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Research Article

Asymptotic Behavior of Stochastic Partly Dissipative Lattice Systems in Weighted Spaces

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We study stochastic partly dissipative lattice systems with random coupled coefficients and multiplicative/additive white noise in a weighted space of infinite sequences. We first show that these stochastic partly dissipative lattice differential equations generate a random dynamical system. We then establish the existence of a tempered random bounded absorbing set and a global compact random attractor for the associated random dynamical system.

1. Introduction

Stochastic lattice differential equations (SLDE's) arise naturally in a wide variety of applications where the spatial structure has a discrete character and random spatiotemporal forcing, called noise, is taken into account. These random perturbations are not only introduced to compensate for the defects in some deterministic models, but are also rather intrinsic phenomena. SLDE's may also arise as spatial discretization of stochastic partial differential equations (SPDE's); however, this need not to be the case, and many of the most interesting models are those which are far away from any SPDE's.

The long term behavior of SLDE's is usually studied via global random attractors. For SLDE's on regular spaces of infinite sequences, Bates et al. initiated the study on existence of a global random attractor for a certain type of first-order SLDE's with additive white noise on 1D lattice $\mathbb{Z}$ [1]. Continuing studies have been made on various types of SLDS's with multiplicative or additive noise, see [2–7].

Note that regular spaces of infinite sequences may exclude many important and interesting solutions whose components are just bounded, considering that a weighted space of infinite sequences can make the study of stochastic LDE's more intensive. More importantly, all existing works on SLDE's consider either a noncoupled additive noise or a multiplicative white noise term at each individual node whereas in a realistic system randomness appears at each node as well as the coupling mode between two nodes. Han et al. initiated the asymptotic study of such SLDE's in a weighted space of infinite sequences, with not only additive/multiplicative noise but also coefficients which are randomly coupled [8].

In this work, following the idea of [8], we will investigate the existence of a global random attractor for the following stochastic partly dissipative lattice systems with random coupled coefficients and multiplicative/additive white noise in weighted spaces:

$$\begin{aligned} \dot{u}_i &= -\lambda u_i + \sum_{j=-q}^{q} \eta_{i,j}(\theta_t\omega)u_{i+j} - f_i(u_i) - \alpha v_i + h_i + u_i \circ \frac{dw(t)}{dt}, \\ \dot{v}_i &= -\sigma v_i + \mu u_i + g_i + u_i \circ \frac{dw(t)}{dt}, \quad i \in \mathbb{Z}, \ t > 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} \dot{u}_i &= -\lambda u_i + \sum_{j=-q}^{q} \eta_{i,j}(\theta_t\omega)u_{i+j} - f_i(u_i) - \alpha v_i + h_i + a_i \frac{dw_i(t)}{dt}, \\ \dot{v}_i &= -\sigma v_i + \mu u_i + g_i + b_i \frac{dw_i(t)}{dt}, \quad i \in \mathbb{Z}, \ t > 0, \end{aligned} \tag{1.2}$$

where $u_i, h_i, g_i, a_i, b_i \in \mathbb{R}$ $f_i \in C^1(\mathbb{R}, \mathbb{R})$; $(i \in \mathbb{Z})$, $\lambda, \alpha, \sigma, \mu > 0$ are positive constants; A is the coupling operator, $\eta_{i,-q}(\omega), \ldots, \ \eta_{i,0}(\omega), \ \ldots, \eta_{i,+q}(\omega)$, $i \in \mathbb{Z}$, $q \in \mathbb{N}$, are random variables, and $w(t)$, $\{w_i(t) : i \in \mathbb{Z}\}$ are two-sided Brownian motions on proper probability spaces.

For deterministic partly dissipative lattice systems without noise, the existence of the global attractor has been studied in [9–13]. For stochastic lattice system (1.2) with additive noises, when $q = 1$, $\eta_{i,\pm 1}(\omega) \equiv 1$, $\eta_{i,0}(\omega) \equiv -2$ for all $i \in \mathbb{Z}$, Huang [4] and Wang et al. [14] proved the existence of a global random attractor for the associated RDS in the regular phase space $l^2 \times l^2$. In this work we will consider the existence of a compact global random attractor in the weighted space $l^2_\rho \times l^2_\rho$, which attracts random tempered bounded sets in pullback sense, for stochastic lattice systems (1.1) and (1.2). Here we choose a positive weight function $\rho : \mathbb{Z} \to (0, M_0]$ such that $l^2 \subset l^2_\rho$. If $\sum_{i\in\mathbb{Z}} \rho(i) < \infty$, then l^2_ρ contains any infinite sequences whose components are just bounded and $l^2 \subset l^\infty \subset l^2_\rho$. Note that when $\rho(i) \equiv 1$, our results recover the results obtained in [4, 14] while l^2_ρ is reduced to the standard l^2. Moreover, the required conditions in this work for the existence of a random attractor for system (1.2)-(1.1) in weighted space $l^2_\rho \times l^2_\rho$ are weaker than those in $l^2 \times l^2$.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results for global random attractors of continuous random dynamical systems in weighted spaces of infinite sequences. We then discuss the existence of random attractors for stochastic lattice systems (1.1) and (1.2) in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we present some concepts related to random dynamical systems (RDSs) and random attractors [1, 8, 15] on weighted space of infinite sequences.

Let ρ be a positive function from $\mathbb{Z}$ to $(0, M_0] \subset \mathbb{R}^+$, where M_0 is a finite positive constant. Define for any $i \in \mathbb{Z}$, $\rho_i = \rho(i)$ and

$$l^2_\rho = \left\{ u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} \rho_i |u_i|^2 < \infty, u_i \in \mathbb{R} \right\}, \tag{2.1}$$

then l_ρ^2 is a separable Hilbert space with the inner product $\langle u,v\rangle_\rho = \sum_{i\in\mathbb{Z}} \rho_i u_i v_i$ and norm $\|u\|_\rho^2 = \langle u,u\rangle_\rho = \sum_{i\in\mathbb{Z}} \rho_i |u_i|^2$ for $u=(u_i)_{i\in\mathbb{Z}}$, $v=(v_i)_{i\in\mathbb{Z}} \in l_\rho^2$. Moreover, define

$$H \doteq l_\rho^2 \times l_\rho^2 \tag{2.2}$$

with inner product

$$\left\langle \left(u^{(1)},v^{(1)}\right),\left(u^{(2)},v^{(2)}\right)\right\rangle_H = \left\langle u^{(1)},u^{(2)}\right\rangle_\rho + \left\langle v^{(1)},v^{(2)}\right\rangle_\rho, \quad \text{for } \left(u^{(1)},v^{(1)}\right),\left(u^{(2)},v^{(2)}\right) \in H, \tag{2.3}$$

and norm

$$\|(u,v)\|_H = \left(\|u\|_\rho^2 + \|v\|_\rho^2\right)^{1/2} \quad \text{for } (u,v) \in H, \tag{2.4}$$

then H is also a separable Hilbert space.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\{\theta_t : \Omega \to \Omega, t\in\mathbb{R}\}$ be a family of measure-preserving transformations such that $(t,\Omega)\mapsto \theta_t\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable, $\theta_0 = \mathrm{Id}_\Omega$ and $\theta_{t+s} = \theta_t\theta_s$ for all $s,t\in\mathbb{R}$. The space $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is called a *metric dynamical system*. In the following, "property (P) holds for a.e. $\omega\in\Omega$ with respect to $(\theta_t)_{t\in\mathbb{R}}$" means that there is $\widetilde{\Omega}\subset\Omega$ with $\mathbb{P}(\widetilde{\Omega})=1$ and $\theta_t\widetilde{\Omega}=\widetilde{\Omega}$ such that (P) holds for all $\omega\in\widetilde{\Omega}$.

Recall the following definitions from existing literature.

(i) A stochastic process $\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ is said to be a *continuous RDS* over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ with state space H, if $S:\mathbb{R}^+\times\Omega\times H \to H$ is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(H),\mathcal{B}(H))$-measurable, and for each $\omega\in\Omega$, the mapping $S(t,\omega): H\to H$, $u\mapsto S(t,\omega)u$ is continuous for $t\geq 0$, $S(0,\omega)u = u$ and $S(t+s,\omega) = S(t,\theta_s\omega)S(s,\omega)$ for all $u\in H$ and $s,t\geq 0$.

(ii) A set-valued mapping $\omega\mapsto D(\omega)\subset H$ (may be written as $D(\omega)$ for short) is said to be a *random set* if the mapping $\omega\mapsto \mathrm{dist}_H(u,D(\omega))$ is measurable for any $u\in H$.

(iii) A random set $D(\omega)$ is called a *closed (compact) random set* if $D(\omega)$ is closed (compact) for each $\omega\in\Omega$.

(iv) A random set $D(\omega)$ is said to be *bounded* if there exist $u_0\in H$ and a random variable $r(\omega)>0$ such that $D(\omega)\subset\{u\in H : \|u-u_0\|_H \leq r(\omega)\}$ for all $\omega\in\Omega$.

(v) A random bounded set $D(\omega)$ is said to be *tempered* if for a.e. $\omega\in\Omega$,

$$\lim_{t\to\infty} e^{-\beta t}\sup\{\|u\|_H : u\in D(\theta_{-t}\omega)\} = 0, \quad \forall\beta>0. \tag{2.5}$$

Denote by $\mathfrak{D}(H)$ the set of all tempered random sets of H.

(vi) A random set $B(\omega)$ is said to be a *random absorbing set in* $\mathfrak{D}(H)$ if for any $D(\omega)\in\mathfrak{D}(H)$ and a.e. $\omega\in\Omega$, there exists $T_D(\omega)$ such that $S(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\subset B(\omega)$ for all $t\geq T_D(\omega)$.

(vii) A random set $A(\omega)$ is said to be a *random attracting set* if for any $D(\omega) \in \mathfrak{D}(H)$, we have

$$\lim_{t\to\infty} \text{dist}_H S(t,\theta_{-t}\omega)D(\theta_{-t}\omega, A(\omega)) = 0, \quad \text{a.e. } \omega \in \Omega, \tag{2.6}$$

in which dist_H is the Hausdorff semidistance defined via $\text{dist}_H(E,F) = \sup_{u\in E}\inf_{v\in F}\|u-v\|_\rho$ for any $E, F \subset l^2_\rho$.

(viii) A random compact set $A(\omega)$ is said to be a *random global* $\mathfrak{D}$ *attractor* if it is a compact random attracting set and $S(t,\omega)A(\omega) = A(\theta_t\omega)$ for a.e. $\omega \in \Omega$ and $t \geq 0$.

