Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2009, Article ID 279818, 33 pages
doi:10.1155/2009/279818

*Research Article*

# Zero Diffusion-Dispersion-Smoothing Limits for a Scalar Conservation Law with Discontinuous Flux Function

**H. Holden,[1,2] K. H. Karlsen,[3] and D. Mitrovic[1]**

[1] *Department of Mathematical Sciences, Norwegian University of Science and Technology, Alfred Getz vei 1, 7491 Trondheim, Norway*
[2] *Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, 0316 Oslo, Norway*
[3] *Department of Mathematics, Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, 0316 Oslo, Norway*

Correspondence should be addressed to H. Holden, holden@math.ntnu.no

Received 2 April 2009; Revised 24 August 2009; Accepted 24 September 2009

Recommended by Philippe G. LeFloch

We consider multidimensional conservation laws with discontinuous flux, which are regularized with vanishing diffusion and dispersion terms and with smoothing of the flux discontinuities. We use the approach of $H$-measures to investigate the zero diffusion-dispersion-smoothing limit.

Copyright © 2009 H. Holden et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## 1. Introduction

We consider the convergence of smooth solutions $u = u_\varepsilon(t,x)$ with $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ of the nonlinear partial differential equation

$$\partial_t u + \operatorname{div}_x f_\varrho(t,x,u) = \varepsilon \operatorname{div}_x b(\nabla u) + \delta \sum_{j=1}^{d} \partial^3_{x_j x_j x_j} u \tag{1.1}$$

as $\varepsilon \to 0$ and $\delta = \delta(\varepsilon), \varrho = \varrho(\varepsilon) \to 0$. Here $f \in C(\mathbf{R}; BV(\mathbf{R}^+_t \times \mathbf{R}^d_x))$ is the Caratheodory flux vector such that

$$\max_{|u|\le l} \left| f_\varrho(t,x,u) - f(t,x,u) \right| \longrightarrow 0, \quad \varrho \longrightarrow 0, \quad \text{in } L^p_{\mathrm{loc}}\left(\mathbf{R}^+ \times \mathbf{R}^d\right), \tag{1.2}$$

for $p > 2$ and every $l > 0$. The aim is to show convergence to a weak solution of the corresponding hyperbolic conservation law:

$$\partial_t u + \mathrm{div}_x f(t,x,u) = 0, \quad u = u(t,x),\ x \in \mathbf{R}^d,\ t \geq 0. \tag{1.3}$$

We refer to this problem as the zero diffusion-dispersion-smoothing limit.

In the case when the flux $f$ is at least Lipschitz continuous, it is well known that the Cauchy problem corresponding to (1.3) has a unique admissible entropy solution in the sense of Kružhkov [1] (or measure valued solution in the sense of DiPerna [2]). The situation is more complicated when the flux is discontinuous and it has been the subject of intensive investigations in the recent years (see, e.g., [3] and references therein). The one-dimensional case of the problem is widely investigated using several approaches (numerical techniques [3, 4], compensated compactness [5, 6], and kinetic approach [7, 8]). In the multidimensional case there are only a few results concerning existence of a weak solution. In [9] existence is obtained by a two-dimensional variant of compensated compactness, while in [10] the approach of $H$-measures [11, 12] is used for the case of arbitrary space dimensions. Still, many open questions remain such as the uniqueness and stability of solutions.

A problem that has not yet been studied in the context of conservation laws with discontinuous flux, and which is the topic of the present paper, is that of zero diffusion-dispersion limits. When the flux is independent of the spatial and temporal positions, the study of zero diffusion-dispersion limits was initiated in [13] and further addressed in numerous works by LeFloch et al. (e.g., [14–17]). The compensated compactness method is the basic tool used in the one-dimensional situation for the so-called limiting case in which the diffusion and dispersion parameters are in an appropriate balance. On the other hand, when diffusion dominates dispersion, the notion of measure valued solutions [2, 18] is used. More recently, in [19] the limiting case has also been analyzed using the kinetic approach and velocity averaging [20].

The remaining part of this paper is organized as follows. In Section 2 we collect some basic a priori estimates for smooth solutions of (1.1). In Section 3 we look into the diffusion-dispersion-smoothing limit for multidimensional conservation laws with a flux vector which is discontinuous with respect to spatial variable. In doing so we rely on the a priori estimates from the previous section in combination with Panov's H-measures approach [10]. Finally, in Section 4 we restrict ourselves to the one-dimensional case for which we obtain slightly stronger results using the compensated compactness method.

## 2. A priori Inequalities

Assume that the flux $f$ in (1.1) is smooth in all variables. Consider a sequence $(u_{\varepsilon,\delta})_{\varepsilon,\delta}$ of solutions of

$$\begin{gathered} \partial_t u + \mathrm{div}_x f(t,x,u) = \varepsilon \mathrm{div}_x b(\nabla u) + \delta \sum_{j=1}^{d} \partial^3_{x_j x_j x_j} u, \\ u(x,0) = u_0(x), \quad x \in \mathbf{R}^d. \end{gathered} \tag{2.1}$$

We assume that $(u_{\varepsilon,\delta})_{\varepsilon,\delta}$ has enough regularity so that all formal computations below are correct. So, following Schonbek [13], we assume that for every $\varepsilon, \delta > 0$ we have $u_{\varepsilon,\delta} \in L^\infty([0,T]; H^4(\mathbf{R}^d))$.

Later on, we will assume that the initial data $u_0$ depends on $\varepsilon$. In this section, we will determine a priori inequalities for the solutions of problem (2.1).

To simplify the notation we will write $u_\varepsilon$ instead of $u_{\varepsilon,\delta}$.

We will need the following assumptions on the diffusion term $b(\lambda) = (b_1(\lambda), \dots, b_n(\lambda))$.

(H1) For some positive constants $C_1, C_2$ we have

$$C_1|\lambda|^2 \le \lambda \cdot b(\lambda) \le C_2|\lambda|^2 \quad \forall\, \lambda \in \mathbf{R}^d. \tag{2.2}$$

(H2) The gradient matrix $Db(\lambda)$ is a positive definite matrix, uniformly in $\lambda \in \mathbf{R}^d$, that is, for every $\lambda, \varrho \in \mathbf{R}^d$, there exists a positive constant $C_3$ such that we have

$$\varrho^T Db(\lambda)\varrho \ge C_3 |\varrho|^2. \tag{2.3}$$

We use the following notation:

$$\left|D^2 u\right|^2 = \sum_{i,k=1}^{d} \left|\partial^2_{x_i x_k} u\right|^2. \tag{2.4}$$

In the sequel, for a vector valued function $g = (g_1, \dots, g_d)$ defined on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$, we denote

$$|g|^2 = \sum_{i=1}^{d} |g_i|^2. \tag{2.5}$$

The partial derivative $\partial_{x_i}$ in the point $(t, x, u)$, where $u$ possibly depends on $(t, x)$, is defined by the formula

$$\partial_{x_i} g(t, x, u(t, x)) = \left(D_{x_i} g(t, x, \lambda)\right)\big|_{\lambda = u(t,x)}. \tag{2.6}$$

In particular, the total derivative $D_{x_i}$ and the partial derivative $\partial_{x_i}$ are connected by the identity

$$D_{x_i} g(t, x, u) = \partial_{x_i} g(t, x, u) + \partial_u g(t, x, u)\partial_{x_i} u. \tag{2.7}$$

Finally we use

$$\begin{gathered} \operatorname{div}_x g(t, x, u) = \sum_{i=1}^{d} D_{x_i} g_i(t, x, u), \quad g = (g_1, \dots, g_d), \\ \Delta_x q(t, x, u) = \sum_{i=1}^{d} D^2_{x_i x_i} q(t, x, u), \quad q \in C^2\left(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}\right). \end{gathered} \tag{2.8}$$

With the previous conventions, we introduce the following assumption on the flux vector $f$.

(H3) The growth of the velocity variable $u$ and the spatial derivative of the flux $f$ are such that for some $C, \alpha > 0$, $p \geq 1$, and every $l > 0$, we have

$$\max_{|\lambda|<l} \left| f_i(t,x,\lambda) \right| \in L^p\left(\mathbf{R}^+ \times \mathbf{R}^d\right), \quad i = 1, \dots, d,$$
$$\sum_{i=1}^{d} \left| \partial_u f_i(t,x,u) \right| \leq C, \qquad \sum_{i,j=1}^{d} \left| \partial_{x_i} f_j(t,x,u) \right| \leq \frac{\mu(t,x)}{1+|u|^{1+\alpha}}, \tag{2.9}$$

where $\mu \in \mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d)$ is a bounded measure (and, accordingly, the above inequality is understood in the sense of measures).

Now, we can prove the following theorem.

**Theorem 2.1.** *Suppose that the flux function $f = f(t,x,u)$ satisfies (H3) and that it is Lipschitz continuous on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$. Assume also that initial data $u_0$ belongs to $L^2(\mathbf{R}^d)$. Under conditions (H1)-(H2) the sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of* (2.1) *for every $t \in [0,T]$ satisfies the following inequalities:*

$$\int_{\mathbf{R}^d} |u_\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \left| \nabla u_\varepsilon(t',x) \right|^2 dx dt'$$
$$\leq C_4 \left( \int_{\mathbf{R}^d} |u_0(x)|^2 dx - \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} \operatorname{div}_x f(t',x,v)\, dv\, dx\, dt' \right), \tag{2.10}$$

$$\varepsilon^2 \int_{\mathbf{R}^d} \left| \nabla u_\varepsilon(t,x) \right|^2 dx + \varepsilon^3 \int_0^t \int_{\mathbf{R}^d} \left| D^2 u_\varepsilon(t',x) \right|^2 dx dt'$$
$$\leq C_5 \left( \varepsilon^2 \int_{\mathbf{R}^d} |\nabla u_0(x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \sum_{k=1}^{d} \left| \partial_{x_k} f(t',x,u_\varepsilon(t',x)) \right|^2 dx dt' + \|\partial_u f\|^2_{L^\infty(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})} \right), \tag{2.11}$$

*for some constants $C_4$ and $C_5$.*

*Proof.* We follow the procedure from [19]. Given a smooth function $\eta = \eta(u)$, $u \in \mathbf{R}$, we define

$$q_i(t,x,u) = \int_0^u \eta'(v) \partial_v f_i(t,x,v)\, dv, \quad i = 1, \dots, d. \tag{2.12}$$

If we multiply (2.1) by $\eta'(u)$, it becomes

$$
\begin{aligned}
&\partial_t\eta(u_\varepsilon)+\sum_{i=1}^{d}\partial_{x_i}q_i(t,x,u_\varepsilon)-\sum_{i=1}^{d}\int_0^{u_\varepsilon}\partial^2_{x_iv}f_i(t,x,v)\eta'(v)dv+\sum_{i=1}^{d}\eta'(u_\varepsilon)\partial_{x_i}f_i(t,x,u_\varepsilon)\\
&\quad=\varepsilon\sum_{i=1}^{d}\partial_{x_i}\left(\eta'(u_\varepsilon)b_i(\nabla u_\varepsilon)\right)-\varepsilon\eta''(u_\varepsilon)\sum_{i=1}^{d}b_i(\nabla u_\varepsilon)\partial_{x_i}u_\varepsilon+\delta\sum_{i=1}^{d}\partial_{x_i}\left(\eta'(u_\varepsilon)\partial^2_{x_ix_i}u_\varepsilon\right)\\
&\qquad-\frac{\delta}{2}\eta''(u_\varepsilon)\sum_{i=1}^{d}\partial_{x_i}(\partial_{x_i}u_\varepsilon)^2.
\end{aligned}
\tag{2.13}
$$

Choosing here $\eta(u)=u^2/2$ and integrating over $[0,t)\times\mathbf{R}^d$, we get

$$
\begin{aligned}
&\int_{\mathbf{R}^d}|u_\varepsilon(t,x)|^2dx+\varepsilon\int_0^t\int_{\mathbf{R}^d}\nabla u_\varepsilon(t',x)\cdot b(\nabla u_\varepsilon(t',x))dxdt'\\
&\quad=\int_{\mathbf{R}^d}|u_0(x)|^2dx+\sum_{j=1}^{d}\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon(t',x)}vD^2_{x_jv}f_j(t',x,v)dvdxdt'\\
&\qquad-\sum_{i=1}^{d}\int_0^t\int_{\mathbf{R}^d}u_\varepsilon(t',x)\partial_{x_i}f_i(t',x,u_\varepsilon(t',x))dxdt'\\
&\quad=\int_{\mathbf{R}^d}|u_0(x)|^2dx-\sum_{i=1}^{d}\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon(t',x)}\partial_{x_i}f_i(t',x,v)dvdxdt',
\end{aligned}
\tag{2.14}
$$

where the second equality sign is justified by the following partial integration:

$$
\begin{aligned}
&\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon}vD^2_{x_jv}f_j(t',x,v)dvdxdt'\\
&\quad=\int_0^t\int_{\mathbf{R}^d}u_\varepsilon\partial_{x_i}f_i(t',x,u_\varepsilon)dxdt'-\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon}\partial_{x_i}f_i(t',x,v)dvdxdt'.
\end{aligned}
\tag{2.15}
$$

Now inequality (2.10) follows from (2.14), using (H1).

