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Research Article

Geometric Integrability of Some Generalizations of the Camassa-Holm Equation

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We study the Camassa-Holm (CH) equation and recently introduced μCH equation from the geometric point of view. We show that Kupershmidt deformations of these equations describe pseudospherical surfaces and hence are geometrically integrable.

1. Introduction

The modern theory of integrable nonlinear partial differential equations arose as a result of the inverse scattering method (ISM) discovered by Gardner et al. [1] for Korteweg de Vries (KdV) equation. Soon after, it was realized that this method can be applied to several important nonlinear equations like nonlinear Shrödinger equation, sine-Gordon, and so forth.

Sasaki [2] gave a natural geometric interpretation for ISM in terms of pseudospherical surfaces. Motivated by Sasaki, Chern and Tenenblat [3] introduce the notion of a scalar equation of pseudospherical type and study systematically the evolution equations that describe pseudospherical surfaces. It appears that almost all important equations and systems in mathematical physics enjoy this property [3–7]. The advantage of this geometric treatment is that most of the ingredients connected with the integrable equations such as Lax pair, zero curvature representation, conservation laws, and symmetries come naturally.

Let us recall some facts about the equations we study in this paper.

The Camassa-Holm equation (CH)

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.1}$$

where $\omega \in \mathbb{R}$, has appeared in [8] as an equation with bi-Hamiltonian structure. Later, in [9], it was considered as a model, describing the unidirectional propagation of shallow water waves over a flat bottom. CH is a completely integrable equation; see, for example, [10–12]. Furthermore, CH is geometrically integrable [6]. Here, we consider the case $\omega = 0$. Then, (1.1) is

$$m_t + um_x + 2mu_x, \quad m = u_{xx} - u. \tag{1.2}$$

The bi-Hamiltonian form of (1.2) is [8, 9]

$$m_t = -\mathcal{B}^1 \frac{\delta H_2[m]}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1[m]}{\delta m}, \tag{1.3}$$

where $\mathcal{B}^1 = \partial - \partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators (∂ stands for $\partial/\partial x$), and the corresponding Hamiltonians are

$$H_1[m] = \frac{1}{2}\int mu\,dx, \qquad H_2[m] = \frac{1}{2}\int\left(u^3 + uu_x^2\right)dx. \tag{1.4}$$

There exists an infinite sequence of conservation laws (multi-Hamiltonian structure) $H_n[m],\ n = 0, \pm1, \pm2, \ldots$ including (1.4) such that

$$\mathcal{B}^1 \frac{\delta H_n[m]}{\delta m} = \mathcal{B}^2 \frac{\delta H_{n-1}[m]}{\delta m}. \tag{1.5}$$

Very recently a “modified” CH equation (mCH) was introduced by analogy of “Miura transform” from the theory of KdV equation in [5].

The μCH equation was derived recently in [13, 14] as

$$-u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \tag{1.6}$$

where $\mu(u) = \int_0^1 u\,dx$ and $u(t,x)$ is a spatially periodic real-valued function of time variable t and space variable $x \in S^1 = [0,1)$. In order to keep a certain symmetry and analogy with CH, one can write the above equation in the form (see also [15])

$$\mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}. \tag{1.7}$$

Note that $\mu(u_t) = 0$ in the periodic case. Introducing $m = Au = \mu(u) - u_{xx}$ (1.7) becomes

$$m_t + um_x + 2mu_x = 0, \quad m = \mu(u) - u_{xx}. \tag{1.8}$$

The μCH is an integrable equation and arises as an asymptotic rotator equation in a liquid crystal with a preferred direction if one takes into account the reciprocal action of dipoles on themselves [13, 14].

