STRUCTURE OF CLEFT RINGS I

BY

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I. INTRODUCTION AND PRELIMINARIES

1A. Introduction

Let R be a ring with radical N and identity element 1. Throughout this paper, we assume that R satisfies the minimum condition on left ideals. Furthermore, we assume that R is a *cleft ring*; that is, as an additive group $R = S \oplus N$ where S is a semisimple ring isomorphic to R/N. This decomposition is called a *cleaving*.

Any cleft ring may be considered to be a direct sum of algebras (Proposition 1.1). We assume, therefore, that R is actually an algebra over a field K. We associate with cleaving $R = S \oplus N$ a concept called a structure which determines the "structure" of the ring R in the ordinary sense. The concept of a structure is developed out of the concept of a structure of an R-module X.

Let $F_1, F_2, \cdots, F_p$ be a complete set of nonisomorphic irreducible R-modules. Of course, these are irreducible S-modules as well. Let X be an R-module. Then X is naturally a completely reducible S-module. To each pair f^*, f where f^* is an S-homomorphism of X onto F_j and f is an S-isomorphism of F_i into X, we will define in the following manner a function $\psi(\alpha)$, α in R, whose values are in the module $\mathrm{Hom}_K(F_i, F_j)$ of K-linear transformations of F_i into F_j. For x in F_i, we set $\psi(\alpha)x = f^*\alpha_L fx$ where α_L denotes left multiplication by α in R. The element $\psi(\alpha)$ is called a structural element of R; it belongs to the module

$$H_{ji} = \mathrm{Hom}_{(S,S)}(R, \ \mathrm{Hom}_K(F_i, F_j)),$$

which we call a structural module.

A structure of an R-module X is the set of functions which describe the dependence of the structural elements on the homomorphisms f^* and f. Theorem 1 shows that if the structures of two R-modules are related in a certain manner, then the modules are isomorphic.

We go on to study the significance of the structural modules themselves. For this purpose we introduce in Part III the concept of a representation module. We show that each element of H_{ji} is a structural element of a particular R-module which is isomorphic to an indecomposable left ideal of R. This leads us to introduce the concept of a structure $\Sigma(R, S)$ of a cleft ring which is defined from the structures of the indecomposable left ideals.

Our principal theorem (Theorem 3) obtains necessary and sufficient conditions for an isomorphism $I_0: S \to S'$ of the semisimple components S and S'

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of cleft rings R and R', respectively, to be extended to an isomorphism $I:R \to R'$. These conditions are given in terms of the structures $\Sigma(R, S)$ and $\Sigma(R', S')$.

In case R/N is a separable algebra of finite rank over a field, it follows from the Wedderburn Principal Theorem that R is a cleft ring. Furthermore, if $R = S \oplus N$ and $R = S' \oplus N$ are any two cleavings for R, it follows from Malcev's Theorem [4 or 8] that there is an inner automorphism $I:R \to R$ such that $I:S \to S'$. In this case, the structures themselves characterize the rings up to isomorphism. We also study extensions of anti-isomorphisms and characterize commutative algebras in terms of their structures. In a subsequent paper, we will investigate an extension of the Malcev theorem stated above.

Certain authors [2, 9, 10, 11] have developed a theory of nonsemisimple algebras in which a basis for the algebra is chosen which exhibits the regular representation in a particularly nice form. Then when the algebra is cleft, certain additive subgroups of R are distinguished. These additive subgroups, called elementary modules when the algebra is of finite rank over an algebraically closed field, may be identified by means of the structural elements of R. The structural elements that we introduce can be used to give an invariant characterization of these modules and enable us to advance the theory. The concept of a structure provides a more flexible technique for handling the structural elements than those used to handle the elementary modules. The concepts of structural modules and representation modules enable us to obtain a more complete theory in the general case of cleft rings, and to study the structural elements independently of the structures and the ring itself.

Our theory extends immediately to rings which are semiprimary in the sense that R is a ring with nil-radical N such that $\bigcap_{r=1}^{\infty} N^r = 0$ and R/N^r is a ring with minimum condition on its set of left (or right) ideals (with the exception of §4C and §4D). More details on this extension will be given in a subsequent paper.

1B. Definitions and conventions

A ring R will always be considered as having an identity and as possessing the minimum condition on its left ideals. We will further assume that R is cleft with cleaving $R = S \oplus N$. We will further assume that R is an algebra of possibly infinite rank over a field K. Then all R-modules will be K-modules. We then assume all module and ring homomorphisms are K-homomorphisms. This is not an essential restriction for the following reason.

PROPOSITION 1.1. *Every cleft ring is a direct sum of algebras over prime fields.*

Proof. Let $R = \oplus_{\mu=1}^{s} R_\mu$ be the decomposition of R into indecomposable ideals.[2] Let $S = \oplus_{i=1}^{k} S_i$ be the decomposition of S into simple ideals. Then

[2] By the term ideal we mean a two-sided ideal.

the projection $p:R \to R_{\mu_0}$ is such that $pS_i = 0$ or p is an isomorphism of S_i. In the latter case, $S_i \cap R_\mu = 0$, $\mu \neq \mu_0$. On the other hand, $1 = \sum_{\mu=1}^{s} \lambda_\mu$ where $\lambda_\mu \in R_\mu$ is the identity of R_μ. Since $\lambda_\mu S_i \subseteqq R_\mu$, $\lambda_\mu S_i = 0$, $\mu \neq \mu_0$ and $\lambda_{\mu_0} S_i \neq 0$. As $1S_i = S_i$, $\lambda_{\mu_0} S_i = S_i$. Hence $S_i \subseteqq R_{\mu_0}$. Thus every simple ideal S_i of S is contained in some indecomposable ideal R_μ of R. From this one may see that $S = \oplus_{\mu=1}^{s} T_\mu$ where $R_\mu = T_\mu \oplus N_\mu$ is a cleaving for R_μ with semisimple component T_μ and radical N_μ, and each T_μ is a sum of some of the ideals S_i. On the other hand, $S_i = Se_i$ where e_i is an idempotent in the center of S contained in S_i. But S_i and S_j are contained in the same indecomposable ideal of R if and only if Re_i and Re_j are contained in the same indecomposable ideal. But if this is the case, $e_i Ne_j \neq 0$ [1, p. 107].[3]

Now if the additive orders p of e_i and q of e_j are finite, they are the characteristics of the fields which are centers of the simple rings S_i and S_j inasmuch as e_i is the identity of S_i and e_j is the identity of S_j. Thus p and q are prime integers. Suppose $p \neq q$. Let a and b be integers such that $1 = ap + bq$. Then $e_i Ne_j = (ap + bq)e_i Ne_j = 0$. Hence Se_i and Se_j belong to distinct indecomposable ideals. Let the additive order of, say, e_j be infinite, and let the additive order of e_i be a prime $p < \infty$. Then $e_j = p\left(\frac{1}{p}\right)e_j$ because e_j is contained in a field of characteristic zero which is the center of S_j. Thus $e_i Ne_j = pe_i N\left(\frac{1}{p} e_j\right) = 0$, and again Se_i and Se_j belong to distinct indecomposable ideals of R. Hence all simple ideals S_i belonging to an ideal T_μ of S may be regarded as algebras over isomorphic prime fields. Then T_μ may be regarded as an algebra over a prime field D_μ. Furthermore, the identity element λ_μ of R_μ is contained in D_μ.

We must show that D_μ is in the center of R_μ. If D_μ is finite, it is generated as an additive group by λ_μ. So in this case, the result follows. If D_μ is isomorphic to the rational numbers, its elements may be represented in the form $\frac{a}{b}\lambda_\mu$ where a and b are integers and $b \neq 0$. Let $\alpha \in R_\mu$; set

$$\beta = \left(\frac{a}{b}\lambda_\mu\right)\alpha - \alpha\left(\frac{a}{b}\lambda_\mu\right)$$

and suppose $\beta \neq 0$. Then

$$\beta = \left(\frac{b}{b}\lambda_\mu\right)\beta = \left(\frac{1}{b}\lambda_\mu\right)b\beta \neq 0.$$

Hence $b\beta \neq 0$. Thus $a\alpha - \alpha a \neq 0$; but then $a(\lambda_\mu \alpha - \alpha\lambda_\mu) \neq 0$, which is a contradiction. Thus in this case also D_μ is in the center of R_μ, and R_μ is an algebra over D_μ.

[3] Actually the result in [1] is stated with primitive idempotents. However, every central idempotent is a sum of primitive idempotents, so the result applies here also.

All modules which we consider will be unitary and (except for K-modules) will have a finite composition series. In general, when we do not otherwise specify, a module will be a left module for the ring being considered. Let X and X' be, respectively, a C-module and a C'-module for rings C and C'. Let $I:C \to C'$ be a homomorphism. A homomorphism $\varphi:X \to X'$ of additive groups such that $\varphi(\alpha x) = \alpha^I\varphi(x)$ will be called an I-homomorphism. If $C = C'$ and I is the identity, then we say that φ is a C-homomorphism or just a homomorphism.

A double module X over rings C and D will be an additive group X which is a left C-module and right D-module such that $(\gamma x)\delta = \gamma(x\delta)$ for $x \in X$, $\gamma \in C$, and $\delta \in D$. Let X be a (C, D)-module and X' a (C', D')-module, and let $I:C \to C'$ and $J:D \to D'$ be ring homomorphisms. A homomorphism $\varphi:X \to X'$ of additive groups such that $\varphi(\gamma x\delta) = \gamma^I(\varphi x)\delta^J$ is called an (I, J)-homomorphism. Again if $C = C', D = D'$, and $I = J = 1$, we say that φ is a (C, D)-homomorphism. We denote the group of such homomorphisms by $\operatorname{Hom}_{(C,D)}(X, X')$.

A left bimodule X over rings C and D will be an additive group X which is both a left C-module and left D-module such that $\gamma(\delta x) = \delta(\gamma x)$ for $\gamma \in C$, $\delta \in D$, and $x \in X$. We designate homomorphisms in the usual manner.

We will make use of the concept of a projective module [3] as well as of the elementary properties of the functor $\operatorname{Hom}_S(X, Y)$ [3, Chapter II]. A homomorphism $\varphi:X \to Y$ of modules X and Y over a ring C will be called a *monomorphism* if it is one-to-one and an *epimorphism* if $\varphi X = Y$.

Let $S = \oplus_{i=1}^k S_i$, where S_i is a simple ideal. Let $F_1, F_2, \cdots F_k$ be a complete set of irreducible R-modules; that is, we take this set of modules so that no two are isomorphic and such that every irreducible R-module is isomorphic to one of them. They also form a complete set of irreducible S-modules. Let K_i, $i = 1, 2, \cdots, k$ denote the endomorphism fields of F_i, $i = 1, 2, \cdots, k$. By U_i, $i = 1, 2, \cdots, k$, we mean left principal indecomposable modules of R. By definition, U_i is isomorphic to an indecomposable left ideal of R. It is well known [1, pp. 98–99] that U_i/NU_i is irreducible, that a principal indecomposable module U_i is determined up to isomorphism by its irreducible factors, and that every irreducible module F_i is isomorphic to an irreducible factor of some principal indecomposable module. Thus we may and will assume that the modules U_i are chosen so that U_i/NU_i is isomorphic to F_i.