As for inequality (2.11), we start by using (2.14), namely,

$$\begin{aligned}
&\int_{\mathbf{R}^d} |u_\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} \nabla u_\varepsilon(t',x) \cdot b(\nabla u_\varepsilon(t',x))\, dx dt' \\
&\quad = \int_{\mathbf{R}^d} |u_0(x)|^2 dx - \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} \partial_{x_i} f_i(t',x,v)\, dv dx dt' \\
&\quad \le \int_{\mathbf{R}^d} |u_0(x)|^2 dx + \sum_{i=1}^d \int_0^t \int_{\mathbf{R}^d} \int_0^{u_\varepsilon(t',x)} \left|\partial_{x_i} f_i(t',x,v)\ \right| dv dx dt' \\
&\quad \le \int_{\mathbf{R}^d} |u_0(x)|^2 dx + \int_0^t \int_{\mathbf{R}^d} \int_{\mathbf{R}} \frac{\mu(t',x)}{1+|v|^{1+\alpha}}\, dv dx dt' \\
&\quad \le \int_{\mathbf{R}^d} |u_0(x)|^2 dx + C \int_0^t \int_{\mathbf{R}^d} \mu(t',x)\, dx dt',
\end{aligned} \tag{2.16}$$

where $C = \int_{\mathbf{R}} (dv/(1+|v|^{1+\alpha}))$.

From here, using (H3), we conclude in particular that

$$\varepsilon \int_0^t \int_{\mathbf{R}^d} \left|\nabla u_\varepsilon(t',x)\right|^2 dx dt' \le C_{11}, \tag{2.17}$$

for some constant $C_{11}$ independent of $\varepsilon$.

Next, we differentiate (2.1) with respect to $x_k$ and multiply the expression by $\partial_{x_k} u$. Integrating over $\mathbf{R}^d$, using integration by parts and then summing over $k = 1, \dots, d$, we get:

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbf{R}^d} \partial_t |\nabla u_\varepsilon|^2 dx - \sum_{k=1}^d \int_{\mathbf{R}^d} (\nabla \partial_{x_k} u_\varepsilon) \cdot \left(\partial_{x_k} f_k(t,x,u_\varepsilon) + \partial_u f_k \partial_{x_k} u_\varepsilon\right) dx \\
&\quad = -\varepsilon \sum_{k=1}^d \int_{\mathbf{R}^d} (\nabla \partial_{x_k} u_\varepsilon)^T Db(\nabla u_\varepsilon)(\nabla \partial_{x_k} u_\varepsilon)\, dx.
\end{aligned} \tag{2.18}$$

Integrating this over $[0,t]$ and using the Cauchy-Schwarz inequality and condition (H2), we find

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbf{R}^d} |\nabla u_\varepsilon(t,\cdot)|^2 dx + \varepsilon C_3 \sum_{k=1}^d \int_0^t \int_{\mathbf{R}^d} |\nabla \partial_{x_k} u_\varepsilon|^2 dx dt' \\
&\quad \le \frac{1}{2} \int_{\mathbf{R}^d} |\nabla u_0|^2 dx + \sum_{k=1}^d \|\nabla(\partial_{x_k} u_\varepsilon)\|_{L^2(\mathbf{R}^+ \times \mathbf{R}^d)} \left\|\partial_{x_k} f_k(\cdot,\cdot,u_\varepsilon) + \partial_u f_k \partial_{x_k} u_\varepsilon\right\|_{L^2(\mathbf{R}^+ \times \mathbf{R}^d)},
\end{aligned} \tag{2.19}$$

where $C_3$ is independent of $\varepsilon$. Then, using Young's inequality (the constant $C_3$ is the same as previously mentioned)

$$ab \leq \frac{C_3\varepsilon}{2}a^2 + \frac{1}{2C_3\varepsilon}b^2, \quad a, b \in \mathbf{R}, \tag{2.20}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^d} |\nabla u_\varepsilon(t,\cdot)|^2 dx + \varepsilon C_3 \sum_{k=1}^{d}\int_0^t\int_{\mathbf{R}^d} |\nabla \partial_{x_k} u_\varepsilon|^2 dx dt' \\
&\quad \leq \frac{1}{2}\int_{\mathbf{R}^d} |\nabla u_0|^2 dx + C_3\frac{\varepsilon}{2}\sum_{k=1}^{d}\int_0^t\int_{\mathbf{R}^d} |\nabla \partial_{x_k} u_\varepsilon|^2 dx dt' \\
&\qquad + \frac{1}{2C_3\varepsilon}\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^{d}\left|\partial_{x_k} f_k(t', x, u_\varepsilon) + \partial_u f_k \partial_{x_k} u_\varepsilon\right|^2 dx dt'.
\end{aligned} \tag{2.21}$$

Multiplying this by $\varepsilon^2$, using $(a+b)^2 \leq 2a^2 + 2b^2$, and applying (2.17), we conclude

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\int_{\mathbf{R}^d} |\nabla u_\varepsilon(t, \ \cdot \ )|^2 dx + C_3\frac{\varepsilon^3}{2}\int_{\mathbf{R}^d}\int_0^t \left|D^2 u_\varepsilon\right|^2 dx dt' \\
&\quad \leq \frac{\varepsilon^2}{2}\int_{\mathbf{R}^d} |\nabla u_0|^2 dx dt' + \frac{\varepsilon}{C_3}\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^{d}\left|\partial_{x_k} f_k(t', x, u_\varepsilon(t', x))\right|^2 dx dt' + \frac{C_{11}}{C_3}\|\partial_u f_k\|^2_{L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})}.
\end{aligned} \tag{2.22}$$

This inequality is actually inequality (2.11) when we take $C_5 = 2\max\{1, 1/C_3, C_{11}/C_3\}/\min\{1, C_3\}$. □

## 3. The Multidimensional Case

Consider the following initial-value problem. Find $u = u(t,x)$ such that

$$\begin{aligned}
&\partial_t u + \operatorname{div}_x f(t, x, u) = 0, \\
&u(x, 0) = u_0(x), \quad x \in \mathbf{R}^d,
\end{aligned} \tag{3.1}$$

where $u_0 \in L^2(\mathbf{R}^d)$ is a given initial data.

For the flux $f = (f_1, \ldots, f_d)$ we need the following assumption, denoted (H4).

(H4a) For the flux $f = f(t, x, u)$, $(t, x, u) \in \mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}$, we assume that $f \in C(\mathbf{R}; BV(\mathbf{R}^+\times\mathbf{R}^d))$ and that for every $l \in \mathbf{R}^+$ we have $\max_{u\in[-l,l]}|f(t, x, u)| \in L^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p > 2$ .

(H4b) There exists a sequence $f_\varrho = (f_{1\varrho}, \dots, f_{d\varrho})$, $\varrho \in (0,1)$, such that $f_\varrho = f_\varrho(t,x,u) \in C^1(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$, satisfying for some $p > 2$ and every $l \in \mathbf{R}^+$:

$$\max_{z\in[-l,l]} \left| f_\varrho(\cdot,\cdot,z) - f(\cdot,\cdot,z) \right| \underset{\varrho\to 0}{\longrightarrow} 0 \quad \text{in } L^p\left(\mathbf{R}^+ \times \mathbf{R}^d\right) = 0, \tag{3.2a}$$

$$\sum_{i=1}^{d} \int_{\mathbf{R}^+\times\mathbf{R}^d} \left| \partial_{x_i} f_{i\varrho}(t,x,u) \right| dx\,dt \le \frac{\widetilde{C}_1}{1+|u|^{1+\alpha}}, \tag{3.2b}$$

$$\varrho \sum_{i=1,k}^{d} \int_{\mathbf{R}^+\times\mathbf{R}^d} \left| \partial_{x_k} f_{i\varrho}(t,x,u) \right|^2 dx\,dt \le \widetilde{C}_2, \tag{3.2c}$$

$$\sum_{i=1}^{d} \left| \partial_u f_{i\varrho}(t,x,u) \right| \le \frac{C}{\beta(\varrho)}, \tag{3.2d}$$

$$\sum_{i=1}^{d} \int_{\mathbf{R}^+\times\mathbf{R}^d} \left| \partial^2_{x_i u} f_{i\varrho}(t,x,u) \right| dx\,dt \le \frac{\widetilde{C}_3}{1+|u|^{1+\alpha}}, \tag{3.2e}$$

where $\widetilde{C}_i$, $i = 1,2,3$, and $C$ are constants, while the function $\beta : \mathbf{R} \to \mathbf{R}$ is such that $\lim_{\rho\to 0}\beta(\rho) = 0$.

In the case when we have only vanishing diffusion, it is usually possible to obtain uniform $L^\infty$ bound for the corresponding sequence of solutions under relatively mild assumptions on the flux and initial data (see, e.g., [9, 10]). In the case when we have both vanishing diffusion and vanishing dispersion, we must assume more on the flux in order to obtain even much weaker bounds (see Theorem 3.2). We remark that demand on controlling the flux at infinity is rather usual in the case of conservation laws with vanishing diffusion and dispersion (see, e.g., [16, 17, 19]).

*Remark 3.1.* For an arbitrary compactly supported, nonnegative $\varphi_1 \in C_0^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$ and $\varphi_2 \in C_0^\infty(\mathbf{R})$ with total mass one denote

$$\varphi_\varrho(z,u) = \frac{1}{\varrho^{d+1}} \varphi_1\left(\frac{z}{\varrho}\right) \frac{1}{\beta(\varrho)} \varphi_2\left(\frac{u}{\beta(\varrho)}\right), \tag{3.3}$$

$z \in \mathbf{R}^+ \times \mathbf{R}^d$ and $u \in \mathbf{R}$, where $\beta$ is a positive function tending to zero as $\varrho \to 0$. In the case when the flux $f \in C(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}^d)\ ) \cap BV(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^d))$ is bounded, straightforward computation shows that the sequence $f_\varrho = f \star \varphi_\varrho = (f_{1\varrho}, \dots, f_{d\varrho})$ satisfies (H4b) with $\beta(\varrho) = \varrho$.

We also need to assume that the flux $f$ is genuinely nonlinear, that is, for every $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ and every $\xi \in \mathbf{R}^d \setminus \{0\}$, the mapping

$$\mathbf{R} \ni \lambda \longmapsto \sum_{i=1}^{d} f_i(t,x,\lambda) \frac{\xi_i}{|\xi|} \tag{3.4}$$

is nonconstant on every nondegenerate interval of the real line.

We will analyze the vanishing diffusion-dispersion-smoothing limit of the problem

$$\partial_t u + \operatorname{div}_x f_\varrho(t,x,u) = \varepsilon \operatorname{div}_x b(\nabla u) + \delta \sum_{j=1}^{d} \partial^3_{x_j x_j x_j} u, \tag{3.5}$$

$$u(x,0) = u_{0,\varepsilon}(x), \quad x \in \mathbf{R}^d, \tag{3.6}$$

where the flux $f_\varrho$ satisfies the conditions (H4b). We denote the solution of (3.5)-(3.6) by $u_\varepsilon = u_\varepsilon(t,x)$. We assume that

$$\|u_{0,\varepsilon} - u_0\|_{L^2(\mathbf{R}^d)} \longrightarrow 0, \qquad \|u_{0,\varepsilon}\|_{L^2(\mathbf{R}^d)} + \varepsilon \|u_{0,\varepsilon}\|_{H^1(\mathbf{R}^d)} \leq C. \tag{3.7}$$

We also assume that $\varrho = \varrho(\varepsilon) \to 0$ and $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. We want to prove that under certain conditions, a sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ of (3.5)-(3.6) converges to a weak solution of problem (3.1) as $\varepsilon \to 0$. To do this in the multidimensional case we use the approach of $H$-measures, introduced in [11] and further developed in [10, 21]. In the one-dimensional case, we use the compensated compactness method, following [13].

In order to accomplish the plan we need the following a priori estimates.

**Theorem 3.2** (a priori inequalities). *Suppose that the flux $f(t,x,u)$ satisfies (H4). Also assume that the initial data $u_0$ satisfies* (3.7). *Under these conditions the sequence of smooth solutions $(u_\varepsilon)_{\varepsilon>0}$ of* (3.5)-(3.6) *satisfies the following inequalities for every $t \in [0,T]$:*

$$\int_{\mathbf{R}^d} |u_\varepsilon(t,x)|^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}^d} |\nabla u_\varepsilon(x,s)|^2 dx ds \leq C_4 \left( \int_{\mathbf{R}^d} |u_{0,\varepsilon}(x)|^2 dx + C_{10} \right), \tag{3.8}$$

$$\varepsilon^2 \int_{\mathbf{R}^d} |\nabla u_\varepsilon(t,x)|^2 dx + \varepsilon^3 \int_0^t \int_{\mathbf{R}^d} \left| D^2 u_\varepsilon(t',x) \right|^2 dx dt' \leq C_5 \left( \varepsilon^2 \int_{\mathbf{R}^d} |\nabla u_{0,\varepsilon}(x)|^2 dx + \frac{\varepsilon}{\varrho} C_{11} + \frac{C_{12}}{\beta(\varrho)^2} \right), \tag{3.9}$$

*for some constants $C_{10}, C_{11}, C_{12}$ (the constants $C_4, C_5$ are introduced in Theorem 2.1).*

*Proof.* For every fixed $\varrho$, the function $f_\varrho = (f_{1\varrho}, \ldots, f_{d\varrho})$ is smooth, and, due to (H4), we see that $f_\varrho$ satisfies (H3). This means that we can apply Theorem 2.1.