Definition 2.1 (see [8]). $\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ is said to be random asymptotically null in $\mathfrak{D}(H)$, if for any $D(\omega) \in \mathfrak{D}(H)$, a.e. $\omega \in \Omega$, and any $\varepsilon > 0$, there exist $T(\varepsilon,\omega,D(\omega)) > 0$ and $I(\varepsilon,\omega,D(\omega)) \in \mathbb{N}$ such that

$$\left(\sum_{|i|>I(\varepsilon,\omega,D(\omega))} \rho_i |S(t,\theta_{-t}\omega)u(\theta_{-t}\omega))_i|^2\right)^{1/2} \leq \varepsilon, \quad \forall t \geq T(\varepsilon,\omega,D(\omega)),\ \forall u(\omega) \in D(\omega). \tag{2.7}$$

Theorem 2.2 (see [8]). *Let $\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ be a continuous RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space H and suppose that*

(a) *there exists a random bounded closed absorbing set $B(\omega) \in \mathfrak{D}(H)$ such that for a.e. $\omega \in \Omega$ and any $D(\omega) \in \mathfrak{D}(H)$, there exists $T_D(\omega) > 0$ yielding $S(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega)$ for all $t \geq T_D(\omega)$;*

(b) *$\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ is random asymptotically null on $B(\omega)$; that is, for a.e. $\omega \in \Omega$ and for any $\varepsilon > 0$, there exist $T(\varepsilon,\omega,B(\omega)) > 0$ and $I(\varepsilon,\omega,B(\omega)) \in \mathbb{N}$ such that*

$$\sup_{u\in B(\omega)} \sum_{|i|>I(\varepsilon,\omega,B(\omega))} \rho_i |(S(t,\theta_{-t}\omega)u(\theta_{-t}\omega))_i|^2 \leq \varepsilon^2, \quad \forall t \geq T(\varepsilon,\omega,B(\omega)). \tag{2.8}$$

Then the RDS $\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ possesses a unique global random $\mathfrak{D}$ attractor $A(\omega)$ given by

$$A(\omega) = \bigcap_{\tau\geq T_B(\omega)} \overline{\bigcup_{t\geq\tau} S(t,\theta_{-t}\omega)B(\theta_{-t}\omega)}. \tag{2.9}$$

3. Stochastic Partly Dissipative Lattice Systems with Multiplicative Noise in Weighted Spaces

This section is devoted to the study of asymptotic behavior for system (1.1) in weighted space $H = l^2_\rho \times l^2_\rho$. We first transform the stochastic lattice system (1.1) to random lattice system in Section 3.1. We then show in Section 3.2 that (1.1) generates random dynamical system in H. Finally we prove in Section 3.3 the existence of a global random attractor for system (1.1).

Throughout the rest of this paper, a positive weight function $\rho : \mathbb{Z} \to \mathbb{R}^+$ is chosen to satisfy

(P0) $0 < \rho(i) \leq M_0$ and $\rho(i) \leq c \cdot \rho(i \pm 1),$ for all $i \in \mathbb{Z}$ for some positive constants M_0 and c.

(e.g., $\rho(x) = 1/(1 + \epsilon^2 x^2)^q,\ q > 1/2$ [16, 17] and $\rho(x) = e^{-\epsilon|x|},\ x \in \mathbb{Z}$ where $\epsilon > 0$).

3.1. Mathematical Setting

Define $\Omega_1 = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} = C_0(\mathbb{R}, \mathbb{R})$, and denote by $\mathcal{F}_1$ the Borel σ-algebra on Ω_1 generated by the compact open topology (see [2, 15]) and $\mathbb{P}_1$ the corresponding Wiener measure on $\mathcal{F}_1$. Defining $(\theta_t)_{t\in\mathbb{R}}$ on Ω_1 via $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for $t \in \mathbb{R}$, then $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Consider the stochastic lattice system (1.1) with random coupled coefficients and multiplicative white noise:

$$\begin{aligned} du &= \left(-\lambda u + A(\theta_t\omega)u - f(u) - \alpha v + h\right)dt + u \circ dw(t), \\ dv &= \left(-\sigma v + \mu u + g\right) + u \circ dw(t), \end{aligned} \quad i \in \mathbb{Z},\ t > 0, \tag{3.1}$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $v = (v_i)_{i\in\mathbb{Z}}$; $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$ with $f_i \in C^1(\mathbb{R}, \mathbb{R})$ $(i \in \mathbb{Z})$, $g = (g_i)_{i\in\mathbb{Z}}$, $h = (h_i)_{i\in\mathbb{Z}}$; λ, α, σ, μ are positive constants; $\eta_{i,-q}(\omega), \ldots, \eta_{i,0}(\omega), \ldots, \eta_{i,+q}(\omega)$ $(q \in \mathbb{N})$ are random variables on the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$; $A(\cdot)$ is a linear operator on l^2_ρ defined by

$$(A(\cdot)u)_i = \sum_{j=-q}^{q} \eta_{i,j}(\cdot)u_{i+j}; \tag{3.2}$$

$w(t)$ is a Brownian motion (Wiener process) on the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$; $\circ$ denotes the Stratonovich sense of the stochastic term.

For convenience, we first transform (3.1) into a random differential equation without white noise. Let

$$\delta(\theta_t\omega) = -\int_{-\infty}^{0} e^s\theta_t\omega(s)ds, \quad t \in \mathbb{R},\ \omega \in \Omega_1, \tag{3.3}$$

then $\delta(\theta_t\omega)$ is an Ornstein-Uhlenbeck process on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, (\theta_t)_{t\in\mathbb{R}})$ that solves the following Ornstein-Uhlenbeck equation (see [2, 15] for details)

$$d\delta + \delta dt = dw(t), \quad t \geq 0, \tag{3.4}$$

where $w(t)(\omega) = w(t, \omega) = \omega(t)$ for $\omega \in \Omega_1$, $t \in \mathbb{R}$, and possesses the following properties.

Lemma 3.1 (see [2, 15]). *There exists a θ_t-invariant set $\widetilde{\Omega}_1 \in \mathcal{F}_1$ of Ω_1 of full $\mathbb{P}_1$ measure such that for $\omega \in \widetilde{\Omega}_1$, one has*

(i) *the random variable $|\delta(\omega)|$ is tempered;*

(ii) *the mapping* $\delta(\theta_t\omega)$

$$(t,\omega) \longmapsto \delta(\theta_t\omega) = -\int_{-\infty}^{0} e^s\omega(t+s)ds + w(t) \tag{3.5}$$

is a stationary solution of Ornstein-Uhlenbeck equation (3.4) *with continuous trajectories;*

(iii)

$$\lim_{t\to\pm\infty}\frac{|\delta(\theta_t\omega)|}{t} = \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t \delta(\theta_s\omega)ds = 0. \tag{3.6}$$

The mapping of θ on $\widetilde{\Omega}_1$ possesses same properties as the original one if we choose the trace σ-algebra with respect to $\widetilde{\Omega}_1$ to be denoted also by $\mathcal{F}_1$. Therefore we can change our metric dynamical system with respect to $\widetilde{\Omega}_1$, still denoted by the symbols $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, (\theta_t)_{t\in\mathbb{R}})$.

Let

$$x(t,\omega) = e^{-\delta(\theta_t\omega)}u(t,\omega), \quad y(t,\omega) = e^{-\delta(\theta_t\omega)}v(t,\omega), \quad \omega\in\Omega_1, \tag{3.7}$$

where $(u(t,\omega), v(t,\omega))$ is a solution of (3.1), then $(u(t,\omega), v(t,\omega)) \mapsto (x(t,\omega), y(t,\omega))$ is a homomorphism in H. System (3.1) can then be transformed to the following random system with random coefficients but without white noise:

$$\begin{aligned} \frac{dx}{dt} &= -\lambda x + A(\theta_t\omega)x - e^{-\delta(\theta_t\omega)}f\left(e^{\delta(\theta_t\omega)}x\right) + \delta(\theta_t\omega)x - \alpha y + e^{-\delta(\theta_t\omega)}h, \\ \frac{dy}{dt} &= -\sigma y + \delta(\theta_t\omega)y + \mu x + e^{-\delta(\theta_t\omega)}g \end{aligned} \qquad t>0, \tag{3.8}$$

Letting $\mathbf{z} = (x, y)$, (3.8) are equivalent to

$$\frac{d\mathbf{z}}{dt} = \mathbf{F}(\mathbf{z}, \theta_t\omega), \quad t>0, \tag{3.9}$$

where

$$\mathbf{F}(\mathbf{z}, \theta_t\omega) = \begin{pmatrix} -\lambda x + A(\theta_t\omega)x - e^{-\delta(\theta_t\omega)}f\left(e^{\delta(\theta_t\omega)}x\right) + \delta(\theta_t\omega)x - \alpha y + e^{-\delta(\theta_t\omega)}h \\ -\sigma y + \delta(\theta_t\omega)y + \mu x + e^{-\delta(\theta_t\omega)}g \end{pmatrix}. \tag{3.10}$$

We now make the following standing assumptions on f_i, g_i, h_i, and $\eta_{i,j}$, $(j = -q,\ldots,q)$ $i\in\mathbb{Z}$ and study in the following subsections asymptotic behavior of system (3.9).

(H1) $g = (g_i)_{i\in\mathbb{Z}}$, $h = (h_i)_{i\in\mathbb{Z}} \in l_\rho^2$.

(H2) *Let*

$$\eta(\omega) = \sup\left\{\left|\eta_{i,-q}(\omega)\right|, \ldots, \left|\eta_{i,0}(\omega)\right|, \ldots, \left|\eta_{i,+q}(\omega)\right| : i\in\mathbb{Z}\right\} \geq 0, \quad q\in\mathbb{N}. \tag{3.11}$$

$\eta(\theta_t\omega)$ $(<\infty)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ with respect to $t\in\mathbb{R}$ for each $\omega\in\Omega_1$.

$$\mathbf{E}(\eta)=\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t\eta(\theta_t\omega)ds<\infty, \tag{3.12}$$

and $\eta(\omega)$ is tempered, that is, there exists a θ_t-invariant set $\Omega_{10}\in\mathcal{F}_1$ of full $\mathbb{P}_1$ measure such that for $\omega\in\Omega_{10}$,

$$\lim_{t\to+\infty}e^{-\beta t}\sup_{t\in\mathbb{R}}\left|\eta(\theta_{-t}\omega)\right|=0,\quad \forall\beta>0. \tag{3.13}$$

In the following, we will consider $\omega\in\Omega_{10}\cap\widetilde{\Omega}_1$ and still write $\Omega_{10}\cap\widetilde{\Omega}_1$ as Ω_1.

(H3) $\min\{\lambda,\sigma\}>\widetilde{q}\mathbf{E}|\eta(\omega)|=\lim_{t\to\pm\infty}(1/t)\int_0^t(q+\sum_{k=0}^q c^k)\eta(\theta_t\omega)ds$, *where* $\widetilde{q}=q+\sum_{k=0}^q c^k$.

(H4) *There exists a function $R\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that*

$$\sup_{i\in\mathbb{Z}}\max_{s\in[-r,r]}\left|f_i'(s)\right|\le R(r),\quad \forall r\in\mathbb{R}^+. \tag{3.14}$$

(H5) *$f_i\in C^1(\mathbb{R},\mathbb{R})$, $f_i(0)=0$, $sf_i(s)\ge -b_i^2$, $b=(b_i)_{i\in\mathbb{Z}}\in l^2_\rho$, and there exists a constant $a\ge 0$ such that $f_i'(s)\ge -a$,* for all $s\in\mathbb{R}$, $i\in\mathbb{Z}$.

3.2. Random Dynamical System Generated by Random Lattice System

In this subsection, we show that the random lattice system (3.9) generates a random dynamical system on H.

Definition 3.2. We call $\mathbf{z}:[0,T)\to H$ a solution of the following random differential equation

$$\frac{d\mathbf{z}}{dt}=\mathbf{F}(\mathbf{z},\theta_t\omega),\quad \mathbf{z}=(\mathbf{z}_i)_{i\in\mathbb{Z}},\qquad \mathbf{F}=(\mathbf{F}_i)_{i\in\mathbb{Z}}, \tag{3.15}$$

where $\omega\in\Omega_0$, if $\mathbf{z}\in C([0,T),H)$ satisfies

$$\mathbf{z}_i(t)=\mathbf{z}_i(0)+\int_0^t\mathbf{F}_i(\mathbf{z}(s),\theta_s\omega)ds,\qquad \text{for } i\in\mathbb{Z},\ t\in[0,T). \tag{3.16}$$

Theorem 3.3. *Let $T>0$ and (P0), (H1), (H2), (H4), and (H5) hold. Then for any $\omega\in\Omega_1$ and any initial data $\mathbf{z}_0=(x(0),y(0))\in H$, (3.9) admits a unique solution $\mathbf{z}(\cdot;\omega,\mathbf{z}_0)=(x(\cdot;\omega,\mathbf{z}_0),y(\cdot;\omega,\mathbf{z}_0))\in C([0,T),H)$ with $\mathbf{z}(0;\omega,\mathbf{z}_0)=\mathbf{z}_0$.*

Proof. (1) Denote $E=l^2\times l^2$, we first show that if $\mathbf{z}_0\in E$ and $(h,g)\in E$, then (3.9) admits a unique solution $\mathbf{z}(t;\omega,\mathbf{z}_0,h,g)\in E$ on $[0,T)$ with $\mathbf{z}(0;\omega,\mathbf{z}_0,g,h)=\mathbf{z}_0$. Given $\mathbf{z}\in E$, $\omega\in\Omega_1$, and $(h,g)\in E$, note that $\mathbf{F}(\mathbf{z},\omega)$ is continuous in $\mathbf{z}$ and measurable in ω from $E\times\Omega_1$ to E.

