_{m\to\infty}(y_m - y_n) \in B$ and therefore $y \geqq y_n$. By hypothesis there exists p such that $[0, y] \subseteq 2^pV$. Then as

$$0 \leqq 2^{-p}x_p \leqq y_p \leqq y, \qquad 2^{-p}x_p \in 2^pV,$$

so $x_p \in 2^{2p}V$, a contradiction. Suppose now that the positive cone of E is closed and that E is monotonically sequentially complete. The former condition insures that the closure of every positive bound set is a positive bound set, and the latter insures that every closed positive bound set is monotonically sequentially complete. Hence by the preceding, V absorbs all positive bound subsets of E.

PROPOSITION 17. *Let $\mathcal{S}$ be a separated, monotonically sequentially complete, locally convex topology on partially ordered space E for which the positive cone is closed. Then for any locally convex topology $\mathfrak{I}$ on E finer than $\mathcal{S}$, the following are equivalent:* (1) *$(E, \mathfrak{I})$ is p-bornological, and every order-bound subset of E is bound for the topology $\mathfrak{I}$.* (2) *The topology $\mathfrak{I}$ is the order-bound topology $\mathfrak{I}_b$.*

Proof. Clearly (2) implies (1). (1) implies (2): The second part of (1) implies $\mathfrak{I} \subseteq \mathfrak{I}_b$. Let V be any convex equilibrated set absorbing all order-bound subsets of E. By Proposition 16, V absorbs every positive bound subset of $(E, \mathcal{S})$. As $\mathcal{S} \subseteq \mathfrak{I}$, V therefore absorbs every positive bound subset of $(E, \mathfrak{I})$ and hence by hypothesis is a neighborhood of zero. Thus $\mathfrak{I} \supseteq \mathfrak{I}_b$.

PROPOSITION 18. *Let E be a partially ordered vector space having the decomposition property. Let $\mathfrak{I}$ be a locally convex topology on E finer than some separated, locally convex, monotonically sequentially complete topology for which the positive cone is closed. Then the following are equivalent:* (1) *$\mathfrak{I}$ is the finest locally o-convex topology on E.* (2) *$\mathfrak{I}$ is the order-bound topology of E.* (3) *$(E, \mathfrak{I})$ is p-bornological, and every order-bound subset of E is bound for topology $\mathfrak{I}$.* (4) *$(E, \mathfrak{I})$ is P-o-bornological. If in addition E is a lattice, the following condition is equivalent to the preceding four:* (5) *$\mathfrak{I}$ is the finest locally solid topology on E.*

Proof. (1) and (2) are equivalent, and if E is a lattice, (1) and (5) are equivalent by Proposition 15. (1) clearly implies (4), (4) implies (3) by Proposition 14, and (3) implies (2) by Proposition 17.

Corollary 1. *Let E be a vector lattice, $\mathfrak{J}$ a locally solid topology on E finer than some separated, locally convex monotonically sequentially complete topology for which the positive cone is closed. The following are equivalent:* (1) *$\mathfrak{J}$ is the finest locally o-convex topology on E.* (2) *$\mathfrak{J}$ is the order-bound topology of E.* (3) *$(E, \mathfrak{J})$ is bornological.* (4) *$(E, \mathfrak{J})$ is o-bornological.* (5) *$\mathfrak{J}$ is the finest locally solid topology on E.*

Corollary 1 is essentially [7, Proposition 8.1] and generalizes the theorem [1, Theorem 10, p. 248; 6, Theorem 2] that the topology of a Banach lattice is the order-bound topology and hence the finest locally solid topology, for the topology of a Banach space is bornological and complete.

It is easy to see that the topology of a bornological locally solid lattice need not be the order-bound topology in general. Indeed, let E be the vector space of the example following Proposition 15, equipped with the lattice ordering defined by the cone $P = [(\alpha_k) : \alpha_k \geqq 0 \text{ for all } k \geqq 1]$ and the locally solid topology defined by the norm $\| (\alpha_k) \| = \max \{ | \alpha_k | \}_{k \geqq 1}$. If

$$V = [(\alpha_k) : | \alpha_k | \leqq k^{-1} \text{ for all } k \geqq 1],$$

V is an absorbing, convex, solid subset of E which is not a neighborhood of zero. Consequently the topology, though locally solid and bornological, is not the finest locally solid topology on E.

We also obtain from Proposition 18 the following result [11, Theorem 3]:

Corollary 2. *Let $(E, \mathfrak{J})$ be a separated, locally solid, monotonically sequentially complete lattice. Then every linear form on E which is bounded on all order-bound subsets of E is bounded for topology $\mathfrak{J}$.*

Proof. As $\mathfrak{J}$ is locally solid, the positive cone of E is closed for $\mathfrak{J}$. A subset of E is bound for $\mathfrak{J}$ if and only if it is bound for the bornological topology $\mathfrak{J}^\star$ associated to $\mathfrak{J}$; consequently, every bound subset of $(E, \mathfrak{J}^\star)$ is contained in the difference of two positive bound subsets of $(E, \mathfrak{J}^\star)$, so $\mathfrak{J}^\star$ is a p-bornological topology on E. By Proposition 18, therefore, as every order-bound set is bound for $\mathfrak{J}$ and hence for $\mathfrak{J}^\star$, $\mathfrak{J}^\star$ is the order-bound topology. Thus if f is a linear form bounded on all order-bound sets, f is continuous for $\mathfrak{J}^\star$ and hence bounded for $\mathfrak{J}$.

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