The bi-Hamiltonian form of (1.8) is

$$m_t = -\mathcal{B}^1 \frac{\delta H_2[m]}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1[m]}{\delta m}, \tag{1.9}$$

where $\mathcal{B}^1 = \partial A = -\partial^3, \mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators, and the corresponding Hamiltonians are

$$H_1[m] = \frac{1}{2}\int mu\, dx, \qquad H_2[m] = \int\left(\mu(u)u^2 + \frac{1}{2}uu_x^2\right)dx. \tag{1.10}$$

Also the μCH equation is geometrically integrable [15] (see there for other geometric descriptions of this equation).

Recently in [16], a new 6th-order wave equation, named KdV6, was derived. After some rescaling, this equation can be presented as the following system;

$$\begin{aligned} u_t &= 6uu_x + u_{xxx} - w_x, \\ w_{xxx} &+ 4uw_x + 2u_x w = 0. \end{aligned} \tag{1.11}$$

This system gives a perturbation to KdV equation ($w = 0$), and since the constrain on w is differential, this is a nonholonomic deformation.

Kupershmidt [17] suggested a general construction applicable to any bi-Hamiltonian systems providing a nonholonomic perturbation on it. This perturbation is conjectured to preserve integrability. In the case of KdV6, the system (1.11) can be converted into

$$\begin{gathered} u_t = \mathcal{B}^1\frac{\delta H_{n+1}}{\delta u} - \mathcal{B}^1(w) = \mathcal{B}^2\frac{\delta H_n}{\delta u} - \mathcal{B}^1(w), \\ \mathcal{B}^2(w) = 0, \end{gathered} \tag{1.12}$$

where $\mathcal{B}^1 = \partial, \mathcal{B}^2 = \partial^3 + 2(m\partial + \partial m)$ are the two standard Hamiltonian operators of the KdV hierarchy and

$$H_1 = \int u\, dx, \qquad H_2 = \frac{1}{2}\int u^2 dx, \ldots. \tag{1.13}$$

In the same paper, Kupershmidt verifies integrability of KdV6, as well as integrability of such nonholonomic deformations for some representative cases: the classical long-wave equation, the Toda lattice (both continuous and discrete), and the Euler top.

In fact, Kersten et al. [18] prove that the Kupershmidt deformation of every bi-Hamiltonian equation is again bi-Hamiltonian system and every hierarchy of conservation laws of the original bi-Hamiltonian system, gives rise to a hierarchy of conservation laws of the Kupershmidt deformation.

The aim of this paper is to show that the CH and the μCH equations have geometrically integrable Kupershmidt deformations. We also show that the KdV6 equation and two-component CH system [19] are also geometrically integrable.

As a matter of fact, Yao and Zeng [20] propose a generalized Kupershmidt deformation and verify that this generalized deformation also preserves integrability in few representative cases: KdV equation, Boussinesq equation, Jaulent-Miodek equation, and CH equation.

Kundu et al. [21] consider slightly generalized form of deformation for the KdV equation and extend this approach to mKdV equation and to AKNS system.

Guha [22] uses Kirillov's theory of coadjoint representation of Virasoro algebra to obtain a large class of KdV6-type equations, equivalent to the original one. Also, applying the Adler-Konstant-Symes scheme, he constructs a new nonholonomic deformation of the coupled KdV equation.

The paper is organized as follows. In Section 2, we recall some facts about equations that describe pseudospherical surfaces. The main results are in Section 3. There we show that the Kupershmidt deformations for the CH equation and for the μCH equation are of pseudospherical type and hence are geometrically integrable. We also derive the corresponding quadratic pseudopotentials which turn out to be very useful in obtaining conservation laws and symmetries. At the end, we make some speculations about "modified" μCH equation.

2. Equations of Pseudospherical Type and Pseudopotentials

In this section, we recall some definitions and facts. One can consult, for example, [3–7] for more details.