By (3.2) and (H2),

$$\|A(\omega)x\| = \left[\sum_{i\in\mathbb{Z}}\left(\sum_{j=-q}^{q}\eta_{i,j}(\omega)x_{i+j}\right)^2\right]^{1/2} \leq (2q+1)\eta(\omega)\cdot\|x\|. \tag{3.17}$$

By (H4),

$$\left\|f\left(e^{\delta(\omega)}x\right)\right\| \leq \max\left\{R\left(e^{\delta(\omega)}\|x\|\right), a\right\}\cdot e^{\delta(\omega)}\|x\|, \tag{3.18}$$

and therefore

$$\begin{aligned}\|\mathbf{F}(\mathbf{z},\omega)\|_E \leq{} & \left(\lambda + (2q+1)\eta(\omega) + \max\left\{R\left(e^{\delta(\omega)}\|x\|\right), a\right\} + |\delta(\omega)| + \mu\right)\cdot\|x\| \\ & + (\alpha+\sigma+|\delta(\omega)|)\cdot\|y\| + \left|e^{-\delta(\omega)}\right|\cdot(\|h\|+\|g\|).\end{aligned} \tag{3.19}$$

For any $\mathbf{z}^{(1)} = (x^{(1)}, y^{(1)})$, $\mathbf{z}^{(2)} = (x^{(2)}, y^{(2)}) \in E$, and for some $\vartheta \in (0,1)$

$$\begin{aligned}\left\|f\left(e^{\delta(\omega)}x^{(1)}\right) - f\left(e^{\delta(\omega)}x^{(2)}\right)\right\| \leq{} & \max\left\{R\left(e^{\delta(\omega)}\left((1-\vartheta)\left\|x^{(1)}\right\| + \vartheta\left\|x^{(2)}\right\|\right)\right), a\right\} \\ & \cdot e^{\delta(\omega)}\left\|x^{(1)} - x^{(2)}\right\|.\end{aligned} \tag{3.20}$$

Also

$$\left\|A(\omega)x^{(1)} - A(\omega)x^{(2)}\right\| = \sum_{i\in\mathbb{Z}}\left\{\sum_{j=-q}^{q}\eta_{i,j}(\omega)\left(x_{i+j}^{(1)} - x_{i+j}^{(2)}\right)\right\}^{1/2} \leq (2q+1)\eta(\omega)\cdot\left\|x^{(1)} - x^{(2)}\right\|. \tag{3.21}$$

It then follows that

$$\begin{aligned}\left\|\mathbf{F}\left(\mathbf{z}^{(1)},\omega\right) - \mathbf{F}\left(\mathbf{z}^{(2)},\omega\right)\right\|_E \leq{} & (\alpha+\sigma+|\delta(\omega)|)\cdot\left\|y^{(1)} - y^{(2)}\right\| \\ & + \Big[\lambda + (2q+1)\eta(\omega) + \mu + |\delta(\omega)| \\ & \quad + e^{\delta(\omega)}\max\left\{R\left(e^{\delta(\omega)}\left((1-\vartheta)\left\|v^{(1)}\right\| + \vartheta\left\|v^{(2)}\right\|\right)\right), a\right\}\Big] \\ & \cdot\left\|x^{(1)} - x^{(2)}\right\|.\end{aligned} \tag{3.22}$$

For any compact set $D \subset E$ with $\sup_{\mathbf{z}\in D}\|\mathbf{z}\| \leq r$, defining random variable $\zeta_D(\omega) \geq 0$ via

$$\begin{aligned}\zeta_D(\omega) ={} & \left(\lambda + (2q+1)\eta(\omega) + \mu + |\delta(\omega)| + \max\left\{R\left(e^{\delta(\omega)}r\right), a\right\}\cdot e^{\delta(\omega)}\right)r \\ & + (\alpha+\sigma+|\delta(\omega)|)r + \left|e^{-\delta(\omega)}\right|\cdot(\|h\|+\|g\|),\end{aligned} \tag{3.23}$$

we have

$$\int_{\tau}^{\tau+1} \zeta_D(\theta_t\omega)dt < \infty, \quad \forall \tau \in \mathbb{R}, \tag{3.24}$$

and for any $\mathbf{z},\ \mathbf{z}^{(1)},\ \mathbf{z}^{(2)} \in D$,

$$\|\mathbf{F}(\mathbf{z},\omega)\|_E \leq \zeta_D(\omega), \qquad \left\|\mathbf{F}\left(\mathbf{z}^{(1)},\omega\right) - \mathbf{F}\left(\mathbf{z}^{(2)},\omega\right)\right\|_E \leq \zeta_D(\omega)\left\|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\right\|_E. \tag{3.25}$$

According to [15, 19, 20], problem (3.9) possesses a unique local solution $\mathbf{z}(\cdot;\omega,\mathbf{z}_0,g,h) \in C([0,T_{\max}),E)$ $(0 < T_{\max} \leq T)$ satisfying the integral equation

$$\mathbf{z}(t) = \mathbf{z}_0 + \int_0^t \mathbf{F}(\mathbf{z}(s),\omega)ds, \quad \text{for } t \in [0,T_{\max}). \tag{3.26}$$

We will next show that $T_{\max} = T$. Since the set $C_0(\mathbb{R})$ of continuous random process in t is dense in the set $L^1(\mathbb{R})$ (see [18, 21]), for each $\omega \in \Omega_1$, there exists a sequence $\{\eta_{i,j}^{(m)}(t,\omega)\}_{m=1}^{\infty}$ of continuous random process in $t \in \mathbb{R}$ such that

$$\begin{gathered}\lim_{m\to\infty}\int_0^T \left(\eta_{i,j}^{(m)}(s,\omega) - \eta_{i,j}(s,\omega)\right)ds = 0, \quad \forall T > 0, \\ \left|\eta_{i,j}^{(m)}(t,\omega)\right| \leq \left|\eta_{i,j}(\theta_t\omega)\right| \leq \left|\eta(\theta_t\omega)\right|, \quad \forall t \in \mathbb{R}.\end{gathered} \tag{3.27}$$

Consider the random differential equation with initial data $\mathbf{z}_0 \in E$:

$$\frac{d\mathbf{z}^{(m)}}{dt} = \begin{pmatrix} -\lambda x^{(m)} + A_m(t,\omega)x^{(m)} - e^{-\delta(\theta_t\omega)}f\left(e^{\delta(\theta_t\omega)}x^{(m)}\right) + \delta(\theta_t\omega)x^{(m)} - \alpha y^{(m)} + e^{-\delta(\theta_t\omega)}h \\ -\sigma y^{(m)} + \delta(\theta_t\omega)y^{(m)} + \mu x^{(m)} + e^{-\delta(\theta_t\omega)}g \end{pmatrix}, \tag{3.28}$$

where

$$\left(A_m(t,\omega)x^{(m)}\right)_i = \sum_{j=-q}^{q}\eta_{i,j}^{(m)}(t,\omega)x_{i+j}. \tag{3.29}$$

Follow the same procedure as above, (3.28) has a unique solution $\mathbf{z}^{(m)}(\cdot;\omega,\mathbf{z}_0,g,h) \in C([0,T_{\max}^{(m)}),E)$, that is,

$$\mathbf{z}_i^{(m)}(t) = \mathbf{z}_0 + \int_0^t \mathbf{F}_i^{(m)}\left(\mathbf{z}^{(m)}(s),\omega\right)ds, \quad \text{for } t \in \left[0,T_{\max}^{(m)}\right), \tag{3.30}$$

and by the continuity of $A_m(s,\omega)$ in s, there holds

$$\frac{d\mathbf{z}_i^{(m)}}{dt} = \begin{pmatrix} -\lambda x_i^{(m)} + \left(A_m(\theta_t\omega)x^{(m)}\right)_i - e^{-\delta(\theta_t\omega)} f_i\left(e^{\delta(\theta_t\omega)} x_i^{(m)}\right) + \delta(\theta_t\omega) x_i^{(m)} - \alpha y_i^{(m)} + e^{-\delta(\theta_t\omega)} h_i \\ -\sigma y_i^{(m)} + \delta(\theta_t\omega) y_i^{(m)} + \mu x_i^{(m)} + e^{-\delta(\theta_t\omega)} g \end{pmatrix}. \tag{3.31}$$

Note that

$$\begin{aligned} \left(A_m(\theta_t\omega)x^{(m)}\right)_i \cdot x_i^{(m)} &\le \eta(\theta_t\omega) \cdot \left| x_i^{(m)} x_{i-q}^{(m)} + \cdots + x_i^{(m)} x_i^{(m)} + \cdots + x_i^{(m)} x_{i+q}^{(m)} \right|, \\ -a e^{2\delta(\theta_t\omega)} \cdot \left(x_i^{(m)}\right)^2 &\le \left(e^{\delta(\theta_t\omega)} x_i^{(m)}\right) \cdot f_i\left(e^{\delta(\theta_t\omega)} x_i^{(m)}\right) \\ &\le R\left(e^{\delta(\theta_t\omega)} \left\| x^{(m)} \right\|\right) \cdot e^{2\delta(\theta_t\omega)} \left(x_i^{(m)}\right)^2, \quad t \in [0,T], \end{aligned} \tag{3.32}$$

multiplying (3.31) by $\begin{pmatrix} \mu x_i^{(m)} & 0 \\ 0 & \alpha y_i^{(m)} \end{pmatrix}$ and sum over $i \in \mathbb{Z}$ results in

$$\begin{aligned} \frac{d}{dt}\left(\mu\left\|x^{(m)}\right\|^2 + \alpha\left\|y^{(m)}\right\|^2\right) \le{}& [-\min\{\lambda,\sigma\} + 2a + 2\delta(\theta_t\omega) + 2(2q+1)\eta(\theta_t\omega)] \\ & \cdot \left(\mu\left\|x^{(m)}\right\|^2 + \alpha\left\|y^{(m)}\right\|^2\right) + \left(\frac{2\mu}{\lambda}\|h\|^2 + \frac{2\alpha}{\sigma}\|g\|^2\right) e^{-2\delta(\theta_t\omega)}. \end{aligned} \tag{3.33}$$

Applying Gronwall's inequality to (3.33) we obtain that

$$\begin{aligned} \mu\left\|x^{(m)}\right\|^2 + \alpha\left\|y^{(m)}\right\|^2 \le{}& e^{2at + \int_0^t [2\delta(\theta_s\omega) + 2(2q+1)\eta(\theta_s\omega)]ds}\left(\mu\|x(0)\|^2 + \alpha\|y(0)\|^2\right) \\ & + \left(\frac{2\mu}{\lambda}\|h\|^2 + \frac{2\alpha}{\sigma}\|g\|^2\right)\left(e^{2at + \int_0^t [2\delta(\theta_s\omega) + 2(2q+1)\eta(\theta_s\omega)]ds}\right) \\ & \cdot \left(\int_0^t e^{(\min\{\lambda,\sigma\} - 2a)s - 2\delta(\theta_s\omega) + \int_0^s [2\delta(\theta_r\omega) + 2(2q+1)\eta(\theta_r\omega)]dr} ds\right) \\ \doteq{}& \kappa(t), \quad t \in \left[0, T_{\max}^{(m)}\right), \end{aligned} \tag{3.34}$$

where $\kappa(t) \in C([0,T),\mathbb{R})$ is independent of m, which implies that the interval of existence of $\mathbf{z}^{(m)}(t)$ is $[0,T)$, and $\mathbf{z}^{(m)}(\cdot;\omega,\mathbf{z}_0,g,h) \in C^1([0,T),E)$.