Replacing the flux $f$ by $f_\varrho$ from (3.5) and $u_0$ by $u_{0,\varepsilon}$ from (3.6) in (2.10) and (2.11), we get

$$\begin{aligned}&\int_{\mathbf{R}^d}|u_\varepsilon(t,x)|^2dx+\varepsilon\int_0^t\int_{\mathbf{R}^d}|\nabla u_\varepsilon(x,s)|^2dxds\\&\quad\le C_3\left(\int_{\mathbf{R}^d}|u_{0,\varepsilon}(x)|^2dx-\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon(t',x)}\operatorname{div}_x f_\varrho(t',x,v)\,dv\,dx\,dt'\right),\end{aligned}\tag{3.10}$$

$$\begin{aligned}&\varepsilon^2\int_{\mathbf{R}^d}|\nabla u_\varepsilon(t,x)|^2dx+\varepsilon^3\int_0^t\int_{\mathbf{R}^d}\left|D^2u_\varepsilon(t',x)\right|^2dxdt'\\&\quad\le C_4\Bigg(\varepsilon^2\int_{\mathbf{R}^d}|\nabla u_{0,\varepsilon}(x)|^2dx+\|\partial_u f_\varrho\|^2_{L^\infty(\mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R})}\\&\qquad+\varepsilon\int_0^t\int_{\mathbf{R}^d}\sum_{k=1}^d\sum_{i=1}^d\left[\partial_{x_k}f_{i\varrho}(t',x,u_\varepsilon(t',x))\right]^2dxdt'\Bigg).\end{aligned}\tag{3.11}$$

To proceed, we use assumption (H4). We have

$$\begin{aligned}\int_0^t\int_{\mathbf{R}^d}\int_0^{u_\varepsilon(t',x)}\operatorname{div} f_{i\varrho}(t',x,v)\,dv\,dx\,dt'&\le\int_0^t\int_{\mathbf{R}^d}\int_{\mathbf{R}}\sum_{i=1}^d\left|\partial_{x_i}f_{i\varrho}(t',x,v)\right|dv\,dx\,dt\\&\le\int_{\mathbf{R}}\frac{\widetilde{C}_1}{1+|v|^{1+\alpha}}dv\le C_{10},\end{aligned}\tag{3.12}$$

which together with (3.10) immediately gives (3.8).

Similarly, combining (H4) and (3.11), and arguing as in (3.12), we get (3.9). □

In this section, we will inspect the convergence of a family $(u_\varepsilon)_{\varepsilon>0}$ of solutions to (3.5)-(3.6) in the case when

$$b(\lambda_1,\dots,\lambda_d)=(\lambda_1,\dots,\lambda_d)\tag{3.13}$$

for the function $b$ appearing in the right-hand side of (3.5). This is not an essential restriction, but we will use it in order to simplify the presentation.

Thus, we use the following theorem which can be proved using the $H$-measures approach (see, e.g., [10, Corollary 2 and Remark 3]). We let $\theta$ denote the Heaviside function.

**Theorem 3.3** (see [10]). *Assume that the vector $f(t,x,u)$ is genuinely nonlinear in the sense of* (3.4). *Then each family $(v_\varepsilon(t,x))_{\varepsilon>0} \subset L^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$ such that for every $c \in \mathbf{R}$ the distribution*

$$\partial_t(\theta(v_\varepsilon - c)(v_\varepsilon - c)) + \operatorname{div}_x\big(\theta(v_\varepsilon - c)\big(f(t,x,v_\varepsilon) - f(t,x,c)\big)\big) \tag{3.14}$$

*is precompact in $H^{-1}_{\mathrm{loc}}$ contains a subsequence convergent in $L^1_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}^d)$.*

We can now prove the following theorem.

**Theorem 3.4.** *Assume that the flux vector $f$ is genuinely nonlinear in the sense of* (3.4) *and that it satisfies (H4). Furthermore, assume that*

$$\varrho = \varepsilon, \quad \delta = \varepsilon^2\rho^2(\varepsilon) \quad \text{with } \rho(\varepsilon) = \mathcal{O}(\beta(\varepsilon)), \tag{3.15}$$

*and that $u_{0,\varepsilon}$ satisfies* (3.7). *Then, there exists a subsequence of the family $(u_\varepsilon)_{\varepsilon>0}$ of solutions to* (3.5)–(3.6) *that converges to a weak solution of problem* (3.1).

*Proof.* We will use Theorem 3.3. Since it is well known that the family $(u_\varepsilon)_{\varepsilon>0}$ of solutions of problem (3.5)–(3.6) is not uniformly bounded, we cannot directly apply the conditions of Theorem 3.3.

Take an arbitrary $C^2$ function $S = S(u)$, $u \in \mathbf{R}$, and multiply the regularized equation (3.5) by $S'(u_\varepsilon)$. As usual, put

$$q(t,x,u) = \int_0^u S'(v)\partial_u f_\varrho \, dv, \quad q = (q_1,\dots,q_d). \tag{3.16}$$

We easily find that

$$\begin{aligned}
&\partial_t S(u_\varepsilon) + \operatorname{div}_x q(t,x,u_\varepsilon) - \operatorname{div}_x q(t,x,v)|_{v=u_\varepsilon} + S'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon} \\
&\quad = \varepsilon\operatorname{div}_x\big(S'(u_\varepsilon)\nabla u_\varepsilon\big) - \varepsilon S''(u_\varepsilon)|\nabla u_\varepsilon|^2 + \delta\sum_{j=1}^d D_{x_j}\Big(S'(u_\varepsilon)\partial^2_{x_jx_j}u_\varepsilon\Big) - \delta\sum_{j=1}^d S''(u_\varepsilon)\partial_{x_j}u_\varepsilon\partial^2_{x_jx_j}u_\varepsilon.
\end{aligned} \tag{3.17}$$

We will apply this formula repeatedly with different choices for $S(u)$.

In order to apply Theorem 3.3, we will consider a truncated sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$, where the truncation function $T_l$ is defined for every fixed $l \in \mathbf{N}$ as

$$T_l(u) = \begin{cases} -l, & u \le -l, \\ u, & -l \le u \le l, \\ l, & u \ge l. \end{cases} \tag{3.18}$$

We will prove that the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$ is precompact for every fixed $l$. Denote by $u_l$ a subsequential limit (in $L^1_{\mathrm{loc}}$) of the family $(T_l(u_\varepsilon))_{\varepsilon>0}$, which gives raise to a new sequence $(u_l)_{l>1}$ that we prove converges to a weak solution of (3.1).

To carry out this plan, we must replace $T_l$ by a $C^2$ regularization $T_{l,\sigma} : \mathbf{R} \to \mathbf{R}$. We define $T_{l,\sigma} : \mathbf{R} \to \mathbf{R}$ by $T_{l,\sigma}(0) = 0$ and

$$T'_{l,\sigma}(u) = \begin{cases} 1, & |u| < l, \\ \dfrac{l - |u| + \sigma}{\sigma}, & l < |u| < l + \sigma, \\ 0, & |u| > l + \sigma. \end{cases} \tag{3.19}$$

Next, we want to estimate $\|T''_{l,\sigma}(u_\varepsilon)\nabla u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}$. To accomplish this, we insert the functions $T^{\pm}_{l,\sigma}$ for $S$ in (3.17) where $T^{\pm}_{l,\sigma}$ are defined by $T^{\pm}_{l,\sigma}(0) = 0$ and

$$\left(T^{+}_{l,\sigma}\right)'(u) = \begin{cases} 1, & u < l, \\ \dfrac{l + \sigma - u}{\sigma}, & l < u < l + \sigma, \\ 0, & u > l + \sigma, \end{cases} \tag{3.20}$$

$$\left(T^{-}_{l,\sigma}\right)'(u) = \begin{cases} 1, & u > -l, \\ \dfrac{l + \sigma + u}{\sigma}, & -l - \sigma < u < -l, \\ 0, & u < -l - \sigma. \end{cases} \tag{3.21}$$

Notice that

$$\begin{gathered} \left(T^{\pm}_{l,\sigma}(u)\right)' \leq 1, \qquad \left|T^{\pm}_{l,\sigma}(u)\right| \leq |u| + \frac{\sigma}{2}, \\ T^{+}_{l,\sigma}(u) = T^{-}_{l,\sigma}(u) \quad \text{for } -l \leq u \leq l. \end{gathered} \tag{3.22}$$

By inserting $S(u) = -T^{+}_{l,\sigma}(u)$, $q = q_+(t,x,u) = -\int_0^u (T^{+}_{l,\sigma})'(v)\partial_u f_\varrho \, dv$ in (3.17) and integrating over $\Pi_t = [0,t] \times \mathbf{R}^d$, we get

$$\begin{aligned} &-\int_{\mathbf{R}^d} T^{+}_{l,\sigma}(u_\varepsilon)dx + \int_{\mathbf{R}^d} T^{+}_{l,\sigma}(u_0)dx + \frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<u_\varepsilon<l+\sigma\}} |\nabla u_\varepsilon|^2 dxdt \\ &\quad = \iint_{\Pi_t} \operatorname{div}_x q_+(t,x,v)|_{v=u_\varepsilon} dxdt + \iint_{\Pi_t} \left(T^{+}_{l,\sigma}\right)'(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon} dxdt \\ &\qquad - \frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l<u_\varepsilon<l+\sigma\}} \sum_{j=1}^{d} \partial_{x_j} u_\varepsilon \partial^2_{x_j x_j} u_\varepsilon dxdt. \end{aligned} \tag{3.23}$$

Similarly, for $S(u) = T^-_{l,\sigma}(u)$, $q = q_-(t,x,u) = \int_0^u (T^-_{l,\sigma})'(v)\partial_u f_Q \, dv$, we have from (3.17)

$$\begin{aligned}
&\int_{\mathbf{R}^d} T^-_{l,\sigma}(u_\varepsilon)dx - \int_{\mathbf{R}^d} T^-_{l,\sigma}(u_0)dx + \frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{-l-\sigma<u_\varepsilon<-l\}} |\nabla u_\varepsilon|^2 dxdt \\
&\quad = \iint_{\Pi_t} \mathrm{div}_x q_-(t,x,v)|_{v=u_\varepsilon} dxdt - \iint_{\Pi_t} \left(T^-_{l,\sigma}\right)'(u_\varepsilon)\mathrm{div}_x f_Q(t,x,v)|_{v=u_\varepsilon} dxdt \\
&\qquad + \frac{\delta}{\sigma}\iint_{\Pi_t\cap\{-l-\sigma<u_\varepsilon<-l\}} \sum_{j=1}^d \partial_{x_j} u_\varepsilon \partial^2_{x_j x_j} u_\varepsilon dxdt.
\end{aligned} \tag{3.24}$$

Adding (3.23) to (3.24), we get

$$\begin{aligned}
&\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}} |\nabla u_\varepsilon|^2 dxdt \\
&\quad = -\int_{\mathbf{R}^d}\left(T^-_{l,\sigma}(u_\varepsilon) - T^+_{l,\sigma}(u_\varepsilon)\right)dx + \int_{\mathbf{R}^d}\left(T^-_{l,\sigma}(u_0) - T^+_{l,\sigma}(u_0)\right)dx \\
&\qquad + \iint_{\Pi_t} \mathrm{div}_x q_-(t,x,v)|_{v=u_\varepsilon} dxdt + \iint_{\Pi_t} \mathrm{div}_x q_+(t,x,v)|_{v=u_\varepsilon} dxdt \\
&\qquad - \iint_{\Pi_t} \left(T^-_{l,\sigma}\right)'(u_\varepsilon)\mathrm{div}_x f_Q(t,x,v)|_{v=u_\varepsilon} dxdt + \iint_{\Pi_t} \left(T^+_{l,\sigma}\right)'(u_\varepsilon)\mathrm{div}_x f_Q(t,x,v)|_{v=u_\varepsilon} dxdt \\
&\qquad + \frac{\delta}{\sigma}\iint_{\Pi_t\cap\{-l-\sigma<u_\varepsilon<-l\}} \sum_{j=1}^d \partial_{x_j} u_\varepsilon \partial^2_{x_j x_j} u_\varepsilon dxdt - \frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l<u_\varepsilon<l+\sigma\}} \sum_{j=1}^d \partial_{x_j} u_\varepsilon \partial^2_{x_j x_j} u_\varepsilon dxdt.
\end{aligned} \tag{3.25}$$

From (3.22) and the definition of $q_-$ and $q_+$, it follows

$$\begin{aligned}
\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}} |\nabla u_\varepsilon|^2 dxdt \le{}& \int_{|u_\varepsilon|>l} 2|u_\varepsilon|dx + \int_{|u_0|>l} 2|u_0|dx \\
&+ 2\iint_{\Pi_t}\int_{\mathbf{R}} \sum_{i=1}^d \left| D^2_{x_i v} f_{iQ}(t,x,v)\right| dvdxdt \\
&+ 2\iint_{\Pi_t} \sum_{i=1}^d \left|\partial_{x_i} f_{iQ}(t,x,u_\varepsilon)\right| dxdt \\
&+ 2\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l-\sigma<|u_\varepsilon|<l\}} \sum_{j=1}^d \left|\partial_{x_j} u_\varepsilon \partial^2_{x_j x_j} u_\varepsilon\right| dxdt.
\end{aligned} \tag{3.26}$$