As in the proof of Proposition 1.1, let $e_1, e_2, \cdots, e_k$ denote the central idempotents of S which are contained in the respective simple ideals $S_1, S_2, \cdots, S_k$. Then set $e_i Re_j = R_{ij}$, $i, j = 1, 2, \cdots, k$. We have

$$R = \oplus_{i,j=1}^k R_{ij}$$

because $1 = \sum_{i=1}^k e_i$ and the idempotents e_i belong to an orthogonal family. The modules R_{ij} are (S, S)-modules, of course. Since $S_\xi R_{ij} = 0$ and

$R_{ij} S_\eta = 0$ for $\xi \neq i$ or $\eta \neq j$, R is isomorphic to R_{ij} as an (S_i, S_j)-module. We will call these modules *Cartan submodules.*

1C. Direct families of homomorphisms

Of fundamental importance in what follows is the representation of a direct decomposition of an S-module by means of homomorphisms. We will review this in order to establish our terminology and to adapt the concept to our needs.

We form the right K_i-modules $\mathrm{Hom}_S(F_i, X)$ and the left K_i-modules $\mathrm{Hom}_S(X, F_i)$, $i = 1, 2, \cdots, k$. Because $f \to f^*f$ is, for each $f^* \in \mathrm{Hom}_S(X, F_i)$, an element of the dual module $\mathrm{Hom}_S^*(F_i, X)$ of $\mathrm{Hom}_S(F_i, X)$, we identify $\mathrm{Hom}_S^*(F_i, X)$ and $\mathrm{Hom}_S(X, F_i)$. The elements $f \in \mathrm{Hom}_S(F_i, X)$ are monomorphisms, which we call *injections*, and the elements $f^* \in \mathrm{Hom}_S^*(F_i, X)$ are epimorphisms, which we call *projections.*

A *direct family* of homomorphisms representing X as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$ is a family of homomorphisms $\{f_\mu^*, f_\mu \mid \mu = 1, 2, \cdots, t\}$ with $f_\mu^* \in \mathrm{Hom}_S^*(F_{i_\mu}, X)$, $f_\mu \in \mathrm{Hom}_S(F_{i_\mu}, X)$, and

$$(1.1) \qquad f_\mu^* f_\nu = 0, \quad \mu \neq \nu; \qquad f_\mu^* f_\mu = 1_{F_{i_\mu}}; \qquad \textstyle\sum_{\mu=1}^t f_\mu f_\mu^* = 1$$

where 1_M is the identity endomorphism of the corresponding module M.

If $\{f_\mu^* \mid \mu = 1, 2, \cdots, t\}$ is a family of projections belonging to a direct family, we say that $\{f_\mu^*\}$ is a *direct family of projections.* Similarly, we define a *direct family of injections.* The direct family of injections and the direct family of projections which belong to a given direct family of homomorphisms will be said to be *complementary.* Of course, given any direct family of homomorphisms $\{f_\mu^*, f_\mu\}$ representing a module as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$, we have the direct decomposition $X = \oplus_{\mu=1}^t f_\mu F_{i_\mu}$.

PROPOSITION 1.2. *Let $\{f_\mu \mid \mu = 1, 2, \cdots, t\}$ be a family of injections with $f_\mu \in \mathrm{Hom}_S(F_{i_\mu}, X)$. Then f_μ is a direct family of injections if, and only if, those elements f_μ which are in a given module* $\mathrm{Hom}_S(F_i, X)$ *form a K_i-basis for* $\mathrm{Hom}_S(F_i, X)$.

Proposition 1.2′ is the dual proposition which may be stated for families of projections. We will prove only Proposition 1.2.

Proof. Necessity. Let $f_{\mu_1}, f_{\mu_2}, \cdots, f_{\mu_s}$ be the elements of a direct family of injections which belong to $\mathrm{Hom}_S(F_i, X)$ for some arbitrary $i = 1, 2, \cdots, k$. Let $\{f_\mu^* \mid \mu = 1, 2, \cdots, t\}$ be the complementary direct family of projections. Suppose $f \in \mathrm{Hom}_S(F_i, X)$. Then $f = \sum_{\mu=1}^t f_\mu(f_\mu^* f)$, and

$$f_\mu^* f \in \mathrm{Hom}_S(F_i, F_i) = K_i .$$

So $\sigma_\mu = f_\mu^* f = 0$, unless $F_{i_\mu} = F_i$; then $\sigma_\mu \in K_i$ and $f_\mu \in \mathrm{Hom}_S(F_i, X)$. Hence $f = \sum_{j=1}^s f_{\mu_j} \sigma_{\mu_j}$ where $f_{\mu_j} \in \mathrm{Hom}_S(F_i, X)$. If $\sum_{j=1}^s f_{\mu_j} \sigma_{\mu_j} = 0$, multiplying by any projection $f_{\mu_j}{}^*$ of the complementary family, we obtain that $\sigma_{\mu_j} = 0$. This proves the necessity.

Sufficiency. Let $\{f_{\mu_1}, f_{\mu_2}, \cdots, f_{\mu_s}\}$ be a K_i-basis for $\mathrm{Hom}_S(F_i, X)$. Then the modules $f_{\mu_1} F_i, f_{\mu_2} F_i, \cdots, f_{\mu_s} F_i$ are S-irreducible, and we may suppose that $f_{\mu_1} F_i, f_{\mu_2} F_i, \cdots, f_{\mu_r} F_i$, $r \leqq s$, form a maximal independent set of irreducible modules.[4]

Should $r < s$ and $x \in F_i$, $f_{\mu_{r+1}} x = \sum_{j=1}^{r} f_{\mu_j} x_j$ where $x_j \in F_j$ is uniquely determined by x. Then $\sigma_j : x \to x_j = \sigma_j x$ may be verified to be in K_i. This means that $f_{\mu_{r+1}} = \sum_{j=1}^{r} f_{\mu_j} \sigma_j$, which is a contradiction. Thus

$$X_i = \sum_{j=1}^{s} f_{\mu_j} F_i = \oplus_{j=1}^{s} f_{\mu_j} F_i .$$

Now X_i is the homogeneous component[5] of X corresponding to F_i. Furthermore, one may find a complementary family $\{f_{\mu_j}{}^* \mid j = 1, 2, \cdots, s\}$ of projections to $\{f_{\mu_j}\}$. Then from all the homogeneous components of X, we may obtain direct families of homomorphisms which together yield a direct family $\{f_\mu{}^*, f_\mu \mid \mu = 1, 2, \cdots, t\}$ for X.

II. Structures of Modules

2A. Isomorphisms of modules

Let X be a given left R-module. Then if $f^* \in \mathrm{Hom}_S{}^*(F_j, X)$ and $f \in \mathrm{Hom}_S(F_i, X)$, $i, j = 1, 2, \cdots, k$, we define the function

$$\psi[f^*, f] : R \to \mathrm{Hom}_K(F_i, F_j)$$

by

$$\psi[f^*, f](\alpha) = f^* \alpha_L f \tag{2.1}$$

for $f^* \in \mathrm{Hom}_S{}^*(F_j, X)$, $f \in \mathrm{Hom}_S(F_i, X)$, $i, j = 1, 2, \cdots, k$, and $\alpha \in R$; α_L designates left multiplication by $\alpha \in R$. We will call these the *structural elements* of the module X. Of course, similar definitions hold for right modules.

The modules F_i are $(K_i - S_i)$-modules as well as $(K_i - S)$-modules. Hence $\mathrm{Hom}_K(F_i, F_j)$ is a $(K_j - S_j, K_i - S_i)$-module as well as a $(K_j - S, K_i - S)$-module. But then

$$H_{ji} = \mathrm{Hom}_{(S,S)}(R, \mathrm{Hom}_K(F_i, F_j)) \tag{2.2}$$

is a (K_j, K_i)-module. But as $S_\mu \mathrm{Hom}_K(F_i, F_j) = \mathrm{Hom}_K(F_i, F_j) S_\nu = 0$ for $\mu \neq j$ and $\nu \neq i$, $\mathrm{Hom}_{(S,S)}(R_{\nu\mu}, \mathrm{Hom}_K(F_i, F_j)) = 0$ for $\mu \neq j$ or $\nu \neq i$. Hence we may identify H_{ji} with

$$H_{ji} = \mathrm{Hom}_{(S_j, S_i)}(R_{ji}, \mathrm{Hom}_K(F_i, F_j)).$$

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[4] By an independent set of modules, we mean a set of submodules $X_1, X_2, \cdots, X_s$ of a module X such that if X' is generated by the elements of all the modules X_i, $i = 1, 2, \cdots, s$, $X' = \oplus_{i=1}^{s} X_i$.

[5] Cf. [7, p. 63].

The (K_j, K_i)-module H_{ji} will be called a *structural module* for R and will be studied in Part III. The structural elements $\psi[f^*, f]$, $f^* \epsilon \operatorname{Hom}_S^*(F_j, X)$, and $f \epsilon \operatorname{Hom}_S(F_i, X)$ all belong to H_{ji}.

A *structure* $|\psi|$ is the set of bilinear mappings $\bar{\psi}$ defined for each pair of indices $i, j = 1, 2, \cdots, k$

$$\bar{\psi}: \operatorname{Hom}_S^*(F_j, X) \times \operatorname{Hom}_S(F_i, X) \to H_{ji}$$

defined by $(f^*, f) \to \psi[f^*, f]$. Actually the bilinear mappings $\bar{\psi}$ should be indexed by the indices i and j, but no confusion will result from our dropping these indices, for we may make the necessary distinction by designating the modules $\operatorname{Hom}_S^*(F_j, X)$ and $\operatorname{Hom}_S(F_i, X)$ to which f^* and f belong.

THEOREM 1. *A necessary and sufficient condition for two R-modules X and X' with structures $|\psi|$ and $|\psi'|$, respectively, to be isomorphic is that there exist K_i-isomorphisms φ and φ^*, which are contragredient*[6] *to each other, such that for $i = 1, 2, \cdots, k$*

$$(2.3) \qquad \begin{aligned} \varphi&: \operatorname{Hom}_S(F_i, X) \to \operatorname{Hom}_S(F_i, X'), \\ \varphi^*&: \operatorname{Hom}_S^*(F_i, X) \to \operatorname{Hom}_S^*(F_i, X'), \end{aligned}$$

such that

$$(2.4) \qquad \psi[f^*, f] = \psi'[\varphi^* f^*, \varphi f]$$

for $f^* \epsilon \operatorname{Hom}_S^*(F_j, X)$, $f \epsilon \operatorname{Hom}_S(F_i, X)$, $i, j = 1, 2, \cdots, k$.