Definition 2.1. A scalar differential equation $\Xi(x,t,u,u_x,\dots,u_{x^n t^m}) = 0$ in two independent variables x,t is of pseudospherical type (or, it describes pseudospherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$

$$\omega^\alpha = f_{\alpha 1}(x,t,u,\dots,u_{x^r t^p})dx + f_{\alpha 2}(x,t,u,\dots,u_{x^s t^q})dt, \quad \alpha = 1,2,3, \tag{2.1}$$

whose coefficients $f_{\alpha\beta}$ are smooth functions which depend on x,t and finite number of derivatives of u, such that the 1-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$ satisfy the structure equations

$$d\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \qquad d\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \qquad d\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2, \tag{2.2}$$

whenever $u = u(x,t)$ is a solution of $\Xi = 0$.

Equations (2.2) can be interpreted as follows. The graphs of local solutions of equations of pseudospherical type can be equipped with structure of pseudospherical surface (see [3, 6, 7]): if $\overline{\omega}^1 \wedge \overline{\omega}^2 \neq 0$, the tensor $\overline{\omega}^1 \otimes \overline{\omega}^1 + \overline{\omega}^2 \otimes \overline{\omega}^2$ defines a Riemannian metric of constant Gaussian curvature -1 on the graph of solution $u(x,t)$, and $\overline{\omega}^3$ is the corresponding metric connection one-form.

An equation of pseudospherical type is the integrability condition for a $\mathrm{sl}(2,\mathbb{R})$-valued problem

$$d\psi = \Omega\psi, \tag{2.3}$$

where Ω is the matrix-valued one-form

$$\Omega = X\,dx + T\,dt = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}. \tag{2.4}$$

Definition 2.2. An equation $\Xi = 0$ is geometrically integrable if it describes a nontrivial one-parameter family of pseudospherical surfaces.

Hence, if $\Xi = 0$ is geometrically integrable, it is the integrability condition of one-parameter family of linear problems $\psi_x = X\psi$, $\psi_t = T\psi$. In fact, this is equivalent to the zero curvature equation

$$X_t - T_x + [X, T] = 0, \tag{2.5}$$

which is an essential ingredient of integrable equations.

Another important property of equations of pseudospherical type is that they admit quadratic pseudopotentials. Pseudopotentials are a generalization of conservation laws.

Proposition 2.3 (see [6]). *Let $\Xi = 0$ be a differential equation describing pseudospherical surfaces with associated one-forms ω^α. The following two Pfaffian systems are completely integrable whenever $u(x,t)$ is a solution of $\Xi = 0$:*

$$-2d\Gamma = \omega^3 + \omega^2 - 2\Gamma\omega^1 + \Gamma^2\left(\omega^3 - \omega^2\right), \tag{2.6}$$

$$2d\gamma = \omega^3 - \omega^2 - 2\gamma\omega^1 + \gamma^2\left(\omega^3 + \omega^2\right). \tag{2.7}$$

Moreover, the one-forms

$$\begin{aligned} \Theta &= \omega^1 - \Gamma\left(\omega^3 - \omega^2\right), \\ \widehat{\Theta} &= -\omega^1 + \gamma\left(\omega^3 + \omega^2\right) \end{aligned} \tag{2.8}$$

are closed whenever $u(x,t)$ is a solution of $\Xi = 0$ and Γ (resp. γ) is a solution of (2.6) *(resp.* (2.7)*).*

Geometrically, Pfaffian systems (2.6) and (2.7) determine geodesic coordinates on the pseudospherical surfaces associated with the equation $\Xi = 0$ [3, 6].

3. Results

In this section, we consider the nonholonomic deformation of CH equation and μCH equation. We show that they are geometrically integrable and consider their quadratic pseudopotentials. The nonlocal symmetries will be studied elsewhere.

3.1. CH Equation

Recall from Introduction the CH equation (1.2) and its bi-Hamiltonian form (1.3) with the corresponding Hamiltonian operators $\mathcal{B}^1$ and $\mathcal{B}^2$.