By (3.34),

$$\left|x_i^{(m)}\right|^2 + \left|y_i^{(m)}\right|^2 \le \frac{\kappa(t)}{\min\{\mu,\alpha\}}, \quad \forall m \in \mathbb{N},\ t \in [0,T). \tag{3.35}$$

Since $|\mathbf{F}_i^{(m)}(\mathbf{z}^{(m)}(t),\theta_t\omega)|^2 \leq K^2(T,\omega)$ for some $K(T,\omega)>0$ and $t\in[0,T)$, then for any $t,\ \tau \in [0,T), m\in\mathbb{N}$,

$$\left|\mathbf{z}_i^{(m)}(t)-\mathbf{z}_i^{(m)}(\tau)\right| = \int_\tau^t \left|\mathbf{F}_i^{(m)}\left(\mathbf{z}^{(m)}(s),\theta_s\omega\right)\right| ds \leq K(T,\omega)\cdot|t-\tau|. \tag{3.36}$$

By the Arzela-Acoli Theorem, there exists a convergent subsequence $\{\mathbf{z}_i^{(m_k)}(t), t\in[0,T)\}$ of $\{\mathbf{z}_i^{(m)}(t), t\in[0,T)\}$ such that

$$\mathbf{z}_i^{(m_k)}(t) \longrightarrow \overline{\mathbf{z}}_i(t) \quad \text{as } k\longrightarrow\infty,\ t\in[0,T),\ i\in\mathbb{Z}, \tag{3.37}$$

and $\overline{\mathbf{z}}_i(t)$ is continuous on $t\in[0,T)$. Moreover, $|\overline{\mathbf{z}}_i|^2 \leq \kappa(t)/\min\{\mu,\alpha\}$ for $t\in[0,T)$. By (3.27), (3.35), assumption (H2), and the Lebesgue Dominated Convergence Theorem we have

$$\begin{gathered}\lim_{k\to\infty}\int_0^t \left(\eta_{i,j}^{(m_k)}(s,\omega)-\eta_{i,j}(\theta_s\omega)\right)ds = \int_0^t \lim_{k\to\infty}\left(\eta_{i,j}^{(m_k)}(s,\omega)-\eta_{i,j}(\theta_s\omega)\right)ds = 0,\\ \lim_{k\to\infty}\left(\eta_{i,j}^{(m_k)}(s,\omega)-\eta_{i,j}(\theta_s\omega)\right) = 0, \quad \text{for a.e. } s\in[0,T],\\ \lim_{k\to\infty}\int_0^t\left(\eta_{i,j}^{(m_k)}(s,\omega)x_i^{(m_k)}(s)-\eta_{i,j}(\theta_s\omega)\overline{x}_i(s)\right)ds = 0.\end{gathered} \tag{3.38}$$

Thus replacing m by m_k in (3.31) and letting $k\to\infty$ give

$$\overline{\mathbf{z}}_i(t) = (\mathbf{z}_0)_i + \int_0^t \mathbf{F}_i(\overline{\mathbf{z}}_i,\theta_s\omega)ds \quad \text{for } t\in[0,T). \tag{3.39}$$

By the uniquness of the solutions of (3.9), we have $\overline{\mathbf{z}}_i(t)=\mathbf{z}_i(t)$ for $t\in[0,T_{\max})$. By (3.34), $\|\mathbf{z}(t)\|_E^2 \leq \kappa(t)/\min\{\mu,\alpha\}$ for $t\in[0,T_{\max})$, which implies that the solution $\mathbf{z}(t)$ of (3.9) exists globally on $t\in[0,T)$.

(2) Next we prove that for any $\mathbf{z}_0\in H$ and $(h,g)\in H$, (3.9) has a solution $\mathbf{z}(t;\omega,\mathbf{z}_0,h,g)$ on $[0,T)$ with $\mathbf{z}(0;\omega,\mathbf{z}_0,h,g)=\mathbf{z}_0$. Let $\mathbf{z}_{1,0},\ \mathbf{z}_{2,0}\in E$ and $h_1=(h_{1,i})_{i\in\mathbb{Z}}$, $h_2=(h_{2,i})_{i\in\mathbb{Z}}, g_1=(g_{1,i})_{i\in\mathbb{Z}}, g_2=(g_{2,i})_{i\in\mathbb{Z}}\in l^2$. Let $\mathbf{z}_1^{(m)}(t,\omega)$, $\mathbf{z}_2^{(m)}(t,\omega)$ be two solutions of (3.28) with initial data $\mathbf{z}_{1,0}$, $\mathbf{z}_{2,0}$ and h,g replaced by h_1, h_2, g_1, g_2, respectively. Set $\mathbf{d}^{(m)}(t)=\mathbf{z}_1^{(m)}(t)-\mathbf{z}_2^{(m)}(t)=(d_1^{(m)}(t),d_2^{(m)}(t))\in E\subset H$. Take inner product $\langle\cdot,\cdot\rangle_H$ of $(d/dt)\mathbf{d}^{(m)}$ with $\mathbf{d}^{(m)}$ and evaluate each term as follows. By (P0), (H1), (H2), and (H4),

$$\begin{gathered}\left|\left\langle A_m(\theta_t\omega)d_1^{(m)},d_1^{(m)}\right\rangle_\rho\right| \leq \tilde{q}\eta(\theta_t\omega)\left\|d_1^{(m)}\right\|_\rho^2,\\ \left\langle f\left(e^{\delta(\theta_t\omega)}x_1^{(m)}\right)-f\left(e^{\delta(\theta_t\omega)}x_2^{(m)}\right),d_1^{(m)}\right\rangle_\rho \geq -ae^{\delta(\theta_t\omega)}\left\|d_1^{(m)}\right\|_\rho^2,\\ \left\langle f\left(e^{\delta(\theta_t\omega)}x_1^{(m)}\right)-f\left(e^{\delta(\theta_t\omega)}x_2^{(m)}\right),d_1^{(m)}\right\rangle_\rho \leq R\left(e^{\delta(\theta_t\omega)}\left(\left\|x_1^{(m)}\right\|+\left\|x_2^{(m)}\right\|\right)\right)e^{\delta(\theta_t\omega)}\left\|d_1^{(m)}\right\|_\rho^2,\\ \left\langle h_1-h_2,d_1^{(m)}\right\rangle_\rho \leq \|h_1-h_2\|_\rho^2\cdot\left\|d_1^{(m)}\right\|_\rho^2; \quad \left\langle g_1-g_2,d_2^{(m)}\right\rangle_\rho \leq \|g_1-g_2\|_\rho^2\cdot\left\|d_2^{(m)}\right\|_\rho^2.\end{gathered} \tag{3.40}$$

It then follows that

$$\frac{d}{dt}\left(\mu\left\|d_1^{(m)}\right\|_\rho^2+\alpha\left\|d_2^{(m)}\right\|_\rho^2\right)\le\left[-\min\{\lambda,\sigma\}+2a+2\delta(\theta_t\omega)+\tilde{q}\eta(\theta_t\omega)\right]\cdot\left(\mu\left\|d_1^{(m)}\right\|_\rho^2+\alpha\left\|d_2^{(m)}\right\|_\rho^2\right)+\left(\frac{2\mu}{\lambda}\|h_1-h_2\|_\rho^2+\frac{2\alpha}{\sigma}\left\|g_1-g_2\right\|_\rho^2\right)e^{-2\delta(\theta_t\omega)}. \tag{3.41}$$

For $T>0$, applying Gronwall's inequality to (3.41) on $[0,T]$ implies that

$$\mu\left\|d_1^{(m)}(t)\right\|_\rho^2+\alpha\left\|d_2^{(m)}(t)\right\|_\rho^2\le C_T\left(\mu\left\|d_1^{(m)}(0)\right\|_\rho^2+\alpha\left\|d_2^{(m)}(0)\right\|_\rho^2+\|h_1-h_2\|t_\rho^2+\left\|g_1-g_2\right\|_\rho^2\right) \tag{3.42}$$

for some constant C_T depending on T, and thus

$$\left\|\mathbf{z}_1^{(m)}(t)-\mathbf{z}_2^{(m)}(t)\right\|_H^2\le\widetilde{C}_T\left(\left\|\mathbf{z}_1^{(m)}(0)-\mathbf{z}_2^{(m)}(0)\right\|_H^2+\|h_1-h_2\|_\rho^2+\left\|g_1-g_2\right\|_\rho^2\right), \tag{3.43}$$

where $\widetilde{C}_T$ is a constant depending on T. Denote by $\widetilde{E}=\widetilde{l}^2\times\widetilde{l}^2$, where $\widetilde{l}^2=l^2$ with the norm $\|\cdot\|_\rho$. By (3.43), there exists a mapping $\Phi^{(m)}\in C(\widetilde{E}\times\widetilde{E},C([0,T],H))$ such that $\Phi^{(m)}(\mathbf{z}_0,g,h)=\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_0,g,h)$, where $\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_0,g,h)$ is the solution of (3.28) on $[0,T)$ with $\mathbf{z}^{(m)}(0;\omega,\mathbf{z}_0,g,h)=\mathbf{z}_0$. Since $\widetilde{l}^2$ is dense in l_ρ^2, the mapping $\Phi^{(m)}$ can be extended uniquely to a continuous mapping $\widetilde{\Phi}^{(m)}:H\times H\to C([0,T],H)$.

For given $\mathbf{z}_0\in H$ and $(g,h)\in H$, $\widetilde{\Phi}^{(m)}(\mathbf{z}_0,g,h)=\mathbf{z}^{(m)}(\cdot;\omega,\mathbf{z}_0,g,h)\in C([0,T],H)$ for $T>0$. There exist sequences $\{\mathbf{z}_{0n}\}\subset\widetilde{E}$, $\{(h_n,g_n)\}\subset\widetilde{E}$ such that

$$\|\mathbf{z}_{0n}-\mathbf{z}_0\|_E\longrightarrow 0,\quad \|h_n-h\|_\rho\longrightarrow 0,\quad \left\|g_n-g\right\|_\rho\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.44}$$

Let $\mathbf{z}_n^{(m)}(t)=\widetilde{\Phi}^{(m)}(\mathbf{z}_{0n},h_n,g_n)=\Phi^{(m)}(\mathbf{z}_{0n},h_n,g_n)=\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_{0n},h_n,g_n)\in\widetilde{E}$ be the solution of (3.28), then it satisfies the integral equation

$$\left(\mathbf{z}_n^{(m)}\right)_i(t)=(\mathbf{z}_{0n})_i+\int_0^t F_i\left(\mathbf{z}_n^{(m)},\theta_s\omega\right)ds. \tag{3.45}$$

By the continuity of $\widetilde{\Phi}^{(m)}$, we have for $t\in[0,T)$,

$$\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_{0n},h_n,g_n)=\widetilde{\Phi}^{(m)}(\mathbf{z}_{0n},h_n,g_n)\overset{n\to\infty}{\longrightarrow}\widetilde{\Phi}^{(m)}(\mathbf{z}_0,h,g)=\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_0,h,g)\in H. \tag{3.46}$$

Thus for each $i\in\mathbb{Z}$,

$$\left(\mathbf{z}_n^{(m)}\right)_i(t)\longrightarrow\mathbf{z}_i^{(m)}(t):=\mathbf{z}_i^{(m)}(t;\omega,\mathbf{z}_0,h,g)\quad\text{as } n\longrightarrow\infty \text{ uniformly on } t\in[0,T). \tag{3.47}$$

Moreover, $\{(\mathbf{z}_n^{(m)})_i(t)\}$ is bounded in n. Let $n \to \infty$, then we have

$$\mathbf{z}_i^{(m)}(t) = (\mathbf{z}_0)_i + \int_0^t \mathbf{F}_i\left(\mathbf{z}^{(m)}, \theta_s\omega\right) ds, \tag{3.48}$$

and $\mathbf{z}_i^{(m)}(t)$ satisfies the differential equation (3.31).