Remark. Again we shall suppress the subscripts on the isomorphisms φ and φ^*.

Proof. Necessity. Let $\Phi: X \to X'$ be an R-isomorphism. Then for each index $i = 1, 2, \cdots, k$, Φ induces a K_i-isomorphism φ of (2.3), and Φ^{-1} induces the contragredient K_i-isomorphism φ^* given by $\varphi f = \Phi f$ and $\varphi^* f^* = f^* \Phi^{-1}$ for $f \epsilon \operatorname{Hom}_S(F_i, X)$ and $f^* \epsilon \operatorname{Hom}_S^*(F_i, X)$. Now we have

$$\psi[f^*, f](\alpha) = f^* \alpha_L f = f^* \Phi^{-1} \alpha_L \Phi f = \psi'[\varphi^* f^*, \varphi f](\alpha)$$

for $f^* \epsilon \operatorname{Hom}_S^*(F_j, X)$, $f \epsilon \operatorname{Hom}_S(F_i, X)$, and $\alpha \epsilon R$. Thus (2.4) is valid.

Sufficiency. Let φ and φ^* be given as in the hypothesis. Let $\{f_\mu^*, f_\mu \mid \mu = 1, 2, \cdots, t\}$ be a direct family of homomorphisms representing X as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$. Let $g_\mu^* = \varphi^* f_\mu^*$ and $g_\mu = \varphi f_\mu$. Since φ and φ^* are contragredient, $\{g_\mu^*, g_\mu \mid \mu = 1, 2, \cdots, t\}$ is an orthogonal family for X'. Since φ is an isomorphism, one may obtain from Proposition 1.2 that $\{g_\mu \mid \mu = 1, 2, \cdots, t\}$ is a direct family of injections. Hence $\{g_\mu^*, g_\mu\}$ is a direct family of homomorphisms.

Define $\Phi: X \to X'$ by setting

$$(2.5) \qquad \Phi(x) = \textstyle\sum_{\mu=1}^{t} g_\mu f_\mu^* x$$

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[6] Here, of course, we need only to assume the existence of one isomorphism φ or φ^* and specify the other to be its contragredient. However, it is slightly more convenient to assume that both exist.

for $x \in X$. Then for $x' \in X'$, we set

$$\Phi'(x') = \sum_{\mu=1}^{t} f_\mu g_\mu^* x'.$$

Since Φ and Φ' are inverse to each other, Φ is an S-isomorphism. Now, if $\alpha \in R$,

$$\Phi(\alpha x) = \sum_{\mu=1}^{t} g_\mu f_\mu^* \alpha x = \sum_{\mu,\nu=1}^{t} g_\mu(f_\mu^* \alpha f_\nu) f_\nu^* x.$$

Using (2.4), we obtain

$$\Phi(\alpha x) = \sum_{\mu,\nu=1}^{t} g_\mu(g_\mu^* \alpha g_\nu) f_\nu^* x = \sum_{\nu=1}^{t} \alpha g_\nu f_\nu x = \alpha\Phi(x).$$

This proves the theorem.

We note here an important formula for a structure of a module.

Proposition 2.1. *If $\alpha, \beta \in R$ and $\{f_\mu^*, f_\mu \mid \mu = 1, 2, \cdots, t\}$ is a direct family representing X as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$, then*

$$\psi[f^*, f](\alpha\beta) = \sum_{\mu=1}^{t} \psi[f^*, f_\mu](\alpha)\psi[f_\mu^*, f](\beta) \tag{2.6}$$

for $f^* \in \mathrm{Hom}_S^*(F_j, X)$ and $f \in \mathrm{Hom}_S(F_i, X)$.

The proof is immediate from the definition of structures and direct families.

2B. Homomorphisms of modules

Let X and X' be R-modules, and let $\Phi: X \to X'$ be an R-homomorphism. Then Φ induces a K_i-homomorphism $\varphi': \mathrm{Hom}_S^*(F_i, X') \to \mathrm{Hom}_S(F_i, X)$ where $\varphi' g = g\Phi$ for $g \in \mathrm{Hom}_S^*(F_i, X) = \mathrm{Hom}_S(X, F_i)$. The kernel of φ' is the K_i-module $\mathrm{Hom}_S^*(F_i, X'/\Phi X) = \mathrm{Hom}_S(X'/\Phi X, F_i)$ consisting of those homomorphisms which vanish on ΦX. We remark that if Φ is an epimorphism [monomorphism], then φ' is a monomorphism [epimorphism].

Similarly, Φ induces a K_i-homomorphism $\varphi: \mathrm{Hom}_S(F_i, X) \to \mathrm{Hom}_S(F_i, X)$ defined by $\varphi f = \Phi f$ for $f \in \mathrm{Hom}_S(F_i, X)$. The kernel of φ is the K_i-submodule $\mathrm{Hom}_S(F_i, X'')$ of $\mathrm{Hom}_S(F_i, X)$ where X'' is the kernel of Φ. Again if Φ is an epimorphism [monomorphism], φ is an epimorphism [monomorphism].

Proposition 2.2. *Let X and X' be R-modules, and let $\Phi: X \to X'$ be an R-homomorphism. Let ψ and ψ' be the structures of X and X', respectively. If φ and φ' are the homomorphisms induced by Φ as above,*

$$\psi'[g^*, \varphi f] = \psi[\varphi' g^*, f] \tag{2.7}$$

for $g^* \in \mathrm{Hom}_S^*(F_i, X)$ and $f \in \mathrm{Hom}_S(F_j, X)$, $i, j = 1, 2, \cdots, k$.

Proof. The proof is immediate from the equation

$$\psi'[g^*, \varphi f](\alpha) = g^* \alpha_L \Phi f = g^* \Phi \alpha_L f = \psi[\varphi' g^*, f](\alpha)$$

for $\alpha \in R$.

III. Representation Modules

3A. Isomorphisms of double modules

We introduced the structural modules

$$H_{ji} = \mathrm{Hom}_{(S_j, S_i)}(R_{ji}, \mathrm{Hom}_K(F_i, F_j)). \tag{3.1}$$

Now we propose to study the (K_j, K_i)-modules

$$M(T) = \mathrm{Hom}_{(S_j, S_i)}(T, \mathrm{Hom}_K(F_i, F_j)) \tag{3.2}$$

associated with a given (S_j, S_i)-module T. We shall call $M(T)$ the *representation module* for the (S_j, S_i)-module T.

The following properties give the principal properties of representation modules.

PROPOSITION 3.1. *Let x_0 be an arbitrary nonzero element of F_i, and let ε be a primitive idempotent of S_i such that $\varepsilon x_0 = x_0$. Let $\psi \in M(T)$. Then the mapping $\omega: M(T) \to \mathrm{Hom}_{S_j}(T\varepsilon, F_j)$ defined by $\omega\psi(\alpha) = \psi(\alpha\varepsilon)x_0 = \psi(\alpha)x_0$ is a K_j-isomorphism. In particular,* $\dim_{S_j} T\varepsilon = \dim_{K_j} M(T)$.

Proof. By definition, $M(T) = \mathrm{Hom}_{(S_j, S_i)}(T, \mathrm{Hom}_K(F_i, F_j))$. Using the associativity isomorphism of functors [3], we obtain that $M(T)$ is isomorphic to $\mathrm{Hom}_{(S_j, K)}(T \otimes_{S_i} F_i, F_j) = \mathrm{Hom}_{S_j}(T \otimes_{S_i} F_i, F_j)$ where to $\psi \in M(T)$ corresponds the homomorphism defined by $\alpha \otimes x \to \psi(\alpha)x$ for $\alpha \in T$, $x \in F_i$.

Let x_0 and ε be given as in the hypothesis of the theorem. Then form the sfield $K_i^* = \varepsilon S_i \varepsilon$, which is anti-isomorphic to K_i. We may identify the S_j-modules $T \otimes_{S_i} F_i$ and $T\varepsilon \otimes_{K_i^*} K_i^* x_0$ because for $\alpha_\mu \in T$ and $x_\mu \in F_i$, $\mu = 1, 2, \cdots, n$, we have

$$\textstyle\sum_{\mu=1}^n \alpha_\mu \otimes_{S_i} x_\mu = \sum_{\mu=1}^n \alpha_\mu \otimes_{S_i} \beta_\mu x_0 = \sum_{\mu=1}^n \alpha_\mu \beta_\mu \otimes_{S_i} x_0$$

where $x_\mu = \beta_\mu x_0 = \beta_\mu \varepsilon x_0$ for $\beta_\mu \in S_i$. Furthermore,

$$\textstyle\sum_{\mu=1}^n \alpha_\mu \otimes_{K_i^*} \lambda_\mu x_0 \to \sum_{\mu=1}^n \alpha_\mu \lambda_\mu, \qquad \lambda_\mu \in K_i^*,$$

determines an S_j-isomorphism of $T\varepsilon \otimes_{K_i^*} K_i^* x_0$ and $T\varepsilon$. Then we obtain the isomorphisms

$$\mathrm{Hom}_{S_j}(T\varepsilon \otimes_{S_i} F_i, F_j) \to \mathrm{Hom}_{S_j}(T\varepsilon \otimes_{K_i^*} K_i^* x_0, F_j) \to \mathrm{Hom}_{S_j}(T\varepsilon, F_j).$$

Composing these with the associativity isomorphism, we obtain the desired isomorphism.

The remainder of the proposition is immediate.

PROPOSITION 3.2. *Let $M(T)$ be a representation module of an (S_j, S_i)-module T. Then given any K_j-basis $\psi_1, \psi_2, \cdots, \psi_{c_{ji}}$ for $M(T)$, any K_i-basis $x_1, x_2, \cdots, x_{n_i}$ for F_i, and $y_1, y_2, \cdots, y_{n_i}$ arbitrary in F_j, there exists an* *element* $\alpha_\lambda \in T$ *such that for each* $\lambda = 1, 2, \cdots, c_{ji}$

$$\psi_\lambda(\alpha_\lambda)x_\mu = y_\mu, \qquad \psi_\xi(\alpha_\lambda)x_\mu = 0, \quad \xi \neq \lambda. \tag{3.3}$$

Proof. Corresponding to the basis $x_1, x_2, \cdots, x_{n_i}$ for F_i is a set of orthogonal primitive idempotents $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ from S_i such that $\varepsilon_\mu x_\mu = x_\mu$, $\mu = 1, 2, \cdots, n_i$. By Proposition 3.1,

$$f_{\xi\mu}^*: \alpha \rightarrow \psi_\xi(\alpha\varepsilon_\mu)x_\mu$$

is in $\mathrm{Hom}_{S_j}(T\varepsilon_\mu, F_j) = \mathrm{Hom}_{S_j}^*(F_j, T\varepsilon_\mu)$, and $\{f_{\xi\mu}^* \mid \xi = 1, 2, \cdots, c_{ji}\}$ is a K_j-basis for $\mathrm{Hom}_{S_j}^*(F_j, T\varepsilon_\mu)$. Then by Proposition 1.2′, $\{f_{\xi\mu}^* \mid \xi = 1, 2, \cdots, c_{ji}\}$ is a direct family of projections representing $T\varepsilon_\mu$ as the S-direct sum of copies of F_j. Hence we may choose $\alpha_{\lambda\mu} = \alpha_{\lambda\mu}\varepsilon_\mu$ in $T\varepsilon_\mu$ such that $f_{\xi\mu}^*(\alpha_{\lambda\mu}) = 0$, $\xi \neq \lambda$, and $f_{\lambda\mu}^*(\alpha_{\lambda\mu}) = y' \neq 0$ where $y' \in F_j$. But $y_\mu = \beta_\mu y'$ where $\beta_\mu \in S_j$. Hence replacing $\alpha_{\lambda\mu}$ by $\beta_\mu \alpha_{\lambda\mu}$, we obtain that $f_{\lambda\mu}^*(\alpha_{\lambda\mu}) = y_\mu$. Thus $\psi_\xi(\alpha_{\lambda\mu})x_\mu = 0$, $\xi \neq \lambda$, $\psi_\lambda(\alpha_{\lambda\mu})x_\mu = y_\mu$, and $\psi_\xi(\alpha_{\lambda\mu})x_\eta = 0$, for $\eta \neq \mu$ and all ξ as $\varepsilon_\mu x_\eta = 0$. We thus obtain the desired element by setting $\alpha_\lambda = \sum_{\mu=1}^{n_i} \alpha_{\lambda\mu}$.