Following Kupershmidt's construction, we introduce the nonholonomic deformation of the CH equation

$$m_t = -um_x - 2mu_x - \mathcal{B}^1(w),$$
$$\mathcal{B}^2(w) = 0. \tag{3.1}$$

Proposition 3.1. *The system* (3.1) *describes pseudospherical surface and hence is geometrically integrable.*

Let us give the corresponding 1-forms

$$\begin{aligned}
\omega^1 &= (m+1)dx + \left[-um + \frac{\eta+1}{\eta}(u_x - u) + \frac{1}{\eta} - w_{xx} + (\eta+1)w_x - \eta w(m+1)\right]dt,\\
\omega^2 &= (\eta+1)dx + \left[\frac{\eta+1}{\eta} - (\eta+1)u + u_x - \eta(\eta+1)w + \eta w_x\right]dt,\\
\omega^3 &= (m+1+\eta)dx\\
&\quad + \left[-um + \frac{\eta-(\eta+1)^2}{\eta}u + \frac{\eta+1}{\eta}u_x + \frac{\eta+1}{\eta} - w_{xx} + (\eta+1)w_x - \eta w(m+1+\eta)\right]dt.
\end{aligned} \tag{3.2}$$

For the proof of Proposition 3.1, we need only to verify the structure equations (2.2) and to check that the parameter η is intrinsic.

For the matrices X and T in, we get

$$X = \frac{1}{2}\begin{pmatrix} \eta+1 & -\eta \\ 2(m+1)+\eta & -(\eta+1) \end{pmatrix}, \qquad T = \frac{1}{2}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned}
T_{11} &= \frac{\eta+1}{\eta} - (\eta+1)u + u_x - \eta(\eta+1)w + \eta w_x,\\
T_{12} &= \eta u + \eta^2 w - 1,\\
T_{21} &= -2um - \frac{(\eta+1)^2+1}{\eta}u + \frac{2(\eta+1)}{\eta}u_x\\
&\quad + \frac{\eta+2}{\eta} - 2w_{xx} + 2(\eta+1)w_x - \eta(\eta+2)w - 2\eta wm.
\end{aligned} \tag{3.4}$$

Hence, we have a zero curvature representation $X_t - T_x + [X,T] = 0$ for the system (3.1). From (3.3), it is straightforward to obtain the corresponding scalar linear problem

$$\psi_{xx} = \left(\frac{1}{4} - \frac{\eta}{2}m\right)\psi,$$
$$\psi_t = \left(-u - \eta w + \frac{1}{\eta}\right)\psi_x + \frac{u_x + \eta w_x}{2}\psi. \tag{3.5}$$

In order to apply Proposition 2.3 to nonholonomic deformation of the CH equation, we consider new 1-forms $\omega^{\alpha}_{\text{new}}$

$$\omega^1_{\text{new}} = \omega^2, \qquad \omega^2_{\text{new}} = -\omega^1, \qquad \omega^3_{\text{new}} = \omega^3. \tag{3.6}$$

With these forms the Pfaffian system (2.7) becomes

$$\begin{aligned} 2\gamma_x &= \eta\gamma^2 - 2(\eta+1)\gamma + 2(m+1) + \eta, \\ 2\gamma_t &= \gamma^2 - 2w_{xx} - 2[\eta\gamma - (\eta+1)]w_x - 2w\eta\gamma_x \\ &\quad - 2\left[\gamma - \frac{\eta+1}{\eta}\right]u_x - 2u\left(\gamma_x + \frac{1}{\eta}\right) + \frac{\eta+2}{\eta} - 2\gamma\frac{\eta+1}{\eta}. \end{aligned} \tag{3.7}$$

Applying the transform

$$\gamma \longmapsto \gamma + \frac{\eta+1}{\eta}, \tag{3.8}$$

after some algebraic manipulations and setting $\lambda = -1/\eta$, we obtain the following result.