Without loss of generality, we can assume that $l > 1$. Having this in mind, we get from (H4) and (3.26)

$$\begin{aligned}
&\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}|\nabla u_\varepsilon|^2 dxdt \\
&\le \int_{|u_\varepsilon|>l} 2|u_\varepsilon|^2 dx + \int_{|u_0|>l} 2|u_0|^2 dx + 2\int_{\mathbf{R}}\sum_{i=1}^{d}\frac{\widetilde{C}_3}{1+|v|^{1+\alpha}}dv \\
&\quad + 2\iint_{\Pi_t}\sum_{i=1}^{d}\left|\partial_{x_i} f_{i\varrho}(t,x,u_\varepsilon)\right| dxdt + 2\frac{\delta}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}\sum_{j=1}^{d}\left|\partial_{x_j}u_\varepsilon \partial^2_{x_j x_j}u_\varepsilon\right| dxdt \\
&\le \int_{\mathbf{R}^d} 2\left(|u_\varepsilon(x,t)|^2 + |u_0(x,t)|^2\right)dx + K_1 + K_2 + 2\frac{\delta}{\sigma\varepsilon^2}\sum_{i=1}^{d}\left\|\varepsilon^{1/2}\partial_{x_i}u_\varepsilon\right\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} \\
&\quad \times \left\|\varepsilon^{3/2}\partial^2_{x_i x_i}u_\varepsilon\right\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} \\
&\le K_5 + \left(\frac{\delta^2}{\sigma^2\varepsilon^4(\beta(\varrho))^2} + \frac{\delta^2}{\sigma^2\varepsilon^4}\right)^{1/2} K_3 K_4,
\end{aligned}\tag{3.27}$$

where $K_i$, $i = 1, \ldots, 5$, are constants such that (cf. (3.8) and (3.9))

$$\begin{gathered}
2\int_{\mathbf{R}}\sum_{i=1}^{d}\frac{\widetilde{C}_3}{1+|v|^{1+\alpha}}dv \le K_1, \\
2\iint_{\Pi_t}\sum_{i=1}^{d}\left|\partial_{x_i} f_{i\varrho}(t,x,u_\varepsilon)\right| dxdt \le K_2, \\
\sum_{i=1}^{d}\left\|\varepsilon^{1/2}\partial_{x_i}u_\varepsilon\right\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} \le K_3, \\
\sum_{i=1}^{d}\left\|\varepsilon^{3/2}\partial^2_{x_i x_i}u_\varepsilon\right\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} \le \left(\frac{1}{(\beta(\varrho))^2} + \frac{\varepsilon}{\varrho}\right)^{1/2} K_4, \\
\int_{\mathbf{R}^d} 2\left(|u_\varepsilon(x,t)|^2 + |u_0(x,t)|^2\right)dx + K_1 + K_2 \le K_5.
\end{gathered}\tag{3.28}$$

These estimates follow from (H4) and the a priori estimates (3.8), (3.9). If in addition we use the assumption $\varepsilon = \varrho$ from (3.15), we conclude

$$\frac{\delta}{\sigma\varepsilon^2}\|\varepsilon^{1/2}\nabla u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}\sum_{i=1}^{d}\|\varepsilon^{3/2}\partial^2_{x_i x_i}u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)} \le \left(\frac{\delta^2}{\sigma^2\varepsilon^4\beta^2(\varepsilon)} + \frac{\delta^2}{\sigma^2\varepsilon^4}\right)^{1/2} K_3 K_4. \tag{3.29}$$

Thus, in view of (3.27),

$$\frac{\varepsilon}{\sigma}\iint_{\Pi_t\cap\{l<|u_\varepsilon|<l+\sigma\}}|\nabla u_\varepsilon|^2 dx dt \le K_5 + \left(\frac{\delta^2}{\sigma^2\varepsilon^4\beta^2(\varepsilon)} + \frac{\delta^2}{\sigma^2\varepsilon^4}\right)^{1/2} K_3K_4, \tag{3.30}$$

which is the sought for estimate for $\|T''_{l,\sigma}(u_\varepsilon)\nabla u_\varepsilon\|_{L^2(\mathbf{R}^+\times\mathbf{R}^d)}$.

Next, take a function $U_\rho(z)$ satisfying $U_\rho(0) = 0$ and

$$U'_\rho(z) = \begin{cases} 0, & z < 0, \\ \dfrac{z}{\rho}, & 0 < z < \rho, \\ 1, & z > \rho. \end{cases} \tag{3.31}$$

Clearly, $U_\rho$ is convex, and $U'_\rho(z) \to \theta(z)$ in $L^p_{\mathrm{loc}}(\mathbf{R})$ as $\rho \to 0$, for any $p < \infty$; as before, $\theta$ denotes the Heaviside function.

Inserting $S(u_\varepsilon) = U_\rho(T_{l,\sigma}(u_\varepsilon) - c)$ in (3.17), we get

$$\begin{aligned}
&\partial_t U_\rho(T_{l,\sigma}(u_\varepsilon) - c) + \mathrm{div}_x \int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v) - c)T'_{l,\sigma}(v)\partial_v f_\varrho(t,x,v)dv \\
&\quad = \int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v) - c)T'_{l,\sigma}(v)\mathrm{div}_x\partial_v f_\varrho(t,x,v)dv \\
&\qquad - U'_\rho(T_{l,\sigma}(u_\varepsilon) - c)T'_{l,\sigma}(u_\varepsilon)\mathrm{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon} \\
&\qquad + \varepsilon\Delta_x U_\rho(T_{l,\sigma}(u_\varepsilon) - c) - \varepsilon D^2_{uu}\left[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\right]|\nabla u_\varepsilon|^2 \\
&\qquad + \delta\sum_{i=1}^d D_{x_i}\left(D_u\left[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\right]\partial^2_{x_ix_i}u_\varepsilon\right) \\
&\qquad - \delta\sum_{i=1}^d D^2_{uu}\left[U_\rho(T_{l,\sigma}(u_\varepsilon) - c)\right]\partial_{x_i}u_\varepsilon\partial^2_{x_ix_i}u_\varepsilon.
\end{aligned} \tag{3.32}$$

We rewrite the previous expression in the following manner:

$$\begin{aligned}
&\partial_t(\theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c)) + \mathrm{div}_x\big(\theta(T_l(u_\varepsilon) - c)\big(f(t,x,T_l(u_\varepsilon)) - f(t,x,c)\big)\big) \\
&\quad = \Gamma_{1,\varepsilon} + \Gamma_{2,\varepsilon} + \Gamma_{3,\varepsilon} + \Gamma_{4,\varepsilon} + \Gamma_{5,\varepsilon} + \Gamma_{6,\varepsilon} + \Gamma_{7,\varepsilon},
\end{aligned} \tag{3.33}$$

where

$$
\begin{aligned}
\Gamma_{1,\varepsilon} &= \partial_t\big(\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)-U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big),\\
\Gamma_{2,\varepsilon} &= \operatorname{div}_x\Bigg(\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)\\
&\qquad -\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\partial_v f_\varrho(t,x,v)dv\Bigg),\\
\Gamma_{3,\varepsilon} &= \int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\operatorname{div}_x\partial_v f_\varrho(t,x,v)dv\\
&\quad -U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T'_{l,\sigma}(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)\big|_{v=u_\varepsilon},\\
\Gamma_{4,\varepsilon} &= \varepsilon\Delta_x U_\rho(T_{l,\sigma}(u_\varepsilon)-c)+\delta\sum_{i=1}^{d}D_{x_i}\Big(D_u\big[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big]\partial^2_{x_ix_i}u_\varepsilon\Big),\\
\Gamma_{5,\varepsilon} &= -\varepsilon U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T''_{l,\sigma}(u\varepsilon)|\nabla u\varepsilon|^2,\\
\Gamma_{6,\varepsilon} &= -\delta\sum_{i=1}^{d}D^2_{uu}\big[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big]\partial_{x_i}u_\varepsilon\partial^2_{x_ix_i}u_\varepsilon,\\
\Gamma_{7,\varepsilon} &= -\varepsilon U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\Big(T'_{l,\sigma}(u_\varepsilon)\Big)^2|\nabla u_\varepsilon|^2.
\end{aligned}
\tag{3.34}
$$

To continue, we assume that $\sigma$ depends on $\varepsilon$ in the following way:

$$\sigma=\rho=\mathcal{O}\big(\beta(\varepsilon)\big). \tag{3.35}$$

From here, we will prove that the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$ satisfies the assumptions of Theorem 3.3. Accordingly, we need to prove that the left-hand side of (3.33) is precompact in $H^{-1}_{\mathrm{loc}}(\mathbf{R}^+\times\mathbf{R}^d)$.

To accomplish this, we use Murat's lemma ([22, Chapter 1, Corollary 1]). More precisely, we have to prove the following.

(i) When the left-hand side of (3.33) is written in the form $\operatorname{div} Q_\varepsilon$, we have $Q_\varepsilon\in L^p_{\mathrm{loc}}(\mathbf{R}^+\times\mathbf{R}^d)$ for $p>2$.

(ii) The right-hand side of (3.33) is of the form $\mathcal{M}_{\mathrm{loc},B}+H^{-1}_{\mathrm{loc},c}$, where $\mathcal{M}_{\mathrm{loc},B}$ denotes a set of families which are locally bounded in the space of measures, and $H^{-1}_{\mathrm{loc},c}$ is a set of families precompact in $H^{-1}_{\mathrm{loc}}$.

First, since $T_l(u_\varepsilon)$ is uniformly bounded by $l$, we see that (i) is satisfied.

To prove (ii), we consider each term on the right-hand side of (3.33). First we prove that

$$\Gamma_{1,\varepsilon}=\partial_t\big(\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)-U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big)\in H^{-1}_{\mathrm{loc},c}. \tag{3.36}$$

We have

$$\begin{aligned}
&\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)-U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\\
&\quad=\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)-\theta(T_{l,\sigma}(u_\varepsilon)-c)(T_{l,\sigma}(u_\varepsilon)-c)\\
&\qquad+\theta(T_{l,\sigma}(u_\varepsilon)-c)(T_{l,\sigma}(u_\varepsilon)-c)-U_\rho(T_{l,\sigma}(u_\varepsilon)-c).
\end{aligned}\tag{3.37}$$

Since the function $\theta(z-c)(z-c)$ is Lipschitz continuous in $z$ with the Lipschitz constant one, and, according to definition of $U_\rho$, it holds $|U_\rho(z)-\theta(z)z|\le 1/2\rho$, we conclude from the last expression

$$\left|\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)-U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\right|\le|T_l(u_\varepsilon)-T_{l,\sigma}(u_\varepsilon)|+\mathcal{O}(\rho)\le\mathcal{O}(\sigma)+\mathcal{O}(\rho).\tag{3.38}$$

From this and assumptions (3.15) and (3.35) on $\sigma=\sigma(\varepsilon)$ and $\rho=\rho(\varepsilon)$, it follows that as $\varepsilon\to 0$

$$\theta(T_l(u_\varepsilon)-c)(T_l(u_\varepsilon)-c)-U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\longrightarrow 0\tag{3.39}$$

in $L^p_{\mathrm{loc}}$ for all $p<\infty$. Thus, (since we can take $p=2$ as well) we see that $\Gamma_{1,\varepsilon}\in H^{-1}_{\mathrm{loc},c}$.

Next, we will prove that

$$\begin{aligned}
\Gamma_{2,\varepsilon}=\operatorname{div}_x\Big(&\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)\\
&-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\partial_v f_\varrho(t,x,v)\,dv\Big)\in H^{-1}_{\mathrm{loc},c}+\mathcal{M}_{\mathrm{loc},B}.
\end{aligned}\tag{3.40}$$

Indeed,

$$\begin{aligned}
&\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\partial_v f_\varrho(t,x,v)\,dv\\
&\quad=\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)-\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f_\varrho(t,x,T_{l,\sigma}(u_\varepsilon))-f_\varrho(t,x,c)\big)\\
&\qquad+\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big)-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_l(v)\partial_v f_\varrho(t,x,v)\,dv\\
&\qquad-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)\Big(T'_{l,\sigma}(v)-T'_l(v)\Big)\partial_v f_\varrho(t,x,v)\,dv.
\end{aligned}\tag{3.41}$$

Since $T_l(u)=u$ if $|u|\le l$ and $T'_l(u)=0$ if $|u|\ge l$,

$$\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_l(v)\partial_v f_\varrho(t,x,v)\,dv=\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_l(v)\partial_v f_\varrho(t,x,T_l(v))\,dv,\tag{3.42}$$

from which we conclude

$$\begin{aligned}
&\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)\\
&\quad-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_{l,\sigma}(v)\partial_v f_\varrho(t,x,v)\,dv\\
&=\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)\\
&\quad-\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big)\\
&\quad+\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big)\\
&\quad-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)T'_l(v)\partial_v f_\varrho(t,x,T_l(v))\,dv\\
&\quad-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)\Big(T'_{l,\sigma}(v)-T'_l(v)\Big)\partial_v f_\varrho(t,x,v)\,dv\\
&=\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)\\
&\quad-\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big)\\
&\quad+\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big)\\
&\quad-\int^{u_\varepsilon}\theta(T_{l,\sigma}(v)-c)D_v\big[f_\varrho(t,x,T_l(v))\big]\,dv\\
&\quad-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)\Big(T'_{l,\sigma}(v)-T'_l(v)\Big)\partial_v f_\varrho(t,x,v)\,dv\\
&\quad-\int^{u_\varepsilon}\Big(U'_\rho(T_{l,\sigma}(v)-c)-\theta(T_{l,\sigma}(v)-c)\Big)T'_l(v)\partial_v f_\varrho(t,x,T_l(v))\,dv\\
&=\Gamma^1_{2,\varepsilon}+\Gamma^2_{2,\varepsilon}+\Gamma^3_{2,\varepsilon},
\end{aligned}\tag{3.43}$$

with

$$\begin{aligned}
\Gamma^1_{2,\varepsilon}&=\theta(T_l(u_\varepsilon)-c)\big(f(t,x,T_l(u_\varepsilon))-f(t,x,c)\big)-\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big),\\
\Gamma^2_{2,\varepsilon}&=\theta(T_{l,\sigma}(u_\varepsilon)-c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon))-f(t,x,c)\big)-\int^{u_\varepsilon}\theta(T_{l,\sigma}(v)-c)D_v\big[f_\varrho(t,x,T_l(v))\big]\,dv,\\
\Gamma^3_{2,\varepsilon}&=-\int^{u_\varepsilon}U'_\rho(T_{l,\sigma}(v)-c)\Big(T'_{l,\sigma}(v)-T'_l(v)\Big)\partial_v f_\varrho(t,x,v)\,dv\\
&\quad-\int^{u_\varepsilon}\Big(U'_\rho(T_{l,\sigma}(v)-c)-\theta(T_{l,\sigma}(v)-c)\Big)T'_l(v)\partial_v f_\varrho(t,x,T_l(v))\,dv.
\end{aligned}\tag{3.44}$$