PROPOSITION 3.3. *Let* $M(T)$ *be a representative module for an* (S_j, S_i)-*module* T. *Let* Ω *be a set of indices, and let* $\{y_\eta \mid \eta \in \Omega\}$ *be a* K-*basis for* F_j. *Let* $x_1, x_2, \cdots, x_{n_i}$ *be a* K_i-*basis for* F_i, *and let* $\psi_1, \psi_2, \cdots, \psi_{c_{ji}}$ *be a* K_j-*basis for* $M(T)$. *Then there is a* K-*basis for* T *uniquely determined by choosing elements* $\alpha_{\mu\nu}^\lambda \in T$ *for each triple* (μ, ν, λ) *of the set* $\Lambda = \{(\mu, \nu, \lambda) \mid \mu \in \Omega; \nu = 1, 2, \cdots, n_i; \lambda = 1, 2, \cdots, c_{ji}\}$ *such that*

$$\psi_\eta(\alpha_{\mu\nu}^\lambda)x_\xi = \delta_{\eta\lambda}\,\delta_{\nu\xi}\,y_\mu, \tag{3.4}$$

where $\delta_{\eta\lambda}$ *and* $\delta_{\nu\xi}$ *are Kronecker deltas.*

Proof. That such elements $\alpha_{\mu\nu}^\lambda$ exist follows from Proposition 3.2. For $\alpha \in T$, let $\psi_\eta(\alpha)x_\xi = 0$ for $\eta = 1, 2, \cdots, c_{ji}$ and $\xi = 1, 2, \cdots, n_i$. This implies $\psi(\alpha)x_\xi = 0$ for all ψ in $M(T)$ and $\xi = 1, 2, \cdots, n_i$. Hence $\psi(\alpha) = 0$ for each $\psi \in M(T)$. By Proposition 3.1, $\alpha\varepsilon = 0$ for all primitive idempotents $\varepsilon \in S_i$. Hence $\alpha = 0$. Thus the elements $\alpha_{\mu\nu}^\lambda$ are uniquely determined by (3.4). We wish to show that they form a basis for T. Suppose that $\sum_\Lambda k_{\mu\nu}^\lambda \alpha_{\mu\nu}^\lambda = 0$ for a finite number of nonzero elements $k_{\mu\nu}^\lambda$ in K. Then for each $\lambda = 1, 2, \cdots, c_{ji}$

$$\sum_{\mu,\nu} \psi_\lambda(k_{\mu\nu}^\lambda \alpha_{\mu\nu}^\lambda)x_\xi = \sum_{\mu,\nu} k_{\mu\nu}^\lambda \psi_\lambda(\alpha_{\mu\nu}^\lambda)x_\xi = 0,$$

where the summation is over all possible values of μ and ν. Hence by (3.4)

$$\sum_{\mu\in\Omega} k_{\mu\nu}^\lambda y_\mu = 0.$$

Since $\{y_\mu \mid \mu \in \Omega\}$ is a K-basis for F_j, we have that $k_{\mu\nu}^\lambda = 0$ for $\mu \in \Omega$. Hence $k_{\mu\nu}^\lambda = 0$ for $(\mu, \nu, \lambda) \in \Lambda$.

Next let $\alpha \in T$. Then for each $\lambda = 1, 2, \cdots, c_{ji}$; $\nu = 1, 2, \cdots, n_i$,

$$\psi_\lambda(\alpha)x_\nu = \sum_{\mu\in\Omega} k_{\mu\nu}^\lambda y_\mu,$$

where $k_{\mu\nu}^{\lambda} \in K$. Set $\alpha' = \sum_{\Lambda} k_{\mu\nu}^{\lambda} \alpha_{\mu\nu}^{\lambda}$. Then one may verify that for $\lambda = 1, 2, \cdots, c_{ji}$, $\nu = 1, 2, \cdots, n_i$, $\psi_\lambda(\alpha)x_\nu = \psi_\lambda(\alpha')x_\nu$. Hence $\psi(\alpha - \alpha') = 0$ for all $\psi \in M(T)$. Thus $\alpha = \alpha'$. This shows that the set of elements $\alpha_{\mu\nu}^{\lambda}$ form a K-basis for T as we desired.

Proposition 3.2 shows that if $\psi \in M(T)$, the set $\psi(T)$ has some of the properties of a complete module of endomorphisms in that one may always find an element $\alpha \in T$ such that $\psi(\alpha)x_\mu$ takes on arbitrary values in F_j for each element x_μ of a K_i-basis for F_i. However, the linearity condition is lacking for the elements $\psi(\alpha)$. But because $M(T)$ is a double (K_j, K_i)-module, we have that $\psi_\lambda \sigma = \sum_{\mu=1}^{c_{ji}} a_{\lambda\mu}(\sigma)\psi_\mu$, $\lambda = 1, 2, \cdots, c_{ji}$, $a_{\lambda\mu}(\sigma) \in K_j$. Thus the mapping $\sigma \to (a_{\lambda\mu}(\sigma))$ is a matrix representation of K_i on the left K_j-module, $M(T)$. As is well known, this representation is determined by the double (K_j, K_i)-module $M(T)$, in the sense that isomorphic modules give rise to similar representations.

The theory of double modules over division algebras of finite rank over K is treated in Hochschild [6] and Jacobson [7, p. 173]. They restrict themselves to the case that K_i maybe identified with K_j or a division subalgebra of K_j. In this case if K_j is separable over K, then M is a completely reducible double (K_j, K_i)-module. If K_j is a galois extension of K_i, then these irreducible modules have K_j-dimension 1. This means that if ψ generates such an irreducible module, there exists an isomorphism $\theta: K_i \to K_j$ such that $\psi\sigma = \theta(\sigma)\psi$ for $\sigma \in K_i$, and ψ is semilinear. If K is the center of K_j, θ is induced by an inner automorphism. If $K = K_j = K_i$, then θ must be the identity, and ψ is linear. This condition will always hold if K is algebraically closed.

Let S_i, S_i', S_j, and S_j' be simple rings with minimum condition. Let $I_i: S_i \to S_i'$ and $I_j: S_j \to S_j'$ be ring isomorphisms. Let F_i be an irreducible S_i-module, F_i' an irreducible S_i'-module, etc. Let $\omega_i: F_i \to F_i'$ be an I_i-isomorphism. Then ω_i induces an isomorphism of K_i onto K_i', which we again denote by I_i, that is defined by

$$\sigma^{I_i} = \omega_i \sigma \omega_i^{-1} \tag{3.5}$$

for $\sigma \in K_i$.

THEOREM 2. *Let $I_i: S_i \to S_i'$ and $I_j: S_j \to S_j'$ be given isomorphisms of simple rings. Let $\omega_i: F_i \to F_i'$ be an I_i-isomorphism of the irreducible modules of S_i and S_i'. Similarly let $\omega_j: F_j \to F_j'$ be an I_j-isomorphism of irreducible modules.*

Let there be given an (S_j, S_i)-module T and an (S_j', S_i')-module T' with representation modules

$$M(T) = \mathrm{Hom}_{(S_j, S_i)}(T, \mathrm{Hom}_K(F_i, F_j)),$$

$$M(T') = \mathrm{Hom}_{(S_j', S_i')}(T', \mathrm{Hom}_K(F_i', F_j'))).$$

For a given (I_j, I_i)-isomorphism $\theta: M(T) \to M(T')$, there is induced an

(I_j^{-1}, I_i^{-1})-isomorphism $J: T' \to T$ satisfying

$$\theta\psi(\alpha') = \omega_j \psi(\alpha'^J)\omega_i^{-1} \tag{3.6}$$

for $\alpha' \in T'$ and $\psi \in M(T)$; conversely, an isomorphism $J: T' \to T$ induces an isomorphism $\theta: M(T) \to M(T')$ satisfying (3.6).

Proof. Let θ be given. As in Proposition 3.3, choose a K-basis $\{y_\mu \mid \mu \in \Omega\}$ for F_j, choose a K_i-basis $\{x_1, x_2, \cdots, x_{n_i}\}$ for F_i, and choose a K_j-basis $\{\psi_1, \psi_2, \cdots, \psi_{c_{ji}}\}$ for $M(T)$. Then $\{\omega_i x_1, \omega_i x_2, \cdots \omega_i x_{n_i}\}$ is a K_i-basis for F_i' as ω_i is a I_0-isomorphism. Likewise $\{\omega_j y_\mu \mid \mu \in \Omega\}$ is a K-basis for F_j'. The set $\{\theta\psi_1, \theta\psi_2, \cdots, \theta\psi_{c_{ji}}\}$ is a K_j'-basis for $M'(T')$. Then by means of Proposition 3.3, choose a K-basis $\{a_{\mu\nu}^{\lambda} \mid (\mu, \nu, \lambda) \in \Omega\}$ for T and a K-basis $\{a_{\mu\nu}^{\lambda'} \mid (\mu, \nu, \lambda) \in \Omega\}$ for T'. Define $J: T' \to T$ by setting $a_{\mu\nu}^{\lambda' J} = a_{\mu\nu}^{\lambda}$ and extending J to T' by requiring it to be K-linear. Then $\theta\psi_\eta(a_{\mu\nu}^{\lambda'})\omega_i x_\xi = \delta_{\eta\lambda}\delta_{\nu\xi}\omega_j y_\mu = \omega_j \psi_\eta(a_{\mu\nu}^{\lambda' J})x_\xi$. From this and the K-linearity of J, θ, and ψ follows

$$\theta\psi_\eta(\alpha')(\omega_i x_\xi) = \omega_j \psi_\eta(\alpha'^J)\omega_i^{-1}(\omega_i x_\xi)$$

for all (μ, ν, λ), $\eta = 1, 2, \cdots, c_{ji}$, $\xi = 1, 2, \cdots, n_i$, and $\alpha' \in T'$.