Proposition 3.2. *The nonholonomic deformation of the CH equation* (3.1) *admits a quadratic pseudo-potential γ, defined by the equations*

$$m = \frac{\gamma^2}{2\lambda} + \gamma_x - \frac{\lambda}{2}, \tag{3.9}$$

$$\gamma_t = \frac{\gamma^2}{2}\left[1 + \frac{1}{\lambda}\left(u - \frac{w}{\lambda}\right)\right] - \gamma\left(u - \frac{w}{\lambda}\right)_x - \left(u - \frac{w}{\lambda}\right)\left(m + \frac{\lambda}{2}\right) + \lambda u - \frac{\lambda^2}{2}, \tag{3.10}$$

where $\lambda \neq 0,\ m = u_{xx} - u$. *Moreover,* (3.1) *possesses the parameter-dependent conservation law*

$$\gamma_t = \lambda\left[(u+w)_x - \gamma - \frac{1}{\lambda}\left(u - \frac{w}{\lambda}\right)\gamma\right]_x. \tag{3.11}$$

Conservation densities can be obtained by expanding (3.9) and (3.11) in powers of λ. Note that the left hand side of (3.11) and (3.9) does not depend on w as it should be. The corresponding expansions are performed in [6].

3.2. μCH Equation

Consider now the μCH equation (1.8), its bi-Hamiltonian form (1.9) with the corresponding Hamiltonian operators $\mathcal{B}^1, \mathcal{B}^2$. Applying Kupershmidt's procedure to (1.8), we obtain the nonholonomic deformation of the μCH equation

$$\begin{gathered} m_t = -um_x - 2mu_x - \mathcal{B}^1(w), \\ \mathcal{B}^2(w) = 0, \end{gathered} \tag{3.12}$$

or

$$\begin{gathered} m_t = -um_x - 2mu_x + w_{xxx}, \\ 2mw_x + wm_x = 0, \quad m = \mu(u) - u_{xx}. \end{gathered} \tag{3.13}$$

Proposition 3.3. *The nonholonomic deformation of the μCH equation* (3.13) *describes pseudospherical surfaces and, hence, is geometrically integrable.*

For validation of Proposition 3.3, we give the 1-forms associated with (3.13)

$$\begin{aligned} \omega^1 &= \frac{1}{2}\left(\eta m - \frac{\eta^2}{2} + 2\right)dx \\ &\quad + \frac{1}{2}\left[\frac{\eta^2}{2}u - \eta\left(u_x + um + \frac{1}{2}\right) + \mu(u) - 2u + \frac{2}{\eta} + \left(\frac{\eta^3}{2} - 2\eta\right)w - \eta^2 w_x + \eta w_{xx} - \eta^2 mw\right]dt, \\ \omega^2 &= \eta dx + \left(1 - \eta u + u_x - \eta^2 w + \eta w_x\right)dt, \\ \omega^3 &= \frac{1}{2}\left(\eta m - \frac{\eta^2}{2} - 2\right)dx \\ &\quad + \frac{1}{2}\left[\frac{\eta^2}{2}u - \eta\left(u_x + um + \frac{1}{2}\right) + \mu(u) + 2u - \frac{2}{\eta} + \left(\frac{\eta^3}{2} + 2\eta\right)w - \eta^2 w_x + \eta w_{xx} - \eta^2 mw\right]dt. \end{aligned} \tag{3.14}$$

For the matrices X and T in, we get

$$X = \frac{1}{2}\begin{pmatrix} \eta & 2 \\ \eta m - \dfrac{\eta^2}{2} & -\eta \end{pmatrix}, \qquad T = \frac{1}{2}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix}, \tag{3.15}$$

where

$$
\begin{aligned}
T_{11} &= 1 - \eta u + u_x - \eta^2 w + \eta w_x, \\
T_{12} &= 2\left(-u + \frac{1}{\eta} - \eta w\right), \\
T_{21} &= \frac{\eta^2}{2}u - \eta\left(u_x + um + \frac{1}{2}\right) + \mu(u) - \eta^2 w_x + \eta w_{xx} - \eta^2 mw + \frac{\eta^3}{2}w.
\end{aligned}
\tag{3.16}
$$

Hence, we have a zero curvature representation $X_t - T_x + [X, T] = 0$ for the system (3.13). From (3.3), it is straightforward to obtain the corresponding scalar linear problem

$$
\begin{aligned}
\psi_{xx} &= \left(\frac{\eta}{2}m\right)\psi, \\
\psi_t &= \left(-u - \eta w + \frac{1}{\eta}\right)\psi_x + \frac{u_x + \eta w_x}{2}\psi,
\end{aligned}
\tag{3.17}
$$

which coincides with those in [13] upon setting $w = 0$ and $\lambda = \eta/2$.