Consider now each term on the right-hand side of (3.43). Since $T_l$ is a continuous function and $T_l(u) \in [-l, l]$, the function $f(t,x,T_l(u))$ is uniformly continuous in $u \in \mathbf{R}$. Therefore, we have pointwise on $\mathbf{R}^+ \times \mathbf{R}^d$:

$$\begin{aligned} \left|\Gamma^1_{2,\varepsilon}\right| = \big|&\theta(T_l(u_\varepsilon) - c)\big(f(t,x,T_l(u_\varepsilon)) - f(t,x,c)\big) \\ &-\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)\big)\big| \longrightarrow 0 \quad \text{as } \sigma \longrightarrow 0. \end{aligned} \tag{3.45}$$

Since $\max_{u\in[-l,l]} f(t,x,u) \in L^p(\mathbf{R}^+ \times \mathbf{R}^d)$, $p > 2$, Lebesgue's dominated convergence theorem yields $|\Gamma^1_{2,\varepsilon}| = o_{\sigma,L^p_{\text{loc}}}(1)$, where $\int_{\mathbf{R}^+\times\mathbf{R}^d} |o_{\sigma,L^p}(1)|^p\,dx\,dt \to 0$ as $\sigma \to 0$. Thus, we conclude

$$\operatorname{div}_x \Gamma^1_{2,\varepsilon} \in H^{-1}_{\text{loc}}\left(\mathbf{R}^+ \times \mathbf{R}^d\right). \tag{3.46}$$

We pass to $\Gamma^2_{2,\varepsilon}$. We have to distinguish between different cases depending on the relative size of $c$ and $l$. Consider first the case when $|c| \le l$, in which case we have $T_l(c) = c$ and $T_{l,\sigma}(c) = c$. Thus,

$$\begin{aligned} \left|\Gamma^2_{2,\varepsilon}\right| &= \bigg|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)\big) \\ &\qquad -\int^{u_\varepsilon} \theta(T_{l,\sigma}(v) - c) D_v\big[f_\varrho(t,x,T_l(v))\big]\,dv\bigg| \\ &= \bigg|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)\big) \\ &\qquad -\theta(T_{l,\sigma}(u_\varepsilon) - c)\int_c^{u_\varepsilon} D_v\big[f_\varrho(t,x,T_l(v))\big]\,dv\bigg| \\ &= \big|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)\big) \\ &\qquad -\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f_\varrho(t,x,T_l(u_\varepsilon)) - f_\varrho(t,x,c)\big)\big| \\ &\le \big|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon))\big)\big| \\ &\quad + \big|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,c) - f_\varrho(t,x,c)\big)\big| \\ &\le \big|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_{l,\sigma}(u_\varepsilon))\big)\big| \\ &\quad + \big|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f_\varrho(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon))\big)\big| \\ &\quad + \big|\theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,c) - f_\varrho(t,x,c)\big)\big| \\ &= o_{\varrho,L^p_{\text{loc}}}(1) + \mathcal{O}\left(\frac{\sigma}{\beta(\varrho)}\right) + o_{\varrho,L^p_{\text{loc}}}(1) = \mathcal{O}(1) + o_{\varrho,L^p_{\text{loc}}}(1), \end{aligned} \tag{3.47}$$

where $o_{\varrho,L^p_{\text{loc}}}(1)$ appears due to (3.2a), and $\mathcal{O}(1)$ comes from (3.35).

For $c > l$ we have $c \geq l + \sigma$ for a $\sigma$ small enough, and therefore $\theta(T_{l,\sigma}(u_\varepsilon) - c) \equiv 0$. On the other hand, for $c < -l$ we have $c \leq -l - \sigma$, and so $\theta(T_{l,\sigma}(u_\varepsilon) - c) \equiv 1$. Thus, the problematic case is when $c < -l$. In this case, we have instead of (3.47)

$$\begin{aligned}\Gamma^2_{2,\varepsilon} &= \theta(T_{l,\sigma}(u_\varepsilon) - c)\big(f(t,x,T_{l,\sigma}(u_\varepsilon)) - f(t,x,c)\big) - \int^{u_\varepsilon} \theta(T_{l,\sigma}(v) - c) D_v\left[f_\varrho(t,x,T_l(v))\right] dv \\ &= f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon)) + f_\varrho(t,x,-l) - f(t,x,c)\end{aligned} \tag{3.48}$$

implying

$$\operatorname{div}_x \Gamma^2_{2,\varepsilon} \in H^{-1}_{\mathrm{loc},c} + \mathcal{M}_{\mathrm{loc},B}, \tag{3.49}$$

since $f(t,x,T_{l,\sigma}(u_\varepsilon)) - f_\varrho(t,x,T_l(u_\varepsilon)) \to 0$ in $L^p_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}^d)$ for $p \geq 2$, and $f_\varrho(t,x,-l) - f(t,x,c) \in BV(\mathbf{R}^+ \times \mathbf{R}^d)$.

It remains to estimate $\Gamma^3_{2,\varepsilon}$. Noticing that $|U'_\rho|, |T'_{l,\sigma}| \leq 1$, we get

$$\begin{aligned}&\left|\int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v) - c)\Big(T'_{l,\sigma}(v) - T'_l(v)\Big)\partial_v f_\varrho(t,x,v)\,dv\right| \\ &\quad\leq \frac{C}{\beta(\rho)}\int_{\mathbf{R}}\left|T'_{l,\sigma}(v) - T'_l(v)\right| dv = \mathcal{O}\left(\frac{\sigma}{\beta(\varrho)}\right) \overset{(3.15),(3.35)}{=} \mathcal{O}(1),\end{aligned} \tag{3.50}$$

where $C$ is the constant given by (3.2d).

Similarly, from (3.2d) and since $|T'_l(v)| \leq 1$, we have

$$\begin{aligned}&\left|\int^{u_\varepsilon}\Big(U'_\rho(T_{l,\sigma}(v) - c) - \theta(T_{l,\sigma}(v) - c)\Big)T'_l(v)\partial_v f_\varrho(t,x,T_l(v))\,dv\right| \\ &\quad\leq \frac{C}{\beta(\rho)}\int_{-l}^{l}\left|U'_\rho(T_{l,\sigma}(v) - c) - \theta(T_{l,\sigma}(v) - c)\right| dv = \mathcal{O}\left(\frac{\varrho}{\beta(\varrho)}\right) \overset{(3.15)}{=} \mathcal{O}(1),\end{aligned} \tag{3.51}$$

from which we conclude that $\Gamma^3_{2,\varepsilon}$ is bounded in $L^2_{\mathrm{loc}}$. From assumptions (3.15) and (3.35), as well as for the estimates (3.46)–(3.51), it follows that the expression from (3.43) is bounded in $L^2_{\mathrm{loc}}$ from which it follows that $\Gamma_{2,\varepsilon} \in H^{-1}_{\mathrm{loc},c}$.

The next term is

$$\begin{aligned}\Gamma_{3,\varepsilon} = &\int^{u_\varepsilon} U'_\rho(T_{l,\sigma}(v) - c)T'_{l,\sigma}(v)\operatorname{div}_x \partial_v f_\varrho(t,x,v)\,dv \\ &- U'_\rho(T_{l,\sigma}(u_\varepsilon) - c)T'_{l,\sigma}(u_\varepsilon)\operatorname{div}_x f_\varrho(t,x,v)|_{v=u_\varepsilon}.\end{aligned} \tag{3.52}$$

According to (H4), it is clear that $\Gamma_{3,\varepsilon} \in \mathcal{M}_{\mathrm{loc},B}$. Indeed, since $|U'_\rho|, |T'_{l,\sigma}| \le 1$ we have from (3.2b) and (3.2e)

$$\iint_{\mathbf{R}^+\times\mathbf{R}^d} |\Gamma_{3,\varepsilon}|dxdt \le \int_{\mathbf{R}} \frac{\widetilde{C}_3}{1+|v|^{1+\alpha}}dv + \widetilde{C}_1 \le K_6 \tag{3.53}$$

for a constant $K_6$, implying the claim.

Next, we claim that

$$\Gamma_{4,\varepsilon} = \sum_{i=1}^{d} D_{x_i}\Big(\varepsilon D_{x_i} U_\rho(T_{l,\sigma}(u_\varepsilon)-c) + \delta D_u\big[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big]\partial^2_{x_i x_i} u_\varepsilon\Big) \in H^{-1}_{\mathrm{loc},c}. \tag{3.54}$$

Due to a priori estimates (3.8) and (3.9) and, again, the fact that $|T'_{l,\sigma}|, |U'_\rho| \le 1$, we see that for every $i = 1,\dots,d$

$$\varepsilon D_{x_i} U_\rho(T_{l,\sigma}(u_\varepsilon)-c) + \delta D_u\big[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big]\partial^2_{x_i x_i} u_\varepsilon \longrightarrow 0 \tag{3.55}$$

in $L^2(\mathbf{R}^+\times\mathbf{R}^d)$. Therefore, $\Gamma_{4,\varepsilon} \in H^{-1}_{\mathrm{loc},c}$.

Further, we claim that

$$\Gamma_{5,\varepsilon} = \varepsilon U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T''_{l,\sigma}(u_\varepsilon)|\nabla u_\varepsilon|^2 \in \mathcal{M}_{\mathrm{loc},B}. \tag{3.56}$$

Since $|U'_\rho| \le 1$ and $|T''_{l,\sigma}| \le 1\sigma$ we have from (3.30) (recall (3.28) for the definition of the constants $K_j$ for $j = 3,4,5$)

$$\begin{aligned} &\varepsilon\int_{\mathbf{R}^+\times\mathbf{R}^d} \Big|U'_\rho(T_{l,\sigma}(u_\varepsilon)-c)T''_{l,\sigma}(u_\varepsilon)\Big|\;|\nabla u_\varepsilon|^2 dxdt \\ &\quad\le \frac{\varepsilon}{\sigma}\int_{l<|u_\varepsilon|<l+\sigma} |\nabla u_\varepsilon|^2 dxdt \overset{(3.30)}{\le} K_5 + \left(\frac{\delta^2}{\sigma^2\varepsilon^4} + \frac{\delta^2}{\sigma^2(\beta(\varrho))^2\varepsilon^4}\right)^{1/2} K_3K_4 \le K_6, \end{aligned} \tag{3.57}$$

for some constant $K_6$, according to assumptions (3.15) and (3.35) on $\delta = \delta(\varepsilon)$, $\sigma = \sigma(\varepsilon)$, $\varrho = \varrho(\varepsilon)$, and $\beta(\varrho) = \beta(\varepsilon^3)$. Thus, we see that $\Gamma_{5,\varepsilon} \in \mathcal{M}_{\mathrm{loc},B}$.

Next, we need to show

$$\Gamma_{6,\varepsilon} = \delta\sum_{i=1}^{d} D^2_{uu}\big[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\big]\partial_{x_i} u_\varepsilon \partial^2_{x_i x_i} u_\varepsilon \in \mathcal{M}_{\mathrm{loc},B}. \tag{3.58}$$

In view of a priori estimates (3.8) and (3.9), and assumptions (3.15) and (3.35), it holds

$$\begin{aligned}
&\iint_{\mathbf{R}^+\times\mathbf{R}^d}\delta\left|\sum_{i=1}^{d}D^2_{uu}\left[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)\right]\partial_{x_i}u_\varepsilon\partial^2_{x_ix_i}u_\varepsilon\right|dxdt\\
&\quad\le\sum_{i=1}^{d}\left(\frac{\delta}{\sigma}+\frac{\delta}{\rho}\right)\left(\iint_{\mathbf{R}^+\times\mathbf{R}^d}|\partial_{x_i}u_\varepsilon|^2dxdt\right)^{1/2}\left(\iint_{\mathbf{R}^+\times\mathbf{R}^d}\left|\partial^2_{x_ix_i}u_\varepsilon\right|^2dxdt\right)^{1/2}\\
&\quad\le\left(\frac{\delta}{\varepsilon^2\sigma}+\frac{\delta}{\varepsilon^2\rho}\right)\sum_{i=1}^{d}\left(\varepsilon\int_{\mathbf{R}^+\times\mathbf{R}^d}|\partial_{x_i}u_\varepsilon|^2dxdt\right)^{1/2}\left(\varepsilon^3\int_{\mathbf{R}^+\times\mathbf{R}^d}\left|\partial^2_{x_ix_i}u_\varepsilon\right|^2dxdt\right)^{1/2}\\
&\quad\le K_7\left(\frac{\delta}{\varepsilon^2\sigma}+\frac{\delta}{\varepsilon^2\rho}\right)\left(\frac{\varepsilon}{\varrho}+\frac{1}{(\beta(\varrho))^2}\right)^{1/2}\le K_8,
\end{aligned}\tag{3.59}$$

for some constants $K_7$ and $K_8$. The second estimate holds since $U''_\rho\le 1/\rho$ and $T''_{l,\sigma}\le 1/\sigma$ implying $|D_{uu}[U_\rho(T_{l,\sigma}(u_\varepsilon)-c)]|\le(1/\sigma+1/\rho)$. Therefore, $\Gamma_{6,\varepsilon}\in\mathcal{M}_{\mathrm{loc},B}$.