Let $x' = \sum_{\xi=1}^{n_i} \tau_\xi x_\xi$. Then

$$\begin{aligned}(\theta\psi_\eta(\alpha'))(\omega_i \tau_\xi x_\xi) &= (\theta\psi_\eta(\alpha')\tau_\xi{}^{I_i}\omega_i x_\xi = (\theta\psi_\eta(\alpha')\tau_\xi)\omega_i x_\xi \\ &= \textstyle\sum_{\zeta=1}^{c_{ji}} a_{\eta\zeta}(\tau_\xi)^{I_j}\theta\psi_\zeta(\alpha')(\omega_i x_\xi)\end{aligned}$$

where $a_{\eta\zeta}(\tau_\xi) \in K_j$. Thus

$$\begin{aligned}(\theta\psi_\eta(\alpha'))(\omega_i \tau_\xi x_\xi) &= \textstyle\sum_{\zeta=1}^{c_{ji}} a_{\eta\zeta}(\tau_\xi)^{I_j}\theta\psi_\zeta(\alpha')\omega_i x_\xi \\ &= \textstyle\sum_{\zeta=1}^{c_{ji}} a_{\eta\zeta}(\tau_\xi)^{I_j}\omega_j \psi_\zeta(\alpha'^J)\omega_i^{-1}\omega_i x_\xi \\ &= \textstyle\sum_{\zeta=1}^{c_{ji}} \omega_j a_{\eta\zeta}(\tau_\xi)\psi_\zeta(\alpha'^J)\omega_i^{-1}\omega_i x_\xi \\ &= \omega_j \psi_\eta(\alpha'^J)\tau_\xi \omega_i^{-1}\omega_i x_\xi \\ &= \omega_j \psi_\eta(\alpha'^J)\omega_i^{-1}\omega_i \tau_\xi x_\xi .\end{aligned}$$

Thus

$$\theta\psi_\eta(\alpha')x' = \omega_j \psi(\alpha'^J)\omega_i^{-1}x' \tag{3.7}$$

for all $x' \in F_i'$. Let $\psi = \sum_{\eta=1}^{c_{ji}} \sigma_\eta \psi_\eta$. Then because θ and ω_j are I_j-isomorphisms, (3.6) follows from (3.7).

Now let $\beta' \in S_j'$ and $\gamma' \in S_i'$ and $\alpha' \in T'$. Let $\psi \in M(T)$. We compute

$$\begin{aligned}\omega_j \psi((\beta'\alpha'\gamma')^J)\omega_i^{-1} &= (\theta\psi)(\beta'\alpha'\gamma') = \beta'\theta\psi(\alpha')\gamma', \\ \omega_j \psi((\beta'\alpha'\gamma')^J)\omega_i^{-1} &= \beta'(\omega_j \psi(\alpha'^J)\omega_i^{-1}\gamma',\end{aligned}$$

$$\omega_j \psi((\beta'\alpha'\gamma')^J)\omega_i^{-1} = \omega_j \psi(\beta'^{I_j^{-1}}\alpha'^J\gamma'^{I_i^{-1}})\omega_i^{-1}. \tag{3.8}$$

Since (3.8) holds for all $\psi \in M(T)$, $(\beta'\alpha'\gamma')^J = \beta^{I_j^{-1}}\alpha'\gamma'^{I_i^{-1}}$, and J is an (I_j^{-1}, I_i^{-1})-isomorphism.

One may verify that (3.6) defines an (I_j, I_i)-isomorphism of $M(T)$ onto $M(T')$ to prove the converse statement and the theorem.

COROLLARY 3.4. *With the same notation as Theorem 2, let G_i' and G_j' be irreducible S_i'- and S_j'-modules, respectively, and form the representation module*

$$M'(T') = \mathrm{Hom}_{(S_j, S_i)}(T', \mathrm{Hom}_K(G_i', G_j)).$$

Let $\theta': M(T) \rightarrow M'(T')$ be an (I_j, I_i)-isomorphism. Then a necessary and sufficient condition for θ' to induce the same (I_j^{-1}, I_i^{-1})-isomorphism $J: T' \rightarrow T$ as that induced by $\theta: M(T) \rightarrow M(T')$, $\omega_i: F_i \rightarrow F_i'$, and $\omega_j: F_j \rightarrow F_j'$ is that there exists an S_i-isomorphism $\mu_i: F_i' \rightarrow G_i'$ and an S_j'-isomorphism $\mu_j: F_j' \rightarrow G_j'$ such that

$$\theta'\psi = \mu_j \theta\psi\mu_i^{-1}. \tag{3.9}$$

Proof. Sufficiency. Let (3.9) hold. Let $J': T' \rightarrow T$ be determined by $\omega_i' = \mu_i \omega_i$, $\omega_j' = \mu_j \omega_j$, and θ'. Then for $\alpha' \in T'$, we have from (3.7),

$$\psi(\alpha'^{J'}) = \theta'\psi(\alpha') = \mu_j \theta\psi(\alpha')\mu_i^{-1} = \mu_j \omega_j \psi(\alpha'^{J})\omega_i^{-1}\mu_i^{-1} = \omega_j' \psi(\alpha'^{J})\omega_i'^{-1}.$$

Hence $\psi(\alpha'^{J'}) = \psi(\alpha'^{J})$ for all $\psi \in M(T)$. Thus $\alpha'^{J'} = \alpha'^{J}$ and $J = J'$.

Necessity. If $\theta\psi(\alpha') = \omega_j \psi(\alpha'^{J})\omega_i^{-1}$ and $\theta'\psi(\alpha') = \omega_j' \psi(\alpha'^{J})\omega_i'$, set $\mu_i = \omega_i' \omega_i^{-1}$ and $\mu_j = \omega_j' \omega_j^{-1}$, and verify (3.9).

3B. Structural modules as representation modules

We have seen that the structural elements of a given module X belong to a structural module H_{ji} (cf. (3.1)), and this module is a representation module for the Cartan submodule R_{ji} of R. What we next wish to show is that every element of H_{ji} is a structural element derived from the structure of a principal indecomposable module.

Let U_i be a principal indecomposable module. Then U_i is isomorphic to a left ideal $R\varepsilon$ of R, where ε is a primitive idempotent of R. Thus U_i is a cyclic left R-module with generator x_0. As an S-module, U_i is the direct sum of its homogeneous components: $U_i = \oplus_{k=1}^{j} U_{ij}$ where each U_{ij} is a direct sum of copies of F_j. Since U_i/NU_i is isomorphic to F_i, NU_i, considered as an S-module, contains U_{ij}, $j \neq i$. But if x_0 is a generator for U_i, so is $x_0 + n$, where n is any element of the maximal submodule NU_i. But also, $x_0 = \sum_{j=1}^{k} x_j$ where $x_j \in U_{ij}$. Consequently, x_i is also a generator for U_i. As x_i is contained in a homogeneous S-submodule U_{ii} of U_i, $Sx_i = A$ is irreducible, and $RA = Rx_i = U_i$. If B is another irreducible submodule of U_{ii} such that $B \cap NU_i = 0$, then $B \oplus NU_i = U_i$, and there exists in B an element $x_i' \equiv x_i \pmod{NU_i}$. Hence also x_i' is a generator for U_i. As before, $B = Sx_i'$ and $U_i = RB$. We have, therefore, proved

PROPOSITION 3.5. *Let $f \in \mathrm{Hom}_S(F_i, U_i)$ be such that $fF_i \cap NU_i = 0$ and $f \neq 0$; then $RfF_i = U_i$.*

We shall call such an element $f \in \mathrm{Hom}_S(F_i, U_i)$ a *generating element* of U_i in $\mathrm{Hom}_S(F_i, U_i)$.

Proposition 3.6. *Let f_0 be a generating element of U_i in $\mathrm{Hom}_S(F_i, U_i)$. Let $|\psi_i|$ be the structure of U_i. Then the K_j-homomorphism of $\mathrm{Hom}_S{}^*(F_j, U_i)$ into H_{ji} defined by $f^* \to \psi_i[f^*, f_0]$ is an isomorphism.*

Proof. It is clear that the mapping is a homomorphism. If $\psi_i[f^*, f_0] = 0$ for $f^* \in \mathrm{Hom}_S{}^*(F_j, U_i)$, then $(f^*Rf_0)F_i = f^*U_i = 0$. Hence $f^* = 0$, and the mapping is a monomorphism.

Now let ε be a primitive idempotent in S_i such that $R\varepsilon$ is isomorphic to U_i. But $R\varepsilon = Re_i\,\varepsilon$ and $Re_i\,\varepsilon = \oplus_{\mu=1}^{k} e_\mu\, Re_i\,\varepsilon = \oplus_{\mu=1}^{k} R_{\mu i}\,\varepsilon$. Since $S_\nu\, R_{\mu i} = S_\nu\, e_\nu\, e_\mu\, R_{\mu i} = 0$ when $\mu \neq \nu$,

$$\mathrm{Hom}_S(R_i\,\varepsilon, F_j) = \oplus_{\mu=1}^{k} \mathrm{Hom}_{S_j}(R_{\mu i}\,\varepsilon, F_j) = \mathrm{Hom}_{S_j}(R_{ji}\,\varepsilon, F_j).$$

Hence $\dim_{K_j} \mathrm{Hom}_{S_j}(R_{ji}\,\varepsilon, F_j) = \dim_{K_j} \mathrm{Hom}_S{}^*(F_j, U_i)$. By Proposition 3.1, $\dim_{K_j} \mathrm{Hom}_S{}^*(F_j, U_i) = \dim_{K_j} H_{ji}$.

Let $H'_{ji} = \{\psi[f^*, f_0] \mid f^* \in \mathrm{Hom}_S{}^*(F_j, U_i)\}$. Then H'_{ji} is a K_j-submodule of H_{ji} which is isomorphic to $\mathrm{Hom}_S{}^*(F_j, U_i)$. Thus $\dim_{K_j} H'_{ji} = \dim_{K_j} H_{ji}$ and $H'_{ji} = H_{ji}$. Thus the given mapping is an epimorphism and hence an isomorphism.

IV. Characterization of Cleft Rings

4A. Isomorphisms of cleft rings

Let R and R' be rings with minimum condition and identity elements which possess cleavings

$$(4.1) \qquad R = S \oplus N, \qquad R' = S' \oplus N',$$

where S and S' are semisimple rings and N and N' are the radicals of R and R', respectively. Let $I_0: S \to S'$ be a given isomorphism. We are interested in determining when one can extend the isomorphism I_0 to an isomorphism $I: R \to R'$.