In order to find pseudopotentials for the nonholonomic deformation of the μCH equation, we proceed as before denoting

$$
\omega^1_{\text{new}} = \omega^2, \qquad \omega^2_{\text{new}} = -\omega^1, \qquad \omega^3_{\text{new}} = \omega^3. \tag{3.18}
$$

With these forms, the Pfaffian system (2.7) becomes

$$
2\gamma_x = -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \tag{3.19}
$$

$$
\begin{aligned}
2\gamma_t = &-\frac{2\gamma^2}{\eta} + 2\gamma^2(u + \eta w) - 2\gamma\left(1 - \eta u + u_x - \eta^2 w + \eta w_x\right) \\
&+ \left[\frac{\eta^2}{2}u - \eta\left(u_x + m + \frac{1}{2}\right) + \mu(u) + \frac{\eta^3}{2}w - \eta^2 w_x + \eta w_{xx} - \eta^2 mw\right].
\end{aligned}
\tag{3.20}
$$

After some manipulations, the above system obtains the form

$$
\begin{aligned}
2\gamma_x &= -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \\
2\gamma_t &= -\frac{2}{\eta}\gamma^2 + \eta w_{xx} - \left[(2\gamma + \eta)(u + \eta w)\right]_x + \mu(u) - 2\gamma - \frac{\eta}{2}.
\end{aligned}
\tag{3.21}
$$

Applying the transform $\gamma \mapsto \gamma - \eta/2$, we get

$$\gamma_x = -\gamma^2 + \frac{\eta}{2}m, \tag{3.22}$$

$$\gamma_t = -\frac{\gamma^2}{\eta} + \frac{\eta}{2}w_{xx} - \left[\gamma(u + \eta w)\right]_x + \frac{\mu(u)}{2}. \tag{3.23}$$

Multiplying the first equation (3.22) by $-1/\eta$ and then adding the result to the second equation (3.23), we get the following result denoting $\lambda = \eta/2$.

Proposition 3.4. *The nonholonomic deformation of the* μ*CH equation* (3.13) *admits a quadratic pseudopotential* γ*, defined by the equations*

$$m = \frac{\gamma^2}{\lambda} + \frac{\gamma_x}{\lambda}, \tag{3.24}$$

$$\gamma_t = -\frac{2\gamma^2}{\lambda}\lambda w_{xx} - \left[\gamma(u + 2\lambda w)\right]_x + \frac{\mu(u)}{2}, \tag{3.25}$$

where $\lambda \neq 0,\ m = \mu(u) - u_{xx}$*. Moreover,* (3.13) *possesses the parameter-dependent conservation law*

$$\gamma_t = \frac{1}{2\lambda}\left[\gamma + \lambda(u + 2\lambda w)_x - 2\lambda(u + 2\lambda w)\gamma\right]_x. \tag{3.26}$$

As the conserved densities for the nonholonomic deformation are the same as for the original bi-Hamiltonian system, we make use of the pseudopotentials to obtain them for the μCH equation. One possible expansion of γ is

$$\gamma = \lambda^{1/2}\gamma_1 + \gamma_0 + \sum_{j=1}^{\infty}\lambda^{-j/2}\gamma_{-j}. \tag{3.27}$$

Substituting this into (3.24) yields

$$\gamma_1 = \sqrt{m}, \qquad \gamma_0 = -\frac{m_x}{4m}, \qquad \gamma_{-1} = \frac{1}{32}\frac{m_x^2}{m^{5/2}} + \frac{1}{8}\left(\frac{m_x}{m^{3/2}}\right)_x, \ldots \text{and so forth.} \tag{3.28}$$

In this way, we can obtain local functionals; see [13].