Multiply equation (3.31) by $\begin{pmatrix} \mu\rho_i x_i^{(m)} & 0 \\ 0 & \alpha\rho_i y_i^{(m)} \end{pmatrix}$ and sum over $i \in \mathbb{Z}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\mu\left\|x^{(m)}\right\|_\rho^2 + \alpha\left\|y^{(m)}\right\|_\rho^2\right) &\le [-\min\{\lambda,\sigma\} + 2a + 2\delta(\theta_t\omega) + \tilde{q}\eta(\theta_t\omega)] \\
&\quad \cdot \left(\mu\left\|x^{(m)}\right\|_\rho^2 + \alpha\left\|y^{(m)}\right\|_\rho^2\right) \\
&\quad + \left(\frac{2\mu}{\lambda}\|h\|_\rho^2 + \frac{2\alpha}{\sigma}\|g\|_\rho^2\right) e^{-2\delta(\theta_t\omega)}, \\
\mu\left\|x^{(m)}\right\|_\rho^2 + \alpha\left\|y^{(m)}\right\|_\rho^2 &\le e^{2at+\int_0^t[2\delta(\theta_s\omega)+\tilde{q}\eta(\theta_s\omega)]ds}\left(\mu\|x(0)\|_\rho^2 + \alpha\|y(0)\|_\rho^2\right) \\
&\quad + \left(\frac{2\mu}{\lambda}\|h\|_\rho^2 + \frac{2\alpha}{\sigma}\|g\|_\rho^2\right)\left(e^{2at+\int_0^t[2\delta(\theta_s\omega)+\tilde{q}\eta(\theta_s\omega)]ds}\right) \\
&\quad \cdot \left(\int_0^t e^{(\lambda+\sigma-2a)s-2\delta(\theta_s\omega)+\int_0^s[2\delta(\theta_r\omega)+\tilde{q}\eta(\theta_r\omega)]dr} ds\right) \\
&\doteq \kappa_\rho(t), \quad t \in [0, T_{\max})
\end{aligned} \tag{3.49}$$

Similar to the process (3.35)–(3.39) in part (1), we obtain the existence of a unique solution $\mathbf{z}(t;\omega,\mathbf{z}_0,g,h) \in H$ of (3.9) with initial data $\mathbf{z}_0 \in H$, which is the limit function of a subsequence of $\{\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_0,g,h)\}$ in H for $t \in [0,T)$. In the latter part of this paper, we may write $\mathbf{z}(t;\omega,\mathbf{z}_0,h,g)$ as $\mathbf{z}(t;\omega,\mathbf{z}_0)$ for simplicity. □

Theorem 3.4. *Assume that (P0), (H1), (H2), (H4), and (H5) hold. Then* (3.9) *generates a continuous RDS* $\{\psi(t,\omega)\}_{t\ge 0,\omega\in\Omega_1}$ *over* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, (\theta_t)_{t\in\mathbb{R}})$ *with state space* H*:*

$$\psi(t,\omega)\mathbf{z}_0 := \mathbf{z}(t;\omega,\mathbf{z}_0) \quad \text{for } \mathbf{z}_0 \in H,\ t \ge 0,\ \omega \in \Omega_1. \tag{3.50}$$

Moreover,

$$\varphi(t,\omega)(u_0,v_0) := e^{\delta(\theta_t\omega)}\psi(t,\omega)e^{-\delta(\omega)}(u_0,v_0) \quad \text{for } (u_0,v_0) \in H,\ t \ge 0,\ \omega \in \Omega_1, \tag{3.51}$$

defines a continuous RDS $\{\varphi(t,\omega)\}_{t\ge 0,\omega\in\Omega_1}$ *over* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1, (\theta_t)_{t\in\mathbb{R}})$ *associated with* (3.1).

Proof. By Theorem 3.3, the solution $\mathbf{z}(t;\omega,\mathbf{z}_0)$ of (3.9) with $\mathbf{z}(0;\omega,\mathbf{z}_0) = \mathbf{z}_0$ exists globally on $[0,\infty)$. It is then left to show that $\mathbf{z}(t;\omega,\mathbf{z}_0) = \mathbf{z}(t;\omega,\mathbf{z}_0,h,g)$ is measurable in $(t,\omega,\mathbf{z}_0)$.

In fact, for $\mathbf{z}_0 \in E$ and $(h,g) \in E$, the solution of (3.9) $\mathbf{z}(t;\omega,\mathbf{z}_0,h,g) \in E$ for $t \in [0,\infty)$. In this case, function $\mathbf{F}(\mathbf{z},t,\omega,h,g) = \mathbf{F}(\mathbf{z},t,\omega)$ is continuous in $\mathbf{z}$,h, g and measurable in t,ω, which implies that $\mathbf{z} : [0,\infty) \times \Omega_1 \times E \times E \to E, (t;\omega,\mathbf{z}_0,h,g) \mapsto \mathbf{z}(t;\omega,\mathbf{z}_0,h,g)$ is $(\mathcal{B}([0,\infty) \times \mathcal{F}_1 \times \mathcal{B}(E) \times \mathcal{B}(E), \mathcal{B}(E))$-measurable.

For $\mathbf{z}_0 \in H$ and $(h,g) \in H$, the solution $\mathbf{z}(t;\omega,\mathbf{z}_0,h,g) \in H$ for $t \in [0,\infty)$. For any given $N > 0$, define $T_N : H \to E$, $(u,v) = ((u_i),(v_i))_{i\in\mathbb{Z}} \to T_N(u,v) = ((T_N(u,v))_i)_{i\in\mathbb{Z}}$ by

$$(T_N(u,v))_j = \begin{cases} (u_j, v_j) & \text{if } |j| \le N, \\ 0 & \text{if } |j| > N, \end{cases} \tag{3.52}$$

and write

$$\mathbf{z}_N(t;\omega,\mathbf{z}_0,h,g) = \mathbf{z}(t;\omega,T_N\mathbf{z}_0,T_N(h,g)). \tag{3.53}$$

Then T_N is continuous and for any $\mathbf{z}_0 \in H$, $(h,g) \in H$, and

$$\mathbf{z}(t;\omega,\mathbf{z}_0,h,g) = \lim_{N\to\infty} \mathbf{z}(t;\omega,T_N\mathbf{z}_0,T_N(h,g)). \tag{3.54}$$

Thus $\mathbf{z} : [0,\infty)\times\Omega_1\times E\times E \to H$ is $(\mathcal{B}([0,\infty))\times\mathcal{F}_0\times\mathcal{B}(E)\times\mathcal{B}(E),\mathcal{B}(H))$-measurable. Observe also that $(\mathrm{Id},\mathrm{Id},T_N,T_N) : [0,\infty)\times\Omega_1\times H\times H \to [0,\infty)\times\Omega_0\times E\times E$ is $(\mathcal{B}([0,\infty))\times\mathcal{F}_0\times\mathcal{B}(H)\times\mathcal{B}(H),\mathcal{B}([0,\infty))\times\mathcal{F}_0\times\mathcal{B}(E)\times\mathcal{B}(E))$-measurable. Hence $\mathbf{z}_N = \mathbf{z}\circ(\mathrm{Id},\mathrm{Id},T_N,T_N) : [0,\infty)\times\Omega_1\times H\times H \to H$ is $(\mathcal{B}([0,\infty))\times\mathcal{F}_0\times\mathcal{B}(H)\times\mathcal{B}(H),\mathcal{B}(H))$-measurable. It then follows from (3.54) that $\mathbf{z} : [0,\infty)\times\Omega_1\times H\times H \to H$ is $(\mathcal{B}([0,\infty))\times\mathcal{F}_1\times\mathcal{B}(\mathrm{H})\times\mathcal{B}(H),\mathcal{B}(H))$-measurable. Therefore, fix $(h,g) \in H$ we have that $\mathbf{z}(t;\omega,\mathbf{z}_0) = \mathbf{z}(t;\omega,\mathbf{z}_0,h,g)$ is measurable in $(t,\omega,\mathbf{z}_0)$. The other statements then follow directly. □

Remark 3.5. If $(h,g) \in E$, system (3.1) defines a continuous RDS $\{\varphi(t)\}_{t\ge0}$ over $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1,(\theta_t)_{t\in\mathbb{R}})$ in both state spaces E and H.

3.3. *Existence of Tempered Random Bounded Absorbing Sets and Global Random Attractors in Weighted Space*

In this subsection, we study the existence of a tempered random bounded absorbing set and a global random attractor for the random dynamical system $\{\varphi(t,\omega)\}_{t\ge0,\omega\in\Omega_1}$ generated by (3.1) in weighted space H.

Theorem 3.6. *Assume that (P0), (H1)–(H5) hold, then there exists a closed tempered random bounded absorbing set $B_{1\rho}(\omega) \in \mathfrak{D}(H)$ of $\{\varphi(t,\omega)\}_{t\ge0,\omega\in\Omega_1}$ such that for any $D(\omega) \in \mathfrak{D}(H)$ and each $\omega \in \Omega_1$, there exists $T_D(\omega) > 0$ yielding $\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B_{1\rho}(\omega)$ for all $t \ge T_D(\omega)$. In particular, there exists $T_{1\rho}(\omega) > 0$ such that $\varphi(t,\theta_{-t}\omega)B_{1\rho}(\theta_{-t}\omega) \subset B_{1\rho}(\omega)$ for all $t \ge T_{1\rho}(\omega)$.*

Proof. (1) For initial condition $\mathbf{z}_0 \in E$ and $(h,g) \in E$, let $\mathbf{z}^{(m)}(t,\omega) = \mathbf{z}^{(m)}(t;\omega,\mathbf{z}_0(\omega),h,g)$ be a solution of (3.28) with $\mathbf{z}_0(\omega) = e^{-\delta(\omega)}\mathbf{z}_0 \in E$, where $\omega \in \Omega_1$, then $\mathbf{z}^{(m)}(t,\omega) \in E$ for all $t \ge 0$. Let $\epsilon_1 > 0$ be such that

$$\lambda_1 = 2\min\{\lambda,\sigma\} - \epsilon_1 > 2\tilde{q}\mathbf{E}|\eta(\omega)|. \tag{3.55}$$

By (H4) and (H5), we have

$$-\|b\|_\rho^2 \le \sum_{i\in\mathbb{Z}} \rho_i\left(e^{\delta(\theta_t\omega)}x_i^{(m)}\right)\cdot f_i\left(e^{\delta(\theta_t\omega)}x_i^{(m)}\right) < \infty, \quad \text{for fixed } t \ge 0, \tag{3.56}$$

$$\begin{aligned}\frac{d}{dt}\left(\mu\left\|x^{(m)}\right\|_\rho^2 + \alpha\left\|y^{(m)}\right\|_\rho^2\right) &\le \left[-\lambda_1 + 2\delta(\theta_t\omega) + 2\tilde{q}\eta(\theta_t\omega)\right]\cdot\left(\mu\left\|x^{(m)}\right\|_\rho^2 + \alpha\left\|y^{(m)}\right\|_\rho^2\right)\\ &\quad + 2e^{-2\delta(\theta_t\omega)}\left(\mu\|b\|_\rho^2 + \frac{\mu}{\epsilon_1}\|h\|_\rho^2 + \frac{\alpha}{\sigma}\|g\|_\rho^2\right).\end{aligned} \tag{3.57}$$