We follow our previous convention in designating the modules and rings F_i, K_i, S_i, U_i, and H_{ji}, $i, j = 1, 2, \cdots, k$, which are associated with the rings R and S. Because of the isomorphism I_0, the corresponding objects associated with R' and S' can and will be designated by F'_i, K'_i, S'_i, U'_i, and H'_{ji}, $i, j = 1, 2, \cdots, k$. We will assume that $I_0: S_i \to S'_i$. Then there are I_0-isomorphisms $\omega_i: F_i \to F'_i$; furthermore, ω_i induces an isomorphism $I_0: K_i \to K'_i$ defined by (3.5).

We will first derive necessary conditions by assuming that the extension I of I_0 exists. Let $J = I^{-1}$. Now I induces an I-isomorphism $\mu_i: U_i \to U'_i$, $i = 1, 2, \cdots, k$, because U_i/NU_i is isomorphic by an I_0-isomorphism to U'_i/NU'_i.

Furthermore, ω_ξ^{-1} and μ_i induce an I_0-isomorphism

$$\varphi: \mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_{S'}(F'_\xi, U'_i)$$

for $\xi, i = 1, 2, \cdots, k$, given by

$$\varphi f = \mu_i f \omega_\xi^{-1} \tag{4.2}$$

for $f \in \mathrm{Hom}_S(F_\xi, U_i)$. Likewise, ω_ξ and μ_i^{-1} induce the I_0-isomorphism

$$\varphi^*: \mathrm{Hom}_S{}^*(F_\xi, U_i) \to \mathrm{Hom}_{S'}{}^*(F'_\xi, U'_i),$$

for $\xi, i = 1, 2, \cdots, k$, given by

$$\varphi^* f^* = \omega_\xi f^* \mu_i^{-1} \tag{4.3}$$

where $f^* \in \mathrm{Hom}_S{}^*(F_\xi, U_i)$. Thus

$$(\varphi^* f^*)(\varphi f) = \omega_\xi f^* f \omega_\xi^{-1} = (f^* f)^{I_0}. \tag{4.4}$$

Two I_0-isomorphisms such as φ and φ^* which satisfy (4.4) will be said to be *contragredient.*

Also ω_ξ, ω_η^{-1}, and J induce (I_0, I_0)-isomorphisms $\theta: H_{\xi\eta} \to H'_{\xi\eta}$ defined for each pair $\xi, \eta = 1, 2, \cdots, k$ by

$$\theta\psi(\alpha') = \omega_\xi \psi(\alpha'^J)\omega_\eta^{-1} \tag{4.5}$$

where $\psi \in H_{\xi\eta}$ and $\alpha' \in R$. Let $f^* \in \mathrm{Hom}_S{}^*(F_\xi, U_i)$, $f \in \mathrm{Hom}_S(F_\eta, U_i)$, and $\alpha' \in R'$. Using (4.2) and (4.3) we obtain

$$\omega_\xi f^* \alpha'^J f \omega_\eta^{-1} = \omega_\xi f^* \mu_i^{-1} \alpha' \mu_i f \omega_\eta^{-1} = (\varphi^* f^*)\alpha'(\varphi f) = \psi'_i[\varphi^* f, \varphi f](\alpha'). \tag{4.6}$$

Therefore, from (4.5) and (4.6)

$$\psi_i[f^*, f] = \psi'_i[\varphi^* f^*, \varphi f]. \tag{4.7}$$

A *(left) structure* $\Sigma(R, S)$ for a cleft ring R with cleaving $R = S \oplus N$ is the set $\{|\psi_i| \mid i = 1, 2, \cdots, k\}$ of structures of the principal indecomposable modules U_i, $i = 1, 2, \cdots, k$, respectively. The structures $|\psi_i|$ are called the *principal structures* of R.

As above, let R and R' be two cleft rings with cleavings (4.1). Let $I_0: S \to S'$ be an isomorphism. Let $\Sigma(R, S)$ and $\Sigma(R', S')$ be the left structures of R and R'. Then $\Sigma(R, S)$ and $\Sigma(R', S')$ are said to be *I_0-conformal* if there exist contragredient[6] I_0-isomorphims φ and φ^*

$$\begin{aligned} &\varphi: \mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_{S'}(F'_\xi, U'_i), && \xi, i = 1, 2, \cdots, k, \\ &\varphi^*: \mathrm{Hom}_S{}^*(F_\xi, U_i) \to \mathrm{Hom}_{S'}{}^*(F'_\xi, U'_i), && \xi, i = 1, 2, \cdots, k, \end{aligned} \tag{4.8}$$

and (I_0, I_0)-isomorphisms θ defined for $\xi, \eta = 1, 2, \cdots k$

$$\theta: H_{\xi\eta} \to H'_{\xi\eta} \tag{4.9}$$

such that

$$\theta\psi_i[f^*, f] = \psi_i'[\varphi^*f^*, \varphi f], \qquad i = 1, 2, \cdots, k, \tag{4.10}$$

where $| \psi_i |$ is the principal structure of R and $| \psi_i' |$ is the corresponding principal structure of R', $f^* \in \mathrm{Hom}_S^*(F_\xi, U_i)$, and $f \in \mathrm{Hom}_S(F_\eta, U_i)$, $\xi, \eta = 1, 2, \cdots, k$.

THEOREM 3. *Let R and R' be cleft rings with cleavings* (4.1). *A necessary and sufficient condition for an isomorphism $I_0: S \to S'$ to be extendable to an isomorphism $I: R \to R'$ is that the left structures $\Sigma(R, S)$ and $\Sigma(R', S')$ of R and R', respectively, be I_0-conformal.*

Proof. We have proved the necessity. We now prove the sufficiency. According to Theorem 2 there exist (S_ξ, S_η)-isomorphisms $J_{\xi\eta}: R'_{\xi\eta} \to R_{\xi\eta}$ induced by the isomorphism θ of (4.9). Clearly the homomorphism $J_{\xi\eta}$ determines an (S, S)-isomorphism $J: R' \to R$. We wish to show that J is a ring isomorphism.

Again by Theorem 2, $\theta\psi(\alpha') = \omega_\xi \, \psi(\alpha'^{J_{\xi\eta}})\omega_\eta^{-1}$ for $\alpha' \in R'_{\xi\eta}$. Thus,

$$\theta\psi_i[f^*, f](\alpha') = \omega_\xi \, \psi_i[f^*, f](\alpha'^J)\omega_\eta^{-1} \tag{4.11}$$

for $f^* \in \mathrm{Hom}_S^*(F_\xi, U_i)$, $f \in \mathrm{Hom}_S(F_\eta, U_i)$, and $\alpha' \in R'$.

Let $\{f_\mu^*, f_\mu \mid \mu = 1, 2, \cdots, t\}$ be a direct family representing U_i as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$. Then $\{\varphi^*f_\mu^*, \varphi f_\mu \mid \mu = 1, 2, \cdots, t\}$ represent U_i' as the S'-direct sum of the modules $F_1', F_2', \cdots, F_k'$ by virtue of the fact φ and φ^* are contragredient isomorphisms. Now for α' and $\beta' \in R'$, $f^* \in \mathrm{Hom}_S^*(F_\xi, U_i)$, and $f \in \mathrm{Hom}_S(F_\eta, U_i)$, we have, using (2.6), (4.10), and (4.11),

$$\begin{aligned}
\omega_\xi \, \psi_i[f^*, f]((\alpha'\beta')^J)\omega_\eta^{-1} &= \psi_i'[\varphi^*f^*, \varphi f](\alpha'\beta') \\
&= \textstyle\sum_{\mu=1}^t \psi_i'[\varphi^*f^*, \varphi f_\mu](\alpha')\psi_i'[\varphi^*f_\mu^*, \varphi f](\beta') \\
&= \textstyle\sum_{\mu=1}^t \omega_\xi \, \psi_i[f^*, f_\mu](\alpha'^J)\psi_i[f_\mu^*, f](\beta'^J)\omega_\eta^{-1} \\
&= \omega_\xi \, \psi_i[f^*, f](\alpha'^J, \beta'^J)\omega_\eta^{-1}.
\end{aligned} \tag{4.12}$$

Therefore

$$\psi_i[f^*, f]((\alpha'\beta')^J) = \psi_i[f^*, f](\alpha'^J, \beta'^J). \tag{4.13}$$

But (4.13) holds for a generating element f of U_i in $\mathrm{Hom}_S(F_i, U_i)$ and any $f^* \in \mathrm{Hom}_S^*(F_\xi, U_i)$. Hence by Proposition 3.6,

$$\psi((\alpha'\beta')^J) = \psi(\alpha'^J\beta'^J)$$

for all $\psi \in H_{\xi i}$. Thus $e_\xi((\alpha'\beta')^J)e_i = e_\xi \, \alpha'^J\beta'^J e_i$, and this must hold for all $\xi, i = 1, 2, \cdots, k$. Hence

$$(\alpha'\beta')^J = \alpha'^J\beta'^J$$

as $1 = \sum_{\xi=1}^k e_\xi$. Then $I = J^{-1}$ is the desired extension of I_0.

We note that if R/N is a separable algebra, and if $R = S \oplus N$ and $R = S' \oplus N$ are two cleavings of R, then there exists an inner automorphism of R mapping S onto S' by Malcev's Theorem [4 or 8]. Thus the structures of R given by any two cleavings are conformal.

4B. Anti-isomorphisms

We here treat the problem of extending an anti-isomorphism of the semi-simple components of two cleft rings. Commutative algebras will afford an interesting interpretation of this theory.

We will adopt the convention of denoting homomorphisms of right modules as left operators. Also we introduce the opposite ring R^0 to a ring R. Then every right R-module X is a left R^0-module, and we have a standard anti-isomorphism such that to $\alpha \in R$ corresponds $\alpha^0 \in R^0$ given by $\alpha^0 x = x\alpha$ for $\alpha \in R$. Similarly a left R-module is a right R^0 module, and we have that $R^{00} = R$ and $\alpha \rightarrow \alpha^{00}$ is the identity mapping.

We let $h:X \rightarrow X'$ be a homomorphism of right R-modules. Then for $x \in X$ and $\alpha \in R$, $h(x\alpha) = h(x)\alpha$ and $h(\alpha^0 x) = \alpha^0(hx)$. Hence h is a homomorphism of left R^0-modules, and conversely. That is, $\mathrm{Hom}_R(X, X') = \mathrm{Hom}_{R^0}(X, X')$. Let $I:R \rightarrow R'$ be an anti-isomorphism. Then the composite mapping I^0 is an isomorphism $I^0:R \rightarrow R'^0$. An I-homomorphism $h:X \rightarrow X'$ of a left R-module X into a right R'-module X' defined by $h(\alpha x) = (hx)\alpha^I$ is also an I^0-homomorphism of the left R-module X into the left R'^0-module X' defined by $h(\alpha x) = \alpha^{I^0}(hx)$. Conversely, every I^0-homomorphism is an I-homomorphism.