Finally, we will prove that

$$\Gamma_{7\varepsilon}=-\varepsilon U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\left(T'_{l,\sigma}(u_\varepsilon)\right)^2|\nabla u_\varepsilon|^2\in\mathcal{M}_{\mathrm{loc},B}.\tag{3.60}$$

First, notice that supp $U''_\rho=(0,\varrho)$, and therefore

$$U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\neq 0\quad\text{for } c\le T_{l,\sigma}(u_\varepsilon)\le c+\rho.\tag{3.61}$$

Assume first that $|c|>l$. Since we can choose $\rho=\sigma$ (see (3.35)) arbitrarily small, we can assume that $|c|>l+\sigma$. In that case $U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\neq 0$ only if $l+\sigma\le T_{l,\sigma}(u_\varepsilon)\le l+\sigma+\rho$ which is never fulfilled according to the definition of $T_{l,\sigma}$ (see (3.22)). So, in this case,

$$\Gamma_{7\varepsilon}\equiv 0\in\mathcal{M}_{\mathrm{loc},B}.\tag{3.62}$$

Next we assume that $|c|<l$. As before, we can assume that $|c|<l-\rho$ since we can choose $\rho=\sigma$ arbitrarily small. From (3.61), we see that $U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)\neq 0$ if $-l\le c\le T_{l,\sigma}(u_\varepsilon)\le c+\rho\le l$ implying that $U''_\rho(T_{l,\sigma}(u_\varepsilon)-c)=U''_\rho(u_\varepsilon-c)$. Thus,

$$\Gamma_{7\varepsilon}=-\varepsilon U''_\rho(u_\varepsilon-c)|\nabla u_\varepsilon|^2\in\mathcal{M}_{\mathrm{loc},B},\tag{3.63}$$

according to (3.30) (we put there $l=c$).

Finally, assume that $|c|=l$. From (3.61) and (3.19), we conclude

$$\begin{aligned}
\left(T'_{l,\sigma}(u_\varepsilon)\right)^2U''_\rho(T_{l,\sigma}(u_\varepsilon)-l)\neq 0\quad&\text{iff }\left(l\le T_{l,\sigma}(u_\varepsilon)\le l+\rho\text{ and }u_\varepsilon\le l+\sigma\right)\\
&\text{iff }\left(l\le u_\varepsilon\le l+\sigma\right).
\end{aligned}\tag{3.64}$$

From here and since $|T'_{l,\sigma}| \le 1$, it follows (recall that we assume $\rho = \sigma$)

$$\iint_{\Pi_t} \varepsilon U''_\rho(T_{l,\sigma}(u_\varepsilon) - c)\left(T'_{l,\sigma}(u_\varepsilon)\right)^2 |\nabla u_\varepsilon|^2 dx dt \le \frac{\varepsilon}{\sigma} \iint_{\Pi_t \cap \{l < |u_\varepsilon| < l+\sigma\}} |\nabla u_\varepsilon|^2 dx dt < \infty, \tag{3.65}$$

according to (3.30). From here, (3.62) and (3.63), we conclude (3.60).

Collecting the previous items, due to the properties of $\Gamma_{i,\varepsilon}$, $i = 1, \dots, 7$, it follows from (3.33) that

$$\begin{aligned} &\partial_t \theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) \\ &\qquad + \operatorname{div}_x \theta(T_l(u_\varepsilon) - c)\left(f(t, x, T_l(u_\varepsilon)) - f(t, x, c)\right) \in \mathcal{M}_{\mathrm{loc},B} + H^{-1}_{\mathrm{loc},c}. \end{aligned} \tag{3.66}$$

Therefore, we see that (ii) is satisfied and we can use Murat's lemma to conclude that

$$\partial_t \theta(T_l(u_\varepsilon) - c)(T_l(u_\varepsilon) - c) + \operatorname{div}_x \theta(T_l(u_\varepsilon) - c)\left(f(t, x, T_l(u_\varepsilon)) - f(t, x, c)\right) \in H^{-1}_{\mathrm{loc},c}. \tag{3.67}$$

Thus we conclude that the conditions of Theorem 3.3 are satisfied, and we find that for every $l > 0$ the sequence $(T_l(u_\varepsilon))_{\varepsilon>0}$ is precompact in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R})$.

Since the sequence $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(\mathbf{R}^+ \times \mathbf{R}^d)$, from [23, Lemma 7], we conclude that $(u_\varepsilon)_{\varepsilon>0}$ is precompact in $L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^d)$. □

## 4. The One-Dimensional Case

We will analyze the convergence of the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions to (3.5)–(3.6) in the one dimensional case. Unlike the situation we had in the previous section, we will assume that the flux is continuously differentiable with respect to $u$. This will enable us to optimize the ratio $\delta/\varepsilon^2$. We will work under the following assumptions on the flux $f = f(t, x, u)$ denoted (H4′).

(H4a′) For the flux $f = f(t, x, u)$ we assume that $f \in C^1(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}_x)) \cap L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$ and $\partial_u f \in L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$.

(H4b′) There exists a sequence $(f_\varrho)_{\varrho>0}$ defined on $\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$, smooth in $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$, and continuously differentiable in $u \in \mathbf{R}$, satisfying for some $p > 2$ and every $l > 0$:

$$\begin{gathered} \max_{|z| \le l} \left| f_\varrho(t, x, z) - f(t, x, z) \right| \longrightarrow 0, \quad \varrho \longrightarrow 0 \text{ in } L^p_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}), \\ \iint_{\mathbf{R}^+ \times \mathbf{R}} \left| \partial_x f_\varrho(t, x, u) \right| dx dt \le \frac{\widetilde{C}_1}{1 + |u|^{1+\alpha}}, \quad \iint_{\mathbf{R}^+ \times \mathbf{R}} \varrho \left| \partial_x f_\varrho(t, x, u) \right|^2 dx dt \le \widetilde{C}_2(t, x), \\ \iint_{\mathbf{R}^+ \times \mathbf{R}} \left| \partial^2_{xu} f_\varrho(t, x, u) \right| dx dt \le \frac{\widetilde{C}_3}{1 + |u|^{1+\alpha}}, \\ \left| \partial_u f_\varrho(t, x, u) \right| \le C, \end{gathered} \tag{4.1}$$

where $\widetilde{C}_i$, $i = 1, 2, 3$, and $C$ are constants independent on $(t, x, u) \in \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}$.

Under these assumptions we will prove the following.

(i) Without assuming nondegeneracy of the flux, the sequence $(u_\varepsilon)_{\varepsilon>0}$ converges along a subsequence to a solution of (3.1)–(11) in the distributional sense when $\delta = \mathcal{O}(\varepsilon^2)$ and $\varrho = \mathcal{O}(\varepsilon)$ (less stringent assumptions than in the multidimensional case).

(ii) If, in addition, we assume $f \in C^2(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}_x)) \cap L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$, and that $f$ is genuinely nonlinear in the sense of (4.12), the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of problem (3.5)–(3.6) is strongly precompact in $L^1_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R})$ when $\delta = \mathcal{O}(\varepsilon^2)$.

*Remark 4.1.* The proof relies on a priori inequalities (3.8) and (3.9). Notice that thanks to (H4a′), we can take $\beta(\varrho) = 1$ in inequality (3.9).

We will need the fundamental theorem of Young measures.

**Theorem 4.2** (see [24]). *Assume that the sequence $(u_{\varepsilon_k})$ is uniformly bounded in $L^p_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}^d))$, $p \geq 1$. Then, there exists a subsequence (not relabeled) $(u_{\varepsilon_k})$ and a sequence of probability measures*

$$\nu_{(t,x)} \in \mathcal{M}(\mathbf{R}), \qquad (t,x) \in \mathbf{R}^+ \times \mathbf{R}^d \tag{4.2}$$

*such that the limit*

$$\overline{g}(t,x) := \lim_{k \to \infty} g(t,x,u_{\varepsilon_k}(t,x)) \tag{4.3}$$

*exists in the distributional sense for all $g$ measurable with respect to $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$, continuous in $u \in \mathbf{R}$ and satisfying uniformly in $(t,x)$:*

$$\left|g(t,x,u)\right| \leq C\left(1 + |u|^q\right), \tag{4.4}$$

*for a constant $C$ independent of $u$, and $q$ such that $0 \leq q < p$. The limit is represented by the expectation value*

$$\overline{g}(t,x) = \int_{\mathbf{R}^+ \times \mathbf{R}^d} g(t,x,\lambda)\, d\nu_{(t,x)}(\lambda), \tag{4.5}$$

*for almost all points $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$.*

*One refers to such a sequence of measures $\nu = (\nu_{(t,x)})$ as the Young measures associated to the sequence $(u_{\varepsilon_k})_{k \in \mathbf{N}}$.*

*Furthermore,*

$$u_{\varepsilon_k} \longrightarrow u \quad \text{in } L^r_{\mathrm{loc}}\left(\mathbf{R}^+ \times \mathbf{R}^d\right),\ 1 \leq r < p, \tag{4.6}$$

*if and only if*

$$\nu_y = \delta_{u(y)} \quad a.e. \tag{4.7}$$

Before we continue, we need to recall the celebrated Div-Curl lemma.

**Lemma 4.3** (Div-Curl). *Let $Q \subset \mathbf{R}^2$ be a bounded domain, and suppose that*

$$\begin{aligned} v_\varepsilon^1 &\rightharpoonup \overline{v}^1, & v_\varepsilon^2 &\rightharpoonup \overline{v}^2, \\ w_\varepsilon^1 &\rightharpoonup \overline{w}^1, & w_\varepsilon^2 &\rightharpoonup \overline{w}^2, \end{aligned} \tag{4.8}$$

*in $L^2(Q)$ as $\varepsilon \downarrow 0$. Assume also that the two sequences $\{\operatorname{div}(v_\varepsilon^1, v_\varepsilon^2)\}_{\varepsilon>0}$ and $\{\operatorname{curl}(w_\varepsilon^1, w_\varepsilon^2)\}_{\varepsilon>0}$ lie in a (common) compact subset of $H^{-1}_{\mathrm{loc}}(Q)$, where $\operatorname{div}(v_\varepsilon^1, v_\varepsilon^2) = \partial_{x_1} v_\varepsilon^1 + \partial_{x_2} v_\varepsilon^2$ and $\operatorname{curl}(w_\varepsilon^1, w_\varepsilon^2) = \partial_{x_1} w_\varepsilon^2 - \partial_{x_2} w_\varepsilon^1$. Then along a subsequence*

$$\left(v_\varepsilon^1, v_\varepsilon^2\right) \cdot \left(w_\varepsilon^1, w_\varepsilon^2\right) \longrightarrow \left(\overline{v}^1, \overline{v}^2\right) \cdot \left(\overline{w}^1, \overline{w}^2\right) \quad \text{in } \mathfrak{D}'(Q) \text{ as } \varepsilon \downarrow 0. \tag{4.9}$$

**Lemma 4.4.** *Assume that $(u_\varepsilon)_{\varepsilon>0} \in L^2(\mathbf{R}^+ \times \mathbf{R})$ converges weakly in $L^2(\mathbf{R}^+ \times \mathbf{R})$ to a function $u \in L^2(\mathbf{R}^+ \times \mathbf{R})$. Assume that $\eta(t,x,\lambda)$, $(t,x,\lambda) \in \mathbf{R}^+ \times \mathbf{R}^2$, is a function satisfying (4.4) with $q = 2$ such that $\eta \in C^2(\mathbf{R}_\lambda; L^\infty \cap BV(\mathbf{R}_t^+ \times \mathbf{R}_x))$.*

*By $\eta_n$ one denotes the truncation of the function $\eta$:*

$$\eta_n(t,x,\lambda) = \begin{cases} \eta(t,x,\lambda), & |\lambda| < n, \\ 0, & |\lambda| > 2n, \end{cases} \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R}, \tag{4.10}$$

*and $q_n(t,x,\lambda)$ the corresponding entropy flux.*

*If for every $n \in \mathbf{N}$ one has*

$$\operatorname{div}_{(t,x)}\left(\eta_n(t,x,u_\varepsilon), q_n(t,x,u_\varepsilon)\right) \in H^{-1}_{\mathrm{loc},c}(\mathbf{R}^+ \times \mathbf{R}), \tag{4.11}$$

*then the limit function $u$ is a weak solution of* (1.3).

*Furthermore, if the flux function $f = f(t,x,\lambda)$ is twice differentiable with respect to $\lambda$, and is genuinely nonlinear, that is, for every $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ the mapping*

$$\mathbf{R} \ni \lambda \longmapsto \partial_\lambda f(t,x,\lambda) \text{ is nonconstant} \tag{4.12}$$

*on nondegenerate intervals, then $(u_\varepsilon)_{\varepsilon>0}$ converges strongly along a subsequence to $u$ in $L^1_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R})$.*

*Proof.* We will apply the method of compensated compactness as in [13].