We finish this section with the geometric integrability of one of the most popular two-component generalization of CH equation and of KdV6 equation.

Another generalization of the Camassa-Holm equation is the following integrable two-component CH system [19]:

$$\begin{gathered} u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} + \sigma\rho\rho_x, \\ \rho_t + (u\rho)_x = 0, \end{gathered} \tag{3.29}$$

where $\sigma = \pm 1$. Introducing a new variable $v = \rho^2/2$, the above system becomes

$$
\begin{gathered}
u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} + \sigma v_x, \\
v_t + 2vu_x + uv_x = 0.
\end{gathered}
\tag{3.30}
$$

The system (3.30) is geometrically integrable. The corresponding 1-forms, satisfying the structure equations (2.2), are the following:

$$
\begin{aligned}
\omega^1 &= (u_{xx} - u - \sigma\eta v + 1)dx + \left[u^2 - uu_{xx} + \frac{\eta+1}{\eta}(u_x - u) + \frac{1}{\eta} - \sigma v + \eta\sigma uv\right]dt, \\
\omega^2 &= (\eta + 1)dx + \left(\frac{\eta+1}{\eta} - (\eta+1)u + u_x\right)dt, \\
\omega^3 &= (u_{xx} - u - \sigma\eta v + \eta + 1)dx \\
&\quad + \left[u^2 - uu_{xx} + \frac{\eta - (\eta+1)^2}{\eta}u + \frac{\eta+1}{\eta}(u_x + 1) - \sigma v + \eta\sigma uv\right]dt.
\end{aligned}
\tag{3.31}
$$

We could easily include two-component Hunter-Saxon system [19] into this picture (see also [7]).

Finally, we note that nonholonomic perturbation of KdV equation, known as KdV6 equation, is also of pseudospherical type, that is, KdV6 equation is geometrically integrable. We just give the corresponding 1-forms

$$
\begin{aligned}
\omega^1 &= (1-u)dx + \left[-u_{xx} + \eta u_x - 2u^2 + \frac{1}{\eta}w_x - \frac{1}{\eta^2}(w_{xx} + 2uw) + \left(2 - \eta^2\right)u + \frac{2}{\eta^2}w + \eta^2\right]dt, \\
\omega^2 &= \eta dx + \left(\eta^3 + 2\eta u + \frac{2}{\eta}w - 2u_x - \frac{2}{\eta^2}w_x\right)dt, \\
\omega^3 &= -(1+u)dx + \left[-u_{xx} + \eta u_x - 2u^2 + \frac{1}{\eta}w_x - \frac{1}{\eta^2}(w_{xx} + 2uw) - \left(2 + \eta^2\right)u - \frac{2}{\eta^2}w - \eta^2\right]dt,
\end{aligned}
\tag{3.32}
$$

which coincide with those for KdV equation [3] when $w \to 0$.

4. Concluding Remarks

In this paper, we study the CH equation and some of its generalizations from the geometric point of view. We show that Kupershmidt deformations for CH and μCH equations preserve integrability and derive some important objects like quadratic pseudopotentials which turn out to be useful for obtaining conservation laws and nonlocal symmetries. It is also shown that the KdV6 equation and two-component CH system are also geometrically integrable.

Having at hand these examples of geometrically integrable Kupershmidt deformations, it is natural to think that maybe there exists a general link in this sense: a Kupershmidt deformation of geometrically integrable system is again geometrically integrable. We have not succeeded in establishing such a link up to now, but we believe that this is true at least for the systems with local Hamiltonian pair of operators as in the above examples.

Let us return, however, to the μCH equation ($w = 0$). It is obvious that pseudo-potentials for the μCH equation (1.8) and parameter-dependent conservation law are obtained from

$$m = \frac{\gamma^2}{\lambda} + \frac