Let F be a left [right] irreducible R-module; then F is a right [left] irreducible R^0-module. Let U be a left [right] principal indecomposable R-module; then U is a right [left] principal indecomposable R^0-module because U is easily seen to be both an indecomposable R^0-module and a projective R^0-module.

Let R be a cleft ring with cleaving $R = S \oplus N$ as before. Let X be a right R-module. A *structure* $|\psi|$ for X is now defined by means of the homomorphisms

$$\psi[g^*, g](\alpha) = g^*\alpha^0 g \tag{4.14}$$

for $g^* \in \mathrm{Hom}_S^*(F_i, X)$ and $g \in \mathrm{Hom}_S(F_j, X)$, $i, j = 1, 2, \cdots, k$. Thus $|\psi|$ is also a structure of the left R^0-module; because of the above remarks, we have $\mathrm{Hom}_S^*(F_j, X) = \mathrm{Hom}_{S^0}^*(F_j, X)$ and $\mathrm{Hom}_S(F_i, X) = \mathrm{Hom}_{S^0}(F_i, X)$.

Likewise, $\mathrm{Hom}_K(F'_i, F'_j)$ is an (S'_i, S'_j)-module where $S'_i = S_i^{I^0}$ and $S'_j = S_j^{I^0}$. But then it is also an (S'^0_j, S'^0_i)-module. For this reason we shall equate

$$H'_{ij} = \mathrm{Hom}_{(S_{i'}, S_{j'})}(R'_{ij}, \mathrm{Hom}_K(F'_i, F'_j) \tag{4.15}$$

with

$$H'^0_{ji} = \mathrm{Hom}_{(S_{j'^0}, S_{i'^0})}(R'^0_{ij}, \mathrm{Hom}_K(F'_i, F'_j)). \tag{4.16}$$

A *right structure* $\Sigma'(R, S)$ for a cleft ring R with minimum condition with cleaving $R = S \oplus N$ is the set of structures $|\psi_i|$, $i = 1, 2, \cdots, k$, of a complete set of right indecomposable modules. Thus a right structure is also a left structure $\Sigma(R^0, S^0)$ for the opposite ring R^0 with cleaving $R^0 = S^0 \oplus N^0$.

Let R and R' be rings with minimum condition which have cleavings

$$R = S \oplus N \quad \text{and} \quad R' = S' \oplus N'. \tag{4.17}$$

Let $I_0: S \rightarrow S'$ be an anti-isomorphism. We say that the left structure $\Sigma(R, S)$ is *I_0-conformal* to the right structure $\Sigma'(R', S')$ if $\Sigma(R, S)$ is $(I_0)^0$-conformal to the left structure $\Sigma(R'^0, S'^0)$ of R'.

COROLLARY 4.1. *Let R and R' be cleft rings with minimum condition with cleavings* (4.17). *Then a necessary and sufficient condition for an anti-isomorphism $I_0: S \rightarrow S'$ to be extendable to R is that the left structure $\Sigma(R, S)$ be I_0-conformal to the right structure $\Sigma'(R', S')$.*

Proof. A necessary and sufficient condition for $I_0: S \rightarrow S'$ to be extendable is that $(I_0)^0: S \rightarrow S'^0$ be extendable. But this is equivalent to having $\Sigma(R, S)$ and $\Sigma(R'^0, S'^0)$ $(I_0)^0$-conformal. By definition, this is equivalent to having $\Sigma(R, S)$ and $\Sigma'(R', S')$ I_0-conformal.

We remark that the extension I of I_0 is characterized by

$$\theta\psi(\alpha') = \omega_\xi \psi(\alpha'^J)\omega_\eta^{-1}, \tag{4.18}$$

where $J = I^{-1}$, θ is given by (4.5), $\omega_\xi: F_\xi \rightarrow F'_\xi$, and $\omega_\eta: F_\eta \rightarrow F'_\eta$ are I_0-isomorphisms and $\psi \in H_{\xi\eta}$.

4C. Dual modules and anti-isomorphisms

Let X be a left R-module. We form the dual module $X^* = \mathrm{Hom}_K(X, K)$. In case $[X:K] < \infty$, a satisfactory duality theory exists because X^{**} may and will be identified with X. Therefore, to assure this, we assume that $[R:K] < \infty$. Hence $[X:K] < \infty$ as X is assumed to have a finite composition series.

As usual. we denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$. Then X^* is a right R-module where

$$\langle \alpha_R\, x^*, x\rangle = \langle x^*\alpha, x\rangle = \langle x^*, \alpha x\rangle = \langle x^*, \alpha_L\, x\rangle.$$

Let $f \in \mathrm{Hom}_R(X, X')$, where X and X' are, say, left R-modules. Denote by $t: \mathrm{Hom}_R(X, X') \rightarrow \mathrm{Hom}_R(X'^*, X^*)$ the transpose mapping defined by $\langle {}^t f x'^*, x\rangle = \langle x'^*, fx\rangle$ for $x \in X$ and $x'^* \in X'^*$. Thus ${}^t\alpha_L = \alpha_R$ and ${}^t\alpha_R = \alpha_L$.

Let F be an irreducible left [right] R-module. Then F^* must be an irreducible right [left] module, for otherwise $F = F^{**}$ would be reducible. Let L be the endomorphism sfield for F. Then F is a left L-module and a right L^0-module. We identify the endomorphism sfield L^* of F^* with L so that F^*

is a right L-module and a left L^0-module. We have for $x^* \epsilon F^*$, $x \epsilon F$, and $\sigma \epsilon L$

$$\langle x^*, \sigma x\rangle = \langle x^*\sigma, x\rangle = \langle \sigma^0 x^*, x\rangle = \langle {}^t\sigma x^*, x\rangle$$

where σ^0 is in L^0, the opposite sfield to L. Thus the transpose mapping may be considered as the anti-isomorphism $\sigma \to \sigma^0$ of L onto L^0.

Consider now F_i to be a left irreducible R-module and X to be an arbitrary R-module. Assume R to be a cleft algebra with cleaving $R = S \oplus N$. Form the module $\mathrm{Hom}_S(F_i, X) = \mathrm{Hom}_{S^0}(F_i, X)$. This is a (K_i^0, K_i)-module, where K_i is the endomorphism sfield of F_i. Then the transpose mapping $t:\mathrm{Hom}_S(F_i, X) \to \mathrm{Hom}_S(X^*, F_i^*) = \mathrm{Hom}_S{}^*(F_i^*, X^*) = \mathrm{Hom}_{S^0}{}^*(F_i^*, X^*)$ is an isomorphism of additive groups. But also $\mathrm{Hom}_S{}^*(F_i^*, X^*)$ is a (K_i^0, K_i)-module just as is F_i^*. Thus if $f \epsilon \mathrm{Hom}_S(F_i, X)$, $\sigma \epsilon K_i$, and $\tau \epsilon K_i^0$, ${}^t(\tau f\sigma) = ({}^t\sigma)({}^tf)({}^t\tau) = \sigma^0({}^tf)\tau^0 = \tau({}^tf)\sigma$. Thus we say that $t:\mathrm{Hom}_S(F_i, X) \to \mathrm{Hom}_S{}^*(F_i^*, X^*)$ is a (K_i^0, K_i)-isomorphism of modules or, sometimes, a K_i-isomorphism, or a K_i^0-isomorphism.

Likewise, if F_i and F_j are irreducible left R-modules, $\mathrm{Hom}_K(F_i, F_j)$ is a (K_j, K_i)-module and $\mathrm{Hom}_K(F_j^*, F_i^*)$ is a (K_i^0, K_j^0)-module and, therefore, also a (K_j, K_i)-module. Then $t:\mathrm{Hom}_K(F_i, F_j) \to \mathrm{Hom}_K(F_j^*, F_i^*)$ may be verified to be a (K_j, K_i)-isomorphism as well as a (K_i^0, K_j^0)-isomorphism.

Now we suppose that R and R' are cleft algebras with cleavings (4.17). As before, associate the left modules F_i, U_i, and the left structural modules H_{ji}, $i, j = 1, 2, \cdots, k$, with R. Also associate F_i', U_i', and the right structural modules H_{ij}' with R'. Of course, F_i', U_i' are right R'^0-modules, and H_{ij}' is a left R'^0-module. Since $\mathrm{Hom}_K(F_i', F_j')$ is a (K_i', K_j')-module, the same is true for

$$H_{ij}' = \mathrm{Hom}_{(S', S')}(R', \mathrm{Hom}_K(F_i', F_j')) = \mathrm{Hom}_{(S_i', S_j')}(R_{ij}', \mathrm{Hom}_K(F_i', F_j')).$$

Then the transpose isomorphism $t:\mathrm{Hom}_K(F_i', F_j') \to \mathrm{Hom}_K(F_j'^*, F_i'^*)$ induces a (K_i', K_j')-isomorphism, which we again denote by t, such that

$$t:H_{ij}' \to H_{ij}'^* = \mathrm{Hom}_{(S,S)}(R', \mathrm{Hom}_K(F_j'^*, F_i'^*))$$

$$= \mathrm{Hom}_{(S_i', S_j')}(R_{ij}', \mathrm{Hom}_K(F_j'^*, F_i'^*)).$$

Now let $I_0:S \to S'$ be an anti-isomorphism. Then if

$$\varphi:\mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S(F_\xi', U_i')$$

is an I_0-isomorphism, of a right K_ξ-module onto a left K_ξ'-module,

$$t\varphi:\mathrm{Hom}_S(F_\xi, U_i') \to \mathrm{Hom}_S{}^*(F_\xi'^*, U_i'^*)$$

is also an I_0-isomorphism of a right K_ξ-module onto a left K_ξ'-module. Conversely, if $\varphi':\mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S{}^*(F_\xi^*, U_i'^*)$ is a given I_0-isomorphism, then $t\varphi':\mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S(F_\xi', U_i')$ is also an I_0-isomorphism. Now also $\varphi^*:\mathrm{Hom}_S{}^*(F_\xi, U_i) \to \mathrm{Hom}_S{}^*(F'_\xi, U'_i)$ is contragredient to φ' if, and only if, $t\varphi^*:\mathrm{Hom}_S{}^*(F_\xi, U_i) \to \mathrm{Hom}_S(F'^*_\xi, U'^*_i)$ is contragredient to $t\varphi'$.

Likewise, if $\theta:H_{ji}\to H'_{ij}$ is an (I_0, I_0)-isomorphism, then $\theta^* = t\theta:H_{ji}\to H'^*_{ij}$ is also an (I_0, I_0)-isomorphism, and conversely.

Therefore, the conditions of Corollary 4.1 translate into the following proposition.