Applying Gronwall's inequality to (3.57), we obtain that for $t > 0$,

$$\begin{aligned}\mu\left\|x^{(m)}(t,\omega)\right\|_\rho^2 + \alpha\left\|y^{(m)}(t,\omega)\right\|_\rho^2 &\le e^{-\lambda_1 t+\int_0^t[2\delta(\theta_s\omega)+2\tilde{q}\eta(\theta_s\omega)]ds}\left(\mu\|x(0)\|_\rho^2 + \alpha\|y(0)\|_\rho^2\right)\\ &\quad + 2\left(\mu\|b\|_\rho^2 + \frac{\mu}{\epsilon_1}\|h\|_\rho^2 + \frac{\alpha}{\sigma}\|g\|_\rho^2\right)e^{-\lambda_1 t+\int_0^t[2\delta(\theta_s\omega)+2\tilde{q}\eta(\theta_s\omega)]ds}\\ &\quad \cdot\int_0^t e^{\lambda_1 s-2\delta(\theta_s\omega)-\int_0^s[2\delta(\theta_r\omega)+2\tilde{q}\eta(\theta_r\omega)]dr}ds.\end{aligned} \tag{3.58}$$

(2) For any $\mathbf{z}_0 \in H$ and $(h,g)\in H$, let $\{\mathbf{z}_{0n}\}\subset E$ and $\{(h_n,g_n)\}\subset E$ be sequences such that

$$\|\mathbf{z}_{0n} - \mathbf{z}_0\|_E \longrightarrow 0, \quad \|h_n - h\|_\rho \longrightarrow 0, \quad \|g_n - g\|_\rho \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.59}$$

Then $\mathbf{z}^{(m)}(t;\omega,\mathbf{z}_{0n},h_n,g_n) \to \mathbf{z}^{(m)}(t;\omega,\mathbf{z}_0,h,g)$ as $n\to\infty$ in H, and (3.58) holds for $\mathbf{z}_0\in H$. Therefore,

$$\begin{aligned}&\mu\left\|x^{(m)}(t,\theta_{-t}\omega,\mathbf{z}_0(\theta_{-t}\omega))\right\|_\rho^2 + \alpha\left\|y^{(m)}(t,\theta_{-t}\omega,\mathbf{z}_0(\theta_{-t}\omega))\right\|_\rho^2\\ &\quad\le e^{-\lambda_1 t+\int_0^t[2\delta(\theta_{s-t}\omega)+2\tilde{q}\eta(\theta_{s-t}\omega)]ds}\left(\mu\|x_0(\theta_{-t}\omega)\|_\rho^2 + \alpha\|y_0(\theta_{-t}\omega)\|_\rho^2\right) + \frac{1}{2}r_{1\rho}^2(\omega)e^{-2\delta(\omega)},\end{aligned} \tag{3.60}$$

where

$$r_{1\rho}^2(\omega) = 4e^{2\delta(\omega)}\left(\mu\|b\|_\rho^2 + \frac{\mu}{\epsilon_1}\|h\|_\rho^2 + \frac{\alpha}{\sigma}\|g\|_\rho^2\right)\int_{-\infty}^0 e^{\lambda_1 s-2\delta(\theta_s\omega)+\int_s^0[2\delta(\theta_r\omega)+2\tilde{q}\eta(\theta_r\omega)]dr}ds < \infty. \tag{3.61}$$

For any $\beta > 0$, since

$$\left(e^{-\beta t} r_{0\rho}(\theta_{-t}\omega)\right)^2 \left(\mu\|b\|_\rho^2 + \frac{\mu}{\epsilon_1}\|h\|_\rho^2 + \frac{\alpha}{\sigma}\|g\|_\rho^2\right)^{-1} = 4e^{-2\beta t + \delta(\theta_{-t}\omega)} \int_{-\infty}^{-t} e^{\lambda_1(s+t) - 2\delta(\theta_s\omega) + \int_s^{-t}[2\delta(\theta_r\omega) + 2\tilde{q}|\eta(\theta_r\omega)|]dr} ds \overset{t \to +\infty}{\longrightarrow} 0, \tag{3.62}$$

then $r_{1\rho}(\omega)$ is tempered.

Let $\mathbf{z}(t,\omega) = \varphi(t,\omega)\mathbf{z}_0(\omega) = \mathbf{z}(t;\omega,\mathbf{z}_0(\omega),h,g)$ be a solution of equation (3.9) with $\mathbf{z}_0(\omega) = e^{-\delta(\omega)}(u_0,v_0) \in H$, where $\omega \in \Omega_1$ and $(h,g) \in H$, then $\mathbf{z}(t,\omega) \in H$, and there exists a subsequence $\mathbf{z}^{(m_k)}(t,\omega)$ converging to $\mathbf{z}(t,\omega)$ as $m_k \to \infty$ for all $t \geq 0$. Inequality (3.60) still holds after replacing $\mathbf{z}^{(m)}(t,\omega)$ by $\mathbf{z}(t,\omega)$ since the right hand of (3.60) is independent of m. Thus for $(u_0,v_0) \in D(\theta_{-t}\omega)$,

$$\begin{aligned}
\mathbf{z}_0(\theta_{-t}\omega) &= \left(x_0(\theta_{-t}\omega), y_0(\theta_{-t}\omega)\right) = e^{-\delta(\theta_{-t}\omega)}(u_0,v_0), \\
&\mu\|u(t,\theta_{-t}\omega,(u_0,v_0))\|_\rho^2 + \alpha\|v(t,\theta_{-t}\omega,(u_0,v_0))\|_\rho^2 \\
&\leq e^{2\delta(\omega)} e^{-\lambda_1 t - 2\delta(\theta_{-t}\omega) + \int_0^t[2\delta(\theta_{s-t}\omega) + 2\tilde{q}\eta(\theta_{s-t}\omega)]ds} \sup_{(u,v)\in D(\theta_{-t}\omega)} \left(\mu\|u\|_\rho^2 + \alpha\|v\|_\rho^2\right) \\
&+ \frac{1}{2} r_{1\rho}^2(\omega).
\end{aligned} \tag{3.63}$$

Let $\epsilon_2 = \lambda_1 - 2\tilde{q}\mathbf{E}|\eta(\omega)| > 0$. By properties of $\eta(\theta_{\pm t}\omega)$ and $D \in \mathfrak{D}(H)$, we have

$$e^{-[\epsilon_2/2 + 2\tilde{q}\mathbf{E}|\eta(\omega)|]t - \delta(\theta_{-t}\omega) + \int_{-t}^0[2\delta(\theta_s\omega) + 2\tilde{q}\eta(\theta_s\omega)]ds} \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty, \tag{3.64}$$

and hence

$$\lim_{t \to +\infty} e^{2\delta(\omega)} e^{-\lambda_1 t - \delta(\theta_{-t}\omega) + \int_{-t}^0[2\delta(\theta_s\omega) + 2\tilde{q}\eta(\theta_s\omega)]ds} \sup_{(u,v)\in D(\theta_{-t}\omega)} \left(\mu\|u\|_\rho^2 + \alpha\|v\|_\rho^2\right) = 0. \tag{3.65}$$

Denote by $\tilde{r}_{1\rho} = r_{1\rho}/\sqrt{\min\{\mu,\alpha\}}$, it follows that

$$B_{1\rho}(\omega) = \overline{\left\{(u,v) \in H : \|(u,v)\|_H \leq \tilde{r}_{1\rho}(\omega)\right\}} = \overline{\mathcal{B}_H\left(0,\tilde{r}_{1\rho}(\omega)\right)} \subset H \tag{3.66}$$

is a tempered closed random absorbing set for $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega_1}$. □

Theorem 3.7. *Assume that (P0), (H1)–(H5) hold, then the RDS* $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega_1}$ *generated by* (3.1) *possesses a unique global random* $\mathfrak{D}$ *attractor given by*

$$A_1(\omega) = \bigcap_{\tau \geq T_{B_{1\rho}}(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}\omega) B_{1\rho}(\theta_{-t}\omega)} \subset H. \tag{3.67}$$

Proof. According to Theorem 2.2, it remains to prove the asymptotically nullness of $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega_1}$; that is, for any $\varepsilon > 0$, there exists $T(\varepsilon,\omega,B_{1\rho}) > T_{1\rho}(\omega)$ and $I(\varepsilon,\omega) \in \mathbb{N}$

such that when $t \geq T(\varepsilon, \omega, B_{1\rho})$, the solution $\varphi(t,\omega)(u_0,v_0) = ((u_i,v_i)(t;\omega,u_0,v_0))_{i\in\mathbb{Z}} \in H$ of (3.1) with $(u_0,v_0) \in B_{1\rho}(\theta_{-t}\omega)$ satisfies

$$\sum_{|i|\geq I(\varepsilon,\omega)} \rho_i \left|\left(\varphi(t,\omega)(u_0,v_0)\right)_i\right|^2 = \sum_{|i|\geq I(\varepsilon,\omega)} \rho_i |(u_i,v_i)(t;\omega,u_0,v_0)|^2 \leq \varepsilon. \tag{3.68}$$

Choose a smooth increasing function $\xi \in C^1(\mathbb{R}^+,[0,1])$ such that

$$\begin{gathered} \xi(s) = 0, \quad 0 \leq s \leq 1, \\ 0 \leq \xi(s) \leq 1, \quad 1 \leq s \leq 2, \qquad \left|\xi'(s)\right| \leq C_0 \text{ (constant)} \quad \text{for } s \in \mathbb{R}^+, \\ \xi(s) = 1, \quad s \geq 2. \end{gathered} \tag{3.69}$$

Let $(u,v)(t;\omega,u_0,v_0,h,g) = ((u_i,v_i)(t;\omega,u_0,v_0,h,g))_{i\in\mathbb{Z}}$ be a solution of (3.1), then

$$\mathbf{z}(t;\omega,\mathbf{z}_0(\omega),h,g) = \left(\mathbf{z}_i(t;\omega,\mathbf{z}_0(\omega),h,g)\right)_{i\in\mathbb{Z}} = e^{-\delta(\theta_t\omega)}\varphi(t,\omega)(u_0,v_0) \tag{3.70}$$

is a solution of (3.9) with $\mathbf{z}_0(\omega) = e^{-\delta(\omega)}(u_0,v_0) \in H$.