First, notice that according to Theorem 4.2 there exist a subsequence $(u_{\varepsilon_k}) \subset (u_\varepsilon)$ and a sequence of probability measures

$$\nu_{(t,x)} \in \mathcal{M}(\mathbf{R}), \quad (t,x) \in \mathbf{R}^+ \times \mathbf{R} \tag{4.13}$$

such that the limit

$$\overline{g}(t,x) := \lim_{k\to\infty} g(t,x,u_{\varepsilon_k}(t,x)) \tag{4.14}$$

exists in the distributional sense for all $g$ measurable with respect to $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$, continuous in $u \in \mathbf{R}$, and satisfying (4.4) for some $q \in \mathbf{R}$ such that $0 \leq q < p$, and is represented by the expectation value

$$\overline{g}(t,x) = \int_{\mathbf{R}^+ \times \mathbf{R}} g(t,x,\lambda) d\nu_{(t,x)}(\lambda), \tag{4.15}$$

for almost all points $(t,x) \in \mathbf{R}^+ \times \mathbf{R}$. Next, notice that the function $f$ satisfies (4.4) for $q = 1$. Indeed, from (H4a′), it follows $\partial_u |f| = \operatorname{sgn}(f)\partial_u f \leq C$, and from here $|f| \leq \widehat{C}(1+|u|)$, for a constant $\widehat{C}$ which depends on the constant $C$ and the $L^\infty$ bound of the function $f$. From this, we conclude that for the flux function $f(t,x,v)$ we have

$$\lim_{k\to\infty} f(t,x,u_{\varepsilon_k}(t,x)) = \int_{\mathbf{R}^+ \times \mathbf{R}} f(t,x,\lambda) d\nu_{(t,x)}(\lambda). \tag{4.16}$$

To continue, notice that

$$u(t,x) = \int \lambda \, d\nu_{(t,x)}(\lambda). \tag{4.17}$$

Take $\eta(u) = I(u) = u$ in (4.10), and consider the vector fields $(I_n(u_\varepsilon), f_n(t,x,u_\varepsilon))$ where $\partial_\lambda f_n(t,x,u_\varepsilon) = I_n'(v)\partial_\lambda f(t,x,u_\varepsilon)$, and $(-\psi_n(t,x,u_\varepsilon), \phi_n(u_\varepsilon))$, where $\phi \in C^1(\mathbf{R})$ is an arbitrary entropy, and $\psi_n$ is the entropy flux corresponding to $\phi_n$. Here $I_n$ and $\phi_n$ denote the smooth truncation functions of $I$ and $\phi$, respectively (cf. (4.10)).

According to (4.11), we can apply the Div-Curl lemma on the given vector fields. Hence, we get after letting $\varepsilon \to 0$ along a subsequence:

$$\begin{aligned}\int &(I_n(\lambda)\psi_n(t,x,\lambda) - \phi_n(\lambda) f_n(t,x,\lambda)) d\nu_{(t,x)}(\lambda) \\ &= \int \left(\overline{u}_n(t,x)\psi_n(t,x,\lambda) - \overline{f}_n(t,x)\phi_n(\lambda)\right) d\nu_{(t,x)}(\lambda),\end{aligned} \tag{4.18}$$

where

$$\overline{f}_n(t,x) = \int f_n(t,x,\lambda) d\nu_{(t,x)}(\lambda), \quad \overline{u}_n(t,x) = \int I_n(\lambda) d\nu_{(t,x)}(\lambda). \tag{4.19}$$

Then, put $\phi(\lambda) = |\lambda - u(t,x)|$. Notice that for $|\lambda| < n$ it holds $\psi_n(t,x,\lambda) = \operatorname{sgn}(\lambda - u(t,x))(f(t,x,\lambda) - f(t,x,u(t,x)))$. Therefore, we have from (4.18)

$$\begin{aligned}
&\int_{-n}^{n}(\lambda \ \operatorname{sgn}(\lambda-u(t,x))(f(t,x,\lambda)-f(t,x,u(t,x)))-|u(t,x)-\lambda|f(t,x,\lambda))d\nu_{(t,x)}(\lambda)\\
&\quad-\int_{-n}^{n}\Big(u(t,x)\ \operatorname{sgn}(\lambda-u(t,x))(f(t,x,\lambda)-f(t,x,u(t,x)))-|u(t,x)-\lambda|\overline{f}_n\Big)d\nu_{(t,x)}(\lambda)\\
&=-\left(\int_{-\infty}^{-n}+\int_{n}^{\infty}\right)(I_n(\lambda)\psi_n(t,x,\lambda)-\phi_n(\lambda)f_n(t,x,\lambda))d\nu_{(t,x)}(\lambda)\\
&\quad+\left(\int_{-\infty}^{-n}+\int_{n}^{\infty}\right)\Big(u(t,x)\psi_n(t,x,\lambda)-\overline{f}_n\phi_n(\lambda)\Big)d\nu_{(t,x)}(\lambda)\\
&\quad+\left(\int_{-\infty}^{-n}+\int_{n}^{\infty}\right)(I_n(\lambda)-\lambda)d\nu_{(t,x)}(\lambda)\int\psi_n(t,x,\lambda)d\nu_{(t,x)}(\lambda).
\end{aligned} \tag{4.20}$$

It is clear that for every fixed $(t,x) \in \mathbf{R}^+ \times \mathbf{R}^d$ the right-hand side of (4.20) tends to zero as $n \to \infty$ implying (due to the Lebesgue dominated convergence theorem)

$$\begin{aligned}
&\int(\lambda \ \operatorname{sgn}(\lambda-u(t,x))(f(t,x,\lambda)-f(t,x,u(t,x)))-|u(t,x)-\lambda|f(t,x,\lambda))d\nu_{(t,x)}(\lambda)\\
&\quad-\int\big(u(t,x)\operatorname{sgn}(\lambda-u(t,x))(f(t,x,\lambda)-f(t,x,u(t,x)))\\
&\quad-|u(t,x)-\lambda|\overline{f}(t,x)\big)d\nu_{(t,x)}(\lambda)=0 \quad \text{in } \mathfrak{D}'\left(\mathbf{R}^+\times\mathbf{R}^d\right).
\end{aligned} \tag{4.21}$$

Now, a standard procedure gives (see, e.g., [6, Remark 2.3])

$$\Big(f(t,x,u(t,x))-\overline{f}(t,x)\Big)\int|\lambda-u(t,x)|d\nu_{(t,x)}(\lambda)=0, \tag{4.22}$$

where $\overline{f}(t,x) = \int f(t,x,\lambda)d\nu_{(t,x)}(\lambda)$. From here, it follows that $u$ is a weak solution to (3.1). This concludes the first part of the lemma. For the details of the procedure one should consult, for example, [13].

Now, assume that $f \in C^2(\mathbf{R}; BV(\mathbf{R}^+ \times \mathbf{R}_x)) \cap L^\infty(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}_x)$, and that it is genuinely nonlinear in the sense of (4.12).

Then, take arbitrary entropies $\eta_1(t,x,u) \in C^1((\mathbf{R}; L^\infty \cap BV(\mathbf{R}_t^+ \times \mathbf{R}_x)))$ and $\eta_2 \in C^1(\mathbf{R})$, and denote by $q_1(t,x,u)$ and $q_2(u)$, respectively, their corresponding entropy fluxes. Assume that $\eta_i$, $q_i$, $\eta_i q_i$, $i = 1,2$, satisfy (4.4) for $q < 2$. Notice that $\partial_u \eta_1$ depends explicitly on $(t,x)$, while $D_u\eta_2$ does not. Denote by $\eta_{1,n}$ and $\eta_{2,n}$ the appropriate smooth truncations (cf. (4.10)) and by $q_{1,n}$ and $q_{2,n}$ the corresponding entropy fluxes, that is,

$$q_{1,n}(t,x,\lambda) = \int^{\lambda} \partial_z \eta_{1,n}(t,x,z)\partial_z f(t,x,z)dz,$$
$$q_{1,n}(t,x,\lambda) = \int^{\lambda} \partial_z \eta_{2,n}(z)\partial_z f(t,x,z)dz. \tag{4.23}$$

Due to (4.11) and the Div-Curl lemma the following commutation relation holds:

$$\int_{\mathbf{R}} (\eta_{1,n}(t,x,\lambda)q_{2,n}(t,x,\lambda) - \eta_2(\lambda)q_{1,n}(t,x,\lambda))d\nu_{(t,x)}$$
$$= \int_{\mathbf{R}} \eta_{1,n}(t,x,\lambda)d\nu_{(t,x)} \int_{\mathbf{R}} q_{2,n}(t,x,\lambda)d\nu_{(t,x)} - \int_{\mathbf{R}} \eta_{2,n}(\lambda)d\nu_{(t,x)} \int_{\mathbf{R}} q_{1,n}(t,x,\lambda)d\nu_{(t,x)}. \tag{4.24}$$

Letting $n \to \infty$ as in (4.20), we get

$$\int_{\mathbf{R}} (\eta_1(t,x,\lambda)q_2(t,x,\lambda) - \eta_2(\lambda)q_1(t,x,\lambda))d\nu_{(t,x)}$$
$$= \int_{\mathbf{R}} \eta_1(t,x,\lambda)d\nu_{(t,x)} \int_{\mathbf{R}} q_2(t,x,\lambda)d\nu_{(t,x)} - \int_{\mathbf{R}} \eta_2(\lambda)d\nu_{(t,x)} \int_{\mathbf{R}} q_1(t,x,\lambda)d\nu_{(t,x)}. \tag{4.25}$$

Next, recall that the function $f$ satisfies (4.4) for $q = 1$. Therefore, the following entropy-entropy fluxes are admissible:

$$\eta_1(t,x,\lambda) = f(t,x,\lambda) - f(t,x,u(t,x)), \qquad q_1(t,x,\lambda) = \int_{u(t,x)}^{\lambda} (\partial_v f(t,x,v))^2 dv,$$
$$\eta_2(\lambda) = \lambda - u(t,x), \qquad q_2(t,x,\lambda) = f(t,x,\lambda) - f(t,x,u(t,x)). \tag{4.26}$$

Then, following [6], we insert the last quantities in (4.25) which yields the following relation:

$$\left(\int_{\mathbf{R}} (f(t,x,\lambda) - f(t,x,u(t,x)))d\nu_{(t,x)}\right)^2$$
$$+ \int_{\mathbf{R}} \left( (\lambda - u(t,x)) \int_{u(t,x)}^{\lambda} (\partial_\varrho f(t,x,\varrho))^2 d\varrho - (f(t,x,\lambda) - f(t,x,u))^2 \right) d\nu_{(t,x)}(\lambda) = 0. \tag{4.27}$$

By the Cauchy–Schwarz inequality

$$\begin{aligned}(f(t,x,\lambda)-f(t,x,u))^2 &= \left(\int_{u(t,x)}^{\lambda}\partial_\varrho f(t,x,\varrho)d\varrho\right)^2\\ &\leq (\lambda-u(t,x))\int_{u(t,x)}^{\lambda}\left[\partial_\varrho f(t,x,\varrho)\right]^2 d\varrho d\nu_{(t,x)}(\lambda),\end{aligned} \tag{4.28}$$

with the equality only if $f(t,x,\varrho)$ is constant for all $\varrho$ between $u(t,x)$ and $\lambda$. Still, this is not possible according to the genuine nonlinearity condition (4.12). Thus, from this and (4.27), we conclude that

$$(\lambda-u(t,x))\int_{u(t,x)}^{\lambda}\left[\partial_\varrho f(t,x,\varrho)\right]^2 d\varrho d\nu_{(t,x)}(\lambda)=0, \tag{4.29}$$

that is, that $\nu_{(t,x)}=\delta_{u(t,x)}$ a.e. on $\mathbf{R}^+\times\mathbf{R}$ implying strong $L^1_{\mathrm{loc}}$ convergence of $(u_\varepsilon)_{\varepsilon>0}$ along a subsequence (see Theorem 4.2). □

Now we are ready to prove the main theorem of the section.

**Theorem 4.5.** *Assume that*

$$\delta=\delta(\varepsilon)=\mathcal{O}\left(\varepsilon^2\right),\quad \varrho=\mathcal{O}(\varepsilon),\ \varepsilon\to 0, \tag{4.30}$$

*and* $u_0\in H^1(\mathbf{R})$.

*Assume that the flux function $f$ from* (3.1) *with $d=1$ satisfies (H4′). Assume also that the function b from* (3.5) *satisfies (H1) and (H2). Then a subsequence of solutions $(u_{\varepsilon_k})\subset(u_\varepsilon)$ of problem* (3.5)–(3.6) *converges in the sense of distributions to a weak solution of problem* (3.1).

*If the flux function $f\in C^2(\mathbf{R};BV(\mathbf{R}^+\times\mathbf{R}_x))\cap L^\infty(\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}_x)$, and if it is genuinely nonlinear in the sense of* (4.12), *then a subsequence of solutions $(u_{\varepsilon_k})\subset(u_\varepsilon)$ of problem* (3.5)–(3.6) *converges strongly in $L^1(\mathbf{R}^+\times\mathbf{R})$ to a weak solution of* (3.1).

*Proof.* Assume that $\eta(t,x,\lambda)$, $(t,x,\lambda)\in\mathbf{R}^+\times\mathbf{R}^2$ is a function such that $\eta\in C^2(\mathbf{R};L^\infty\cap BV(\mathbf{R}^+_t\times\mathbf{R}_x))$. As usual, denote by $\eta_n$ the truncation given by (4.10), and let the entropy flux corresponding to $\eta_n$ and $f$ be

$$q_n(t,x,u)=\int^u\partial_v\eta_n(t,x,v)\partial_v f(t,x,v)dv. \tag{4.31}$$

According to Lemma 4.4, it is enough to prove that for every fixed $n\in\mathbf{N}$ the expression $\operatorname{div}(\eta_n(t,x,u_\varepsilon(t,x)),q_n(t,x,u_\varepsilon(t,x)))$ is precompact in $H^{-1}_{\mathrm{loc}}(\mathbf{R}^+\times\mathbf{R})$.