PROPOSITION 4.2. *Let R and R' be cleft algebras of finite rank over a field K with cleavings*

$$R = S \oplus N, \qquad R' = S' \oplus N'$$

as above. Then a necessary and sufficient condition for an anti-isomorphism $I_0:S \to S'$ to be extendable to an anti-isomorphism $I:R \to R'$ is that there exist contragredient I_0-isomorphisms φ' and φ'^* for ξ, $i = 1, 2, \cdots, k$ such that

$$\varphi':\mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S{}^*(F'^*_\xi, U'^*_i),$$
$$(4.19)\qquad \varphi'^*:\mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S(F'^*_\xi, U'^*_i),$$

and an (I_0, I_0)-isomorphism, for ξ, $\eta = 1, 2, \cdots, k$

$$(4.20)\qquad \theta^*:H_{\xi\eta} \to H'^*_{\xi\eta}$$

such that

$$(4.21)\qquad \theta^*\psi_i[f^*, f] = \psi'^*_i[\varphi' f, \varphi'^* f^*],$$

where $|\psi_i|$ is the structure of U_i, and $|\psi'^*_i|$ is the structure of U'^*_i, $f^* \in \mathrm{Hom}_S{}^*(F_\xi, U_i)$ and $f \in \mathrm{Hom}_S(F_\eta, U_i)$.

Proof. We need only show that (4.21) is equivalent to (4.10). Hence suppose that (4.10) holds. Then for $\alpha' \in R'$, $f^* \in \mathrm{Hom}_S{}^*(F_\xi, U_i)$ and $f \in \mathrm{Hom}_S(F_\eta, U_i)$, we have that

$$\theta\psi_i[f^*, f](\alpha') = \psi'_i[\varphi^* f^*, \varphi f](\alpha') = (\varphi^* f^*)(\alpha'_L)(\varphi f).$$

Thus

$$\theta^*\psi_i[f^*, f](\alpha') = {}^t[(\varphi^* f^*)(\alpha'_R)(\varphi f)] = {}^t(\varphi f)\,{}^t(\alpha'_R)\,{}^t(\varphi^* f^*)$$
$$= (\varphi' f)(\alpha'_L)(\varphi'^* f^*) = \psi'^*_i[\varphi' f, \varphi'^* f^*](\alpha').$$

The argument may be reversed to show also that (4.21) implies (4.10). This completes the proof.

We remark that if $R' = R$ and $S' = S$, we may obtain a condition for extending an anti-automorphism. Then also F'^*_ξ is a left R-module. Hence there exists an R-isomorphism $\gamma_\xi:F'^*_\xi \to F'_\xi$ and $(1', 2', \cdots, k')$ will be a permutation of the sequence $(1, 2, \cdots, k)$.

Furthermore, $V_{i'} = U'^*_i$ is an injective R-module [3] with unique minimal irreducible submodule $(U'_i/NU'_i)^*$. This is an irreducible module isomorphic to F'^*_i and hence is isomorphic to F'_i. We may then reinterpret the conditions (4.19), (4.20), and (4.21) by replacing φ by $\lambda = \varphi\gamma_\xi^{-1}$, φ^* by $\lambda^* = \gamma_\xi\varphi'^*$, and

θ^* by η, where $\eta\psi = \gamma_\xi(\theta^*\psi)\gamma_\eta^{-1}$ for $\psi \in H_{\xi\eta}$. Then we obtain that

$$\begin{aligned} &\lambda: \mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S^*(F_{\xi'}, V_{i'}) \\ &\lambda^*: \mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S^*(F_{\xi'}, V_{i'}) \end{aligned} \tag{4.19a}$$

are contragredient I_0-isomorphisms. Furthermore,

$$\eta: H_{ji} \to H_{i'j'} \tag{4.20a}$$

is an (I_0, I_0)-isomorphism. One may verify that (4.21) is equivalent to

$$\eta\psi_i[f^*, f] = \psi_i'^*[\lambda f, \lambda^* f^*]. \tag{4.21a}$$

4D. Characterization of commutative algebras

THEOREM 4. *Let R be a cleft algebra with $[R:K] < \infty$,[7] and with cleaving $R = S \oplus N$. Suppose that S is contained in the center of R. Let U_1, $U_2, \cdots, U_k$ be the left principal indecomposable R-modules, and let $V_i = U_i'^*$, $i = 1, 2, \cdots, k$. Then a necessary and sufficient condition for R to be commutative is that*

(i) *the structural module $H_{\xi\eta} = 0$ for $\xi \neq \eta$;*

(ii) *there exist contragredient K_i-isomorphisms λ and λ^* for $i = 1, 2, \cdots, k$ such that*

$$\begin{aligned} &\lambda: \mathrm{Hom}_S(F_i, U_i) \to \mathrm{Hom}_S^*(F_i, V_i), \\ &\lambda^*: \mathrm{Hom}_S^*(F_i, U_i) \to \mathrm{Hom}_S(F_i, V_i); \end{aligned} \tag{4.22}$$

and

(iii)

$$\psi_i[f^*, f] = \zeta_i[\lambda f, \lambda^* f^*], \qquad i = 1, 2, \cdots, k, \tag{4.23}$$

where $| \psi_i |$ is the structure of U_i and $| \zeta_i |$ is the structure of V_i.

Proof. First, we note that $H_{\xi\eta} = \mathrm{Hom}_{(S_\xi, S_\eta)}(R_{\xi\eta}, \mathrm{Hom}_K(F_\xi, F_\eta))$. Hence $H_{\xi\eta} = 0$ if and only if $R_{\xi\eta} = 0$. But then $R = \oplus_{i=1}^k R_i$ where $R_i = R_{ii} = e_i R e_i$ is a subideal of R. Thus $R_i \supseteqq S_i$, $i = 1, 2, \cdots, k$. Hence $R_i = S_i \oplus N_i$ where $N_i = e_i N e_i$ is the radical of R_i. Thus (i) is equivalent to having R be the direct sum of primary rings R_i. Since every commutative ring is a direct sum of primary rings, we need only show that (ii) and (iii) are equivalent to having each R_i a commutative ring.

Let now R be a primary cleft ring $R = S \oplus N$ where S is a field in the center of R. Thus R is an algebra over S as well as over K. Furthermore, if A is a left R-module, it is also a right S-module, and we have that [5, p. 6]

$$A^* = \mathrm{Hom}_K(A, K) \cong \mathrm{Hom}_K(A \otimes_S S, K) \cong \mathrm{Hom}_S(A, \mathrm{Hom}_K(S, K));$$

$$A^* \cong \mathrm{Hom}_S(A, S).$$

[7] Of course, if R is indecomposable so that $R = S \oplus N$ where S is a subfield of R contained in the center of R, then we can take $K = S$, and the assumption that $[R:K] < \infty$ will follow from the fact that R possesses the minimum condition.

In this case there is one simple ideal component $S_1 = S$ of S, one irreducible module $F = F_1$, one principal indecomposable module $U = U_1$, and one structural module $H = H_{11}$. We have the dimension $[F:S] = 1$. Hence we may identify F and S as (S, S)-modules. Also since S is commutative, we may identify S and S^0. Thus $F = F'$ where F' is the irreducible S^0-module defined from F.

Let $\delta: S \to F^*$ be defined by $\delta(\sigma) = \sigma x_0$ where x_0 is a fixed vector of S. Now δ is a correlation. Because S is commutative, δ is an S-isomorphism. Thus δ defines a bilinear function $f: F \times F \to S$ by setting $f(x, y) = \langle \delta x, y \rangle$ for $x, y \in F$. Then we have, for $h \in \mathrm{Hom}_S(F, F) = S$,

$$f(\delta^{-1}({}^t h)\delta x, y) = f(x, hy) = hf(x, y) = f(hx, y).$$

Thus ${}^t h = \delta h \delta^{-1}$.

Now we refer to formulas (4.19a), (4.20a), and (4.21a). Observe that in our present case, $H = H_{11} = H_{1'1'}$. Hence by setting $\eta = 1$, the identity isomorphism, we immediately obtain (4.19a), (4.20a), (4.21a) from (4.22) and (4.23). Thus we see that (4.22) and (4.23) imply that the identity automorphism can be extended to an anti-automorphism of R. We wish to see that this anti-automorphism is the identity automorphism. Using the S-isomorphism $\gamma = \delta^{-1}$ where $\gamma: F^* \to F$, we obtain the isomorphisms φ' and φ'^* of Proposition 4.2 by setting $\varphi' = \lambda\gamma$ and $\varphi'^* = \gamma^{-1}\lambda^*$. Likewise, η induces the (K, K)-isomorphism $\theta^*\psi$ of (4.20) when we set $\theta^*\psi = \gamma^{-1}(\eta\psi)\gamma = {}^t(\eta\psi) = {}^t\psi$. But $\theta^* = {}^t(\theta\psi)$ where θ is the (K, K)-isomorphism of (4.18). Thus ${}^t(\theta\psi) = {}^t\psi$ or $\theta = 1$. Hence (4.18) becomes

$$\psi(\alpha') = \omega\psi(\alpha'^J)\omega^{-1} \tag{4.24}$$

where $\omega: F \to F'$ is a K-isomorphism of F onto $F' = F$. But by Corollary 3.4, we may assume that $\omega = 1$. Hence $\psi(\alpha') = \psi(\alpha'^J)$ and $\alpha' = \alpha'^J$. Thus J is the identity automorphism. This means that R must be commutative.

Conversely, if R is commutative, the identity automorphism of R is an anti-automorphism. This means that (4.18) may be replaced by

$$\psi(\alpha') = \psi(\alpha'^J) \tag{4.25}$$

with $\theta = 1$. But then we may establish (4.19), (4.20), and (4.21) with $\theta^* = t$. Again choosing $\gamma = \delta^{-1}$, we obtain (4.19a), (4.20a), and (4.21a) with $\eta = 1$. From this, (4.22) and (4.23) follow directly. This completes the proof.

Bibliography

1. E. Artin, C. Nesbitt, and R. Thrall, *Rings with minimum condition*, Ann Arbor, University of Michigan, 1944.
2. R. Brauer, *Some remarks on associative rings and algebras*, Report of a Conference on Linear Algebras, National Academy of Sciences-National Research Council, Publication 502, Washington, 1957, pp. 4–11.
3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton, 1956.

4. C. W. Curtis, *The structure of non-semisimple algebras*, Duke Math. J., vol. 21 (1954), pp. 79–85.
5. S. Eilenberg and T. Nakayama, *On the dimension of modules and algebras, II (Frobenius algebras and quasi-Frobenius rings)*, Nagoya Math J., vol. 9 (1955), pp. 1–16.
6. G. Hochschild, *Double vector spaces over division rings*, Amer. J. Math., vol. 71 (1949), pp. 443–460.
7. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, 1956.
8. A. Malcev, *On the representation of an algebra as the direct sum of the radical and a semisimple subalgebra*, C. R. (Doklady) Acad. Sci. URSS (N.S.), vol. 36 (1942), pp. 42–45.
9. C. Nesbitt, *On the regular representations of algebras*, Ann. of Math. (2), vol. 39 (1938), pp. 634–658.
10. W. M. Scott, *On matrix algebras over an algebraically closed field*, Ann. of Math. (2), vol. 43 (1942), pp. 147–160.
11. B. Vinograde, *Cleft rings*, Trans. Amer. Math. Soc., vol. 56 (1944), pp. 494–507.

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