Let $\mathbf{z}_{0n} = T_n\mathbf{z}_0, (h_n,g_n) = T_n(h,g)$, where T_n is as in (3.52). Then $\mathbf{z}_{0n} \in E, (h_n,g_n) \in E$ and $\mathbf{z}(t;\omega,\mathbf{z}_{0n},h_n,g_n) \to \mathbf{z}(t;\omega,\mathbf{z}_0,h,g)$ in H. For any $n \geq 1$, let $\mathbf{z}^{(m)}(t) = \mathbf{z}^{(m)}(t;\omega,\mathbf{z}_{0n}(\omega),h_n,g_n)$ be the solution of (3.28), where $\mathbf{z}^{(m)}(0) = \mathbf{z}_{0n}(\omega)$. By Theorem 3.4, $\mathbf{z}^{(m)}(\cdot) \in C([0,\infty),E) \cap C^1((0,\infty),E)$. Let M be a suitable large integer (will be specified later); multiply (3.31) by $\begin{pmatrix} \mu\rho_i\xi(|i|/M)x_i^{(m)} & 0 \\ 0 & \alpha\rho_i\xi(|i|/M)y_i^{(m)} \end{pmatrix}$ and sum over $i \in \mathbb{Z}$, we obtain

$$\begin{aligned} &\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\rho_i\left(\mu\left(x_i^{(m)}\right)^2 + \alpha\left(y_i^{(m)}\right)^2\right) \\ &\quad \leq \left[-\lambda_1 + 2\delta(\theta_t\omega) + 2\tilde{q}\eta(\theta_t\omega)\right] \cdot \sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\rho_i\left(\mu\left(x_i^{(m)}\right)^2 + \alpha\left(y_i^{(m)}\right)^2\right) \\ &\qquad + \eta(\theta_t\omega)\cdot\frac{cC_0}{M}\left\|x^{(m)}\right\|_\rho^2 + 2e^{-2\delta(\theta_t\omega)}\sum_{|i|\geq M}\rho_i\left(\mu b_i^2 + \frac{\mu}{\epsilon_1}(h_n)_i^2 + \frac{\alpha}{\sigma}(g_n)_i^2\right). \end{aligned} \tag{3.71}$$

Applying Gronwall's inequality to (3.71) from $T_{1\rho} = T_{B_{1\rho}}(\omega)$ to t gives

$$\begin{aligned} &\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{M}\right)\rho_i\left(\mu\left(x_i^{(m)}(t;\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2 + \alpha\left(y_i^{(m)}(t;\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2\right) \\ &\quad \leq e^{-\lambda_1(t-T_{1\rho})+\int_{T_{1\rho}}^t[2\delta(\theta_s\omega)+2\tilde{q}\eta(\theta_s\omega)]ds}\cdot\left(\mu\left\|x^{(m)}(T_{1\rho})\right\|_\rho^2 + \alpha\left\|y^{(m)}(T_{1\rho})\right\|_\rho^2\right) \\ &\qquad + \sum_{|i|\geq M}\rho_i\left(\mu b_i^2 + \frac{\mu}{\epsilon_1}(h_n)_i^2 + \frac{\alpha}{\sigma}(g_n)_i^2\right)\int_{T_{1\rho}}^t e^{-\lambda_1(t-\tau)+\int_\tau^t[2\delta(\theta_s\omega)+2\tilde{q}\eta(\theta_s\omega)]ds-2\delta(\theta_\tau\omega)}d\tau \\ &\qquad + \frac{cC_0}{M}\int_{T_{1\rho}}^t e^{-\lambda_1(t-\tau)+\int_\tau^t[2\delta(\theta_s\omega)+2\tilde{q}|\eta(\theta_s\omega)|]ds-2\delta(\theta_\tau\omega)}\eta(\theta_\tau\omega)\left(\mu\left\|x^{(m)}(\tau)\right\|_\rho^2 + \alpha\left\|y^{(m)}(\tau)\right\|_\rho^2\right)d\tau. \end{aligned} \tag{3.72}$$

Therefore for $(u_0, v_0) \in B_{1\rho}(\theta_{-t}\omega) \cap H$,

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \xi\left(\frac{|i|}{M}\right)\rho_i\left(\mu\left(x_i^{(m)}(t;\theta_{-t}\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2 + \alpha\left(y_i^{(m)}(t;\theta_{-t}\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2\right)\\
&\quad\le \underbrace{e^{-\lambda_1(t-T_{1\rho})+\int_{T_{1\rho}}^t [2\delta(\theta_{s-t}\omega)+2\tilde{q}|\eta(\theta_{s-t}\omega)|]ds}\left(\mu\left\|x^{(m)}(T_{1\rho})\right\|_\rho^2 + \alpha\left\|y^{(m)}(T_{1\rho})\right\|_\rho^2\right)}_{(i)}\\
&\qquad + \underbrace{\sum_{|i|\ge M}\rho_i\left(\mu b_i^2 + \frac{\mu}{\epsilon_1}(h_n)_i^2 + \frac{\alpha}{\sigma}(g_n)_i^2\right)\int_{T_{1\rho}}^t e^{-\lambda_1(t-\tau)+\int_\tau^t [2\delta(\theta_{s-t}\omega)+2\tilde{q}|\eta(\theta_{s-t}\omega)|]ds-2\delta(\theta_{\tau-t}\omega)}d\tau}_{(ii)}\\
&\qquad + \underbrace{\frac{cC_0}{M}\int_{T_{1\rho}}^t e^{-\lambda_1(t-\tau)+\int_\tau^t [2\delta(\theta_{s-t}\omega)+2\tilde{q}|\eta(\theta_{s-t}\omega)|]ds-2\delta(\theta_{\tau-t}\omega)}\eta(\theta_{\tau-t}\omega)\left(\mu\left\|x^{(m)}(\tau)\right\|_\rho^2 + \alpha\left\|y^{(m)}(\tau)\right\|_\rho^2\right)d\tau}_{(iii)}.
\end{aligned}
\tag{3.73}
$$

We next estimate terms $(i), (ii), (iii)$ on the right-hand side of (3.73). By (3.61),

$$
e^{-\lambda_1(t-T_{1\rho})+\int_{T_{1\rho}}^t [2\delta(\theta_{s-t}\omega)+2\tilde{q}\eta(\theta_{s-t}\omega)]ds}\left(\mu\left\|x^{(m)}(T_{1\rho})\right\|_\rho^2 + \alpha\left\|y^{(m)}(T_{1\rho})\right\|_\rho^2\right) \longrightarrow 0 \quad \text{as } t \longrightarrow +\infty,
\tag{3.74}
$$

which implies that for all $\varepsilon > 0$, there exists $T_1(\varepsilon, \omega, B_{1\rho}) \ge T_{1\rho}$ such that for $t \ge T_1(\varepsilon, \omega, B_{1\rho})$,

$$
(i) \le \frac{\varepsilon}{3}\min\{\mu, \alpha\}e^{-2\delta(\omega)}.
\tag{3.75}
$$

By $h, g, b \in l_\rho^2$ and

$$
\int_{T_{1\rho}}^t e^{-\lambda_1(t-\tau)+\int_\tau^t [2\delta(\theta_{s-t}\omega)+2\tilde{q}\eta(\theta_{s-t}\omega)]ds-2\delta(\theta_{\tau-t}\omega)}d\tau \le \int_{-\infty}^0 e^{\lambda_1\tau-\int_\tau^0 [\delta(\theta_s\omega)+2\tilde{q}\eta(\theta_s\omega)]ds-2\delta(\theta_\tau\omega)}d\tau < \infty,
\tag{3.76}
$$

there exists $I_1(\varepsilon, \omega) \in \mathbb{N}$ such that for $M > I_1(\varepsilon, \omega)$,

$$
(ii) \le \frac{\varepsilon}{3}\min\{\mu, \alpha\}e^{-2\delta(\omega)}.
\tag{3.77}
$$

Note that $\epsilon_2 = \lambda_1 - 2\tilde{q}\mathbf{E}|\eta(\omega)| > 0$, then by (H2), $\eta(\omega)$ is tempered and it follows that there exists a $T_1' > 0$ such that

$$
\eta(\theta_{\tau-t}\omega) \le e^{(\epsilon_2/3)(t-\tau)}, \quad \forall t-\tau \ge T_1'.
\tag{3.78}
$$

Therefore

$$\begin{aligned}
&\int_{T_{1\rho}}^{t} e^{-\lambda_1(t-\tau)+\int_\tau^t[2\delta(\theta_{s-t}\omega)+2\widetilde{q}\eta(\theta_{s-t}\omega)]ds-2\delta(\theta_{\tau-t}\omega)}\eta(\theta_{\tau-t}\omega)\left(\mu\left\|x^{(m)}(\tau)\right\|_\rho^2+\alpha\left\|y^{(m)}(\tau)\right\|_\rho^2\right)d\tau \\
&\quad\le\left(\mu\|x_0\|_\rho^2+\alpha\|y_0\|_\rho^2\right)e^{-(\lambda_1-(\epsilon_2/3))t-4\delta(\theta_{-t}\omega)+\int_{-t}^0[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)]ds}\int_{T_{1\rho}}^{t} e^{(-\epsilon_2/3)\tau}d\tau \\
&\qquad+\frac{1}{2}r_{1\rho}^2(\omega)e^{-2\delta(\omega)}\int_0^{t-T_{1\rho}} e^{-(\lambda_1-(\varepsilon_2/3))\tau+\int_0^\tau[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)]ds}d\tau.
\end{aligned}\tag{3.79}$$

For $t\gg T_{1\rho}$, by (H2) and (3.6), there exists $0<T_2'=T_2'(\omega)<t-T_{1\rho}$ such that

$$\frac{1}{\tau}\int_0^\tau\left[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)\right]ds<\lambda_1-\frac{2\epsilon_2}{3},\quad \text{for } \tau\ge T_2'.\tag{3.80}$$

Let $T_2=\max\{T_1',T_2'\}<\infty$, and $t>T_2+T_{1\rho}$, write

$$\int_0^{t-T_{1\rho}} e^{-(\lambda_1-\epsilon_2/3)\tau+\int_0^\tau[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)]ds}d\tau=\left(\int_0^{T_2}+\int_{T_2}^{t-T_{1\rho}}\right)e^{-(\lambda_1-\epsilon_2/3)\tau+\int_0^\tau[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)]ds}d\tau,\tag{3.81}$$

of which

$$\begin{gathered}
\int_0^{T_2} e^{-(\lambda_1-\epsilon_2/3)\tau+\int_0^\tau[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)]ds}d\tau<\infty, \\
\int_{T_2}^{t-T_{1\rho}} e^{-(\lambda-\epsilon_2/3)\tau+\int_0^\tau[2\delta(\theta_s\omega)+2\widetilde{q}\eta(\theta_s\omega)]ds}d\tau\le\frac{3}{\epsilon_2}e^{(-\epsilon_2/3)T_2}.
\end{gathered}\tag{3.82}$$

Equation (3.79) together with (3.82) implies that there exist $T_3(\varepsilon,\omega,B_{1\rho})\ge T_2$ and $I_2(\varepsilon,\omega)\in\mathbb{N}$ such that for $M>I_2(\varepsilon,\omega),t\ge T_3(\varepsilon,\omega,B_{1\rho})$,

$$(iii)\le\frac{\varepsilon}{3}\min\{\mu,\alpha\}e^{-2\delta(\omega)}.\tag{3.83}$$

In summary, let

$$\begin{gathered}
T(\varepsilon,\omega,B_{1\rho})=\max\{T_1(\varepsilon,\omega,B_{1\rho}),T_2(\varepsilon,\omega,B_{1\rho}),T_3(\varepsilon,\omega,B_{1\rho})\}, \\
I(\varepsilon,\omega)=\max\{I_1(\varepsilon,\omega),I_2(\varepsilon,\omega)\}
\end{gathered}\tag{3.84}$$

Then for $t>T(\varepsilon,\omega,B_{1\rho})$ and $M\ge I(\varepsilon,\omega)$, we have

$$\sum_{|i|\ge 2M}\rho_i\left(\mu\left(x_i^{(m)}(t;\theta_{-t}\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2+\alpha\left(y_i^{(m)}(t;\theta_{-t}\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2\right)\le\varepsilon\min\{\mu,\alpha\}e^{-2\delta(\omega)}.\tag{3.85}$$

Since $\mathbf{z}_i^{(m_k)}(t;\theta_{-t}\omega,\mathbf{z}_{0n}(\theta_{-t}\omega),h_n,g_n) \to \mathbf{z}_i(t;\theta_{-t}\omega,\mathbf{z}_{0n}(\theta_{-t}\omega),h_n,g_n)$ as $m_k \to \infty$, by (3.85),

$$\begin{aligned}\sum_{|i|\geq 2M}\rho_i\left|(\varphi(t,\omega)(u_0,v_0))_i\right|^2 &= \sum_{|i|\geq 2M}\rho_i e^{2\delta(\omega)}\left|\mathbf{z}_i(t;\theta_{-t}\omega,\mathbf{z}_{0n},g_n,h_n)\right|^2\\ &\leq \frac{e^{2\delta(\omega)}}{\min\{\mu,\alpha\}}\sum_{|i|\geq 2M}\rho_i\left(\mu\left(x_i^{(m)}(t;\theta_{-t}\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2+\alpha\left(y_i^{(m)}(t;\theta_{-t}\omega,\mathbf{z}_{0n},h_n,g_n)\right)^2\right)\\ &\leq \varepsilon.\end{aligned} \tag{3.86}$$

Letting $n \to \infty$ in (3.86), we obtain

$$\sum_{|i|\geq 2M}\rho_i\left|(\varphi(t,\omega)(u_0,v_0))_i\right|^2 \leq \varepsilon \tag{3.87}$$

That is, $\{\varphi(t,\omega)\}_{t\geq 0,\omega\in\Omega_1}$ is asymptotically null on $B_{1\rho}(\omega)$, which completes the proof. □

4. Stochastic Partly Dissipative Lattice Systems with Additive White Noise in Weighted Spaces

This section is devoted to the study of asymptotic behavior for system (1.2) in weighted space $H = l_\rho^2 \times l_\rho^2$. The structure and the idea of proofs are similar to that of Section 3, and we will present our major results without elaborting the details of proofs in this section.