In order to prove the latter, take the following mollifier $\eta_{n,\varepsilon}(t,x,u) = \eta_n(\cdot,\cdot,u) \star (1/\varepsilon^{1/2})\omega(t/\varepsilon^{1/4})\omega(x/\varepsilon^{1/4})$, where $\omega$ is a nonnegative real function with unit mass. Denote the entropy flux corresponding to $\eta_n$ and $f$ by

$$q_{n,\varepsilon}(t,x,u) = \int^u \partial_v \eta_{n,\varepsilon}(t,x,v)\partial_v f_\varrho(t,x,v)dv. \tag{4.32}$$

Recall that here (and in the sequel) we assume that $\varrho = \mathcal{O}(\varepsilon)$. Actually, we can take $\varrho = \varepsilon$ without loss of generality.

Notice that according to the assumptions on $\eta$ and the choice of the mollifier $\eta_{n,\varepsilon}$ we have

$$\begin{gathered}\iint_{\mathbf{R}^+\times\mathbf{R}} \left(\left|\partial_t \eta_{n,\varepsilon}(t,x,u)\right| + \left|\partial_x \eta_{n,\varepsilon}(t,x,u)\right| + \left|\partial_{xv}\eta_{n,\varepsilon}(t,x,u)\right|\right)dx\,dt \le \widetilde{C}_4,\\ \iint_{\mathbf{R}^+\times\mathbf{R}} \left(\left|\partial_x \eta_{n,\varepsilon}(t,x,u)\right|^2 + \left|\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u)\right|^2\right)dx\,dt \le \frac{\widetilde{C}_4}{\varepsilon},\end{gathered} \tag{4.33}$$

for a constant $\widetilde{C}_4$.

Then, applying (3.17) with $S$ replaced by $\eta_{n,\varepsilon}$, we find

$$\begin{aligned}&D_t \eta_n(t,x,u_\varepsilon) + D_x q_n(t,x,u_\varepsilon)\\ &\quad= \int^{u_\varepsilon} \left(\partial^2_{xv} f_\varrho(t,x,v)\partial_v \eta_{n,\varepsilon}(t,x,v) + \partial_v f_\varrho(t,x,v)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,v)\right)dv\\ &\qquad- \partial_v \eta_{n,\varepsilon}(t,x,u_\varepsilon)\partial_x f_\varrho(t,x,u_\varepsilon) + \partial_t \eta_{n,\varepsilon}(t,x,u_\varepsilon)\\ &\qquad+ \varepsilon D_x\left(\partial_v \eta_{n,\varepsilon}(t,x,u_\varepsilon)b(\partial_x u_\varepsilon)\right) - \varepsilon\partial^2_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)b(\partial_x u_\varepsilon)\partial_x u_\varepsilon\\ &\qquad- \varepsilon b(\partial_x u_\varepsilon)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon) - \delta\partial^2_{xx}u_\varepsilon\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)\\ &\qquad+ \delta D_x\left(\partial_v \eta_{n,\varepsilon}(t,x,u_\varepsilon)\partial^2_{xx}u_\varepsilon\right) - \frac{\delta}{2}\partial^2_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)D_x(\partial_x u_\varepsilon)^2\\ &\qquad+ D_x\left(-q_{n,\varepsilon}(t,x,u_\varepsilon) + q_n(t,x,u_\varepsilon)\right)\\ &\qquad+ D_t\left(-\eta_{n,\varepsilon}(t,x,u_\varepsilon) + \eta_n(t,x,u_\varepsilon)\right).\end{aligned} \tag{4.34}$$

Now, we apply a similar procedure as in the multidimensional case.

Combining (H4b′) and (4.33), we get

$$\iint_{\mathbf{R}^+\times\mathbf{R}} \left|\int^{u_\varepsilon} \left(\partial^2_{xv} f_\varrho(t,x,v)\partial_v \eta_{n,\varepsilon}(t,x,v) + \partial_v f_\varrho(t,x,v)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,v)\right)dv\right| dx\,dt \le \widetilde{C}_6, \tag{4.35}$$

for a constant $\widetilde{C}_6$, implying boundedness of the subintegral expression in the sense of measures.

Similarly, for a constant $\tilde{C}_7$

$$\iint_{\mathbf{R}^+\times\mathbf{R}} \left| -\partial_v \eta_{n,\varepsilon}(t,x,u_\varepsilon)\partial_x f_\varrho(t,x,u_\varepsilon) - \partial_t \eta_{n,\varepsilon}(t,x,u_\varepsilon) \right| dx dt \leq \tilde{C}_7, \tag{4.36}$$

implying boundedness of the subintegral expression in the sense of measures.

Then, combining (4.33) with (3.8) and (3.9) we infer (see estimation of $\Gamma_{6\varepsilon}$) that

$$-\varepsilon b(\partial_x u_\varepsilon)\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon) - \delta\partial^2_{xx}u_\varepsilon\partial^2_{xv}\eta_{n,\varepsilon}(t,x,u_\varepsilon) \tag{4.37}$$

is bounded in $\mathcal{M}(\mathbf{R}^+ \times \mathbf{R})$.

Next,

$$D_x\left(\varepsilon\partial_v\eta_n(t,x,u_\varepsilon)b(\partial_x u_\varepsilon) + \delta\partial_v\eta_n(t,x,u_{\varepsilon_k})\partial^2_{xx}u_{\varepsilon_k}\right) \tag{4.38}$$

is precompact in $H^{-1}(\mathbf{R}^+ \times \mathbf{R})$ since $|\eta_n'| < C$, $\delta = \mathcal{O}(\varepsilon^2)$, $\varrho = \mathcal{O}(\varepsilon)$, and from (3.8) and (3.9) (see also Remark 4.1) we have

$$\varepsilon \mathrm{b}(\partial_\mathrm{x}\mathrm{u}_\varepsilon) + \delta\partial^2_{\mathrm{xx}}\mathrm{u}_\varepsilon \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0 \text{ in } L^2(\mathbf{R}^+ \times \mathbf{R}). \tag{4.39}$$

Similarly, by (3.8) and (3.9) (see the estimation of $\Gamma_{6\varepsilon}$ again)

$$\varepsilon\partial_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)b(\partial_x u_\varepsilon)\partial_x u_\varepsilon + \frac{\delta}{2}\partial_{vv}\eta_{n,\varepsilon}(t,x,u_\varepsilon)D_x(\partial_x u_\varepsilon)^2 \in \mathcal{M}_{\mathrm{loc},B}(\mathbf{R}^+ \times \mathbf{R}). \tag{4.40}$$

Next, due to (H4b′) and the well-known properties of the convolution, it holds for a constant $C$ independent on $\varepsilon$:

$$\left|q_{n,\varepsilon}(t,x,u_\varepsilon) - q_n(t,x,u_\varepsilon)\right| \leq C\varepsilon \longrightarrow 0 \text{ in } L^2_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}) \text{ as } \varepsilon \longrightarrow 0 \tag{4.41}$$

for arbitrary $p > 0$ implying

$$D_x\left(q_{n,\varrho}(t,x,u_\varepsilon) - q_n(t,x,u_\varepsilon)\right) \in H^{-1}_{\mathrm{loc},c}. \tag{4.42}$$

Similarly, it is easy to see that

$$\max_{-2n<v<2n}\left(-\eta_{n,\varepsilon}(t,x,u_\varepsilon) + \eta_n(t,x,u_\varepsilon)\right) \longrightarrow 0 \text{ in } L^2(\mathbf{R}^+ \times \mathbf{R}), \tag{4.43}$$

and thus

$$D_t\left(-\eta_{n,\varepsilon}(t,x,u_\varepsilon) + \eta_n(t,x,u_\varepsilon)\right) \in H^{-1}_c(\mathbf{R}^+ \times \mathbf{R}). \tag{4.44}$$

From (4.35)–(4.44) and the fact that $(\eta_n(t,x,u_\varepsilon), q_n(t,x,u_\varepsilon)) \in L^\infty(\mathbf{R}^+ \times \mathbf{R})$, we conclude using Murat's lemma that

$$\operatorname{div}\big(\eta_n(t,x,u_\varepsilon), q_n(t,x,u_\varepsilon)\big) \in H^{-1}_{\mathrm{loc},c}(\mathbf{R}^+ \times \mathbf{R}). \tag{4.45}$$

Finally, relying on Lemma 4.4 we conclude the proof of the theorem. □

## Acknowledgments

The work is supported in part by the Research Council of Norway. The work was supported by the Research Council of Norway through the Projects Nonlinear Problems in Mathematical Analysis, Waves In Fluids and Solids, and an Outstanding Young Investigators Award (KHK). This paper was written as part of the the International Research Program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2008–2009.

## References

[1] S. N. Kružhkov, "First order quasilinear quations in several independent variables," *Mathematics of the USSR-Sbornik*, vol. 10, pp. 217–243, 1970.

[2] R. J. DiPerna, "Measure-valued solutions to conservation laws," *Archive for Rational Mechanics and Analysis*, vol. 88, no. 3, pp. 223–270, 1985.

[3] K. H. Karlsen, N. H. Risebro, and J. D. Towers, "$L_1$ stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients," *Skrifter. Det Kongelige Norske Videnskabers Selskab*, no. 3, pp. 1–49, 2003.

[4] Adimurthi, S. Mishra, and G. D. Veerappa Gowda, "Optimal entropy solutions for conservation laws with discontinuous flux-functions," *Journal of Hyperbolic Differential Equations*, vol. 2, no. 4, pp. 783–837, 2005.

[5] L. Tartar, "Compensated compactness and applications to partial differential equations," in *Nonlinear Analysis & Mechanics: Heriot-Watt Symposium, Vol. IV*, vol. 39 of *Research Notes in Mathematics*, pp. 136–212, Pitman, Boston, Mass, USA, 1979.

[6] K. H. Karlsen, N. H. Risebro, and J. D. Towers, "On a nonlinear degenerate parabolic transport-diffusion equation with a discontinuous coefficient," *Electronic Journal of Differential Equations*, vol. 93, pp. 1–23, 2002.

[7] P.-L. Lions, B. Perthame, and E. Tadmor, "A kinetic formulation of multidimensional scalar conservation laws and related equations," *Journal of the American Mathematical Society*, vol. 7, no. 1, pp. 169–191, 1994.

[8] F. Bachmann and J. Vovelle, "Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients," *Communications in Partial Differential Equations*, vol. 31, no. 1–3, pp. 371–395, 2006.

[9] K. H. Karlsen, M. Rascle, and E. Tadmor, "On the existence and compactness of a two-dimensional resonant system of conservation laws," *Communications in Mathematical Sciences*, vol. 5, no. 2, pp. 253–265, 2007.

[10] E. Yu. Panov, "Existence and strong precompactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux," *Archive for Rational Mechanics and Analysis*. In press.

[11] L. Tartar, "$H$-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations," *Proceedings of the Royal Society of Edinburgh: Section A*, vol. 115, no. 3-4, pp. 193–230, 1990.

[12] P. Gérard, "Microlocal defect measures," *Communications in Partial Differential Equations*, vol. 16, no. 11, pp. 1761–1794, 1991.

[13] M. E. Schonbek, "Convergence of solutions to nonlinear dispersive equations," *Communications in Partial Differential Equations*, vol. 7, no. 8, pp. 959–1000, 1982.

[14] J. M. C. Correia and P. G. Lefloch, "Nonlinear diffusive-dispersive limits for multidimensional conservation laws," in *Advances in Nonlinear Partial Differential Equations and Related Areas (Beijing, 1997)*, pp. 103–123, World Scientific, Edge, NJ, USA, 1998.

[15] P. G. LeFloch, *Hyperbolic Systems of Conservation Laws*, Birkhäuser, Basel, Switzerland, 2002.

[16] P. G. LeFloch and R. Natalini, "Conservation laws with vanishing nonlinear diffusion and dispersion," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 36, no. 2, pp. 213–230, 1999.

[17] P. G. LeFloch and C. I. Kondo, "Zero diffusion-dispersion limits for scalar conservation laws," *SIAM Journal on Mathematical Analysis*, vol. 33, no. 6, pp. 1320–1329, 2002.

[18] A. Szepessy, "An existence result for scalar conservation laws using measure valued solutions," *Communications in Partial Differential Equations*, vol. 14, no. 10, pp. 1329–1350, 1989.

[19] S. Hwang, "Nonlinear diffusive-dispersive limits for scalar multidimensional conservation laws," *Journal of Differential Equations*, vol. 225, no. 1, pp. 90–102, 2006.

[20] B. Perthame and P. E. Souganidis, "A limiting case for velocity averaging," *Annales Scientifiques de l'École Normale Supérieure*, vol. 31, no. 4, pp. 591–598, 1998.

[21] E. Yu. Panov, "A condition for the strong precompactness of bounded sets of measure-valued solutions of a first-order quasilinear equation," *Sbornik: Mathematics*, vol. 190, pp. 427–446, 1999.

[22] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, vol. 74, American Mathematical Society, Providence, RI, USA, 1990.

[23] G. Dolzmann, N. Hungerbühler, and S. Müller, "Uniqueness and maximal regularity for nonlinear elliptic systems of $n$-Laplace type with measure valued right hand side," *Journal für die Reine und Angewandte Mathematik*, vol. 520, pp. 1–35, 2000.

[24] P. Pedregal, *Parametrized Measures and Variational Principles*, vol. 30 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Basel, Switzerland, 1997.