4.1. Mathematical Setting

Define $\Omega_2 = \{\omega \in C(\mathbb{R}, l^2) : \omega(0) = 0\}$, and denote by $\mathcal{F}_2$ the Borel σ-algebra on Ω_2 generated by the compact open topology [1] and $\mathbb{P}_2$ is the corresponding Wiener measure on $\mathcal{F}_2$, then $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system. Let the infinite sequence e^i $(i \in \mathbb{Z})$ denote the element having 1 at position i and 0 for all other components. Write

$$W^1(t,\omega) = \sum_{i\in\mathbb{Z}} a_i w_i(t) e^i, \qquad W^2(t,\omega) = \sum_{i\in\mathbb{Z}} b_i w_i(t) e^i, \tag{4.1}$$

where $\{w_i(t) : i \in \mathbb{Z}\}$ are independent two-sided Brownian motions on probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$; then $W^1(t,\omega)$ and $W^2(t,\omega)$ are Wiener processes with values in l^2 defined on the probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$.

Consider stochastic lattice system (1.2) with random coupled coefficients and additive independent white noises:

$$\begin{aligned}\dot{u} &= -\lambda u + A(\theta_t\omega)u - f(u) + \alpha v + h + \frac{dW^1(t)}{dt},\\ \dot{v} &= -\sigma v + \mu u + g + \frac{dW^2(t)}{dt}, \quad i \in Z,\ t > 0,\end{aligned} \tag{4.2}$$

where $u_i, v_i,\ h_i, g_i, a_i, b_i \in \mathbb{R}$; $u = (u_i)_{i\in\mathbb{Z}}$, $v = (v_i)_{i\in\mathbb{Z}}$, $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$, $g = (g_i)_{i\in\mathbb{Z}}$, $h = (h_i)_{i\in\mathbb{Z}}$, $a = (a_i)_{i\in\mathbb{Z}} \in l^2$, $b = (b_i)_{i\in\mathbb{Z}} \in l^2$, $f_i \in C^1(\mathbb{R}, \mathbb{R})$ $(i \in \mathbb{Z})$; λ, α, σ, μ are positive constants; $\eta_{i,-q}(\omega),\ldots, \eta_{i,0}(\omega),\ldots,\eta_{i,+q}(\omega), i \in \mathbb{Z}, q \in \mathbb{N}$ are random variables; A is defined as in (3.2).

To transform (4.2) into a random equation without white noise, let

$$\delta^1(\theta_t\omega) = -\lambda \int_{-\infty}^{0} e^{\lambda s}\theta_t\omega(s)ds, \quad \delta^2(\theta_t\omega) = -\sigma \int_{-\infty}^{0} e^{\sigma s}\theta_t\omega(s)ds, \quad t \in \mathbb{R},\ \omega \in \Omega_2. \tag{4.3}$$

Then $\delta^1(\theta_t\omega)$, $\delta^2(\theta_t\omega)$ are both Ornstein-Uhlenbeck processes on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ and solve the following Ornstein-Uhlenbeck equations (see [1]), respectively,

$$d\delta^1 + \lambda\delta^1 dt = dW^1(t), \qquad d\delta^2 + \sigma\delta^2 dt = d\mathrm{S}W^2(t), \quad t \geq 0. \tag{4.4}$$

Lemma 4.1 (see [1]). *There exists a θ_t-invariant set $\widetilde{\Omega}_2 \in \mathcal{F}_2$ of Ω_2 of full $\mathbb{P}$ measure such that for $\omega \in \widetilde{\Omega}_2$,*

(i) $\lim_{t\to\pm\infty}\|\omega(t)\|/t = 0$;

(ii) *the random variables $\|\delta^j(\omega)\|$ are tempered and the mappings*

$$(t, \omega) \longrightarrow \delta^j(\theta_t\omega) \in l^2, \quad j = 1, 2, \tag{4.5}$$

are stationary solutions of Ornstein-Uhlenbeck equations (4.4) *in l^2 with continuous trajectories;*

(iii)

$$\lim_{t\to\pm\infty} \frac{\|\delta^j(\theta_t\omega)\|}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \delta^j(\theta_s\omega)ds = 0, \quad j = 1, 2. \tag{4.6}$$

In the following, we consider the completion of the probability space $\omega \in \widetilde{\Omega}_2$, still denoted by $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$,

Let

$$\widetilde{x}(t, \omega) = u(t, \omega) - \delta^1(\theta_t\omega), \quad \widetilde{y}(t, \omega) = v(t, \omega) - \delta^2(\theta_t\omega), \quad \omega \in \Omega_2,\ t \in \mathbb{R}. \tag{4.7}$$

then system (4.2) becomes the following random system with random coefficients but without white noise:

$$\begin{aligned}\frac{d\widetilde{x}}{dt} &= -\lambda\widetilde{x} + A(\theta_t\omega)\widetilde{x} - f\left(\widetilde{x} + \delta^1(\theta_t\omega)\right) - \alpha\widetilde{x} + A(\theta_t\omega)\delta^1(\theta_t\omega) - \alpha\delta^2(\theta_t\omega) + h,\\ \frac{d\widetilde{y}}{dt} &= -\sigma\widetilde{y} + \mu\widetilde{x} + \mu\delta^1(\theta_t\omega) + g, \quad i \in \mathbb{Z},\ t > 0.\end{aligned} \tag{4.8}$$

In addition, we make the following assumptions on functions $f_i, i \in \mathbb{Z}$:

(H6) $f_i \in C^1(\mathbb{R},\mathbb{R})$ *satisfy*

$$sf_i(s) \geq \mu s^{2(p+1)} - d_i^2, \quad \left|f_i(s)\right| \leq d_f|s|\left(|s|^{2p}+1\right), \quad \forall s \in \mathbb{R},\ i \in \mathbb{Z}, \tag{4.9}$$

where μ, d_i, d_f *are positive constants,* $p \in \mathbb{N}$, *and* $d = (d_i)_{i\in\mathbb{Z}} \in l^2$.

4.2. Random Dynamical System Generated by Random Lattice System

Denote by $\widetilde{\mathbf{z}} = (\widetilde{x}, \widetilde{y})$, we have the following.

Theorem 4.2. *Let* $T > 0$ *and assume that (P0), (H1), (H2), (H4), and (H6) hold. Then for every* $\omega \in \Omega_2$ *and any initial data* $\widetilde{\mathbf{z}}_0 = (\widetilde{x}_0, \widetilde{y}_0) \in H$, *problem* (4.8) *admits a unique solution* $\widetilde{\mathbf{z}}(\cdot;\omega,\widetilde{\mathbf{z}}_0) \in C([0,T),H)$ *with* $\widetilde{\mathbf{z}}(0;\omega,\widetilde{\mathbf{z}}_0) = \widetilde{\mathbf{z}}_0$.

Proof. Similar to the proof of Theorem 3.3. □

Theorem 4.3. *Assume that (P0), (H1), (H2), (H4), and (H6) hold. Then system* (4.8) *generates a continuous RDS* $\{\widetilde{\varphi}(t,\omega)\}_{t\geq 0,\omega\in\Omega_2}$ *over* $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2, (\theta_t)_{t\in\mathbb{R}})$ *with state space* H:

$$\widetilde{\varphi}(t,\omega)\widetilde{\mathbf{z}}_0 := \widetilde{z}(t;\omega,\ \widetilde{\mathbf{z}}_0), \quad \text{for } \widetilde{\mathbf{z}}_0 \in H,\ t \geq 0,\ \omega \in \Omega_2. \tag{4.10}$$

Moreover,

$$\widetilde{\varphi}(t,\omega)(u_0,v_0) := \widetilde{\varphi}(t,\omega)\begin{pmatrix} u_0 - \delta^1(\omega) \\ v_0 - \delta^2(\omega) \end{pmatrix} + \begin{pmatrix} \delta^1(\theta_t\omega) \\ \delta^2(\theta_t\omega) \end{pmatrix}, \tag{4.11}$$

where $(u_0,v_0) \in H, t \geq 0$, $\omega \in \Omega_2$, *defines a continuous RDS* $\{\widetilde{\varphi}(t,\omega)\}_{t\geq 0,\omega\in\Omega_2}$ *over* $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2, (\theta_t)_{t\in\mathbb{R}})$ *associated with* (4.2).

Proof. It follows immediately from similar arguments to the proof of Theorem 3.4. □

4.3. Existence of Tempered Bounded Random Absorbing Set and Random Attractor in Weighted Space

In this subsection, we study the existence of a tempered random bounded absorbing set and a global random attractor for the random dynamical system $\{\widetilde{\varphi}(t,\omega)\}_{t\geq 0,\omega\in\Omega_2}$ generated by (4.2) in weighted space H.

Theorem 4.4. *Assume that (P0), (H1)–(H4), and (H6) hold. Then*

(a) *there exists a closed tempered bounded random absorbing set* $B_{2\rho}(\omega) \in \mathfrak{D}(H)$ *of RDS* $\{\widetilde{\varphi}(t,\omega)\}_{t\geq 0,\omega\in\Omega_2}$ *such that for any* $D \in \mathfrak{D}(H)$ *and each* $\omega \in \Omega_2$, *there exists* $\widetilde{T}_D(\omega) > 0$ *yielding* $\widetilde{\varphi}(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \subset B_{2\rho}(\omega)$, *for all* $t \geq \widetilde{T}_D(\omega)$. *In particular, there exists* $T_{2\rho}(\omega) > 0$ *such that* $\widetilde{\varphi}(t,\theta_{-t}\omega)B_{2\rho}(\theta_{-t}\omega) \subset B_{2\rho}(\omega)$, *for all* $t \geq T_{2\rho}(\omega)$;

(b) *the RDS* $\{\tilde{\varphi}(t,\omega_2)\}_{t\geq 0,\omega\in\Omega_2}$ *generated by equations* (4.2) *possesses a unique global random* $\mathfrak{D}$ *attractor given by*

$$A_{2\rho}(\omega) = \bigcap_{\tau\geq T_{2\rho}(\omega)} \overline{\bigcup_{t\geq\tau} \tilde{\varphi}(t,\theta_{-t}\omega)B_{2\rho}(\theta_{-t}\omega)} \in H. \tag{4.12}$$

Proof. Similar to the proofs of Theorems 3.6 and 3.7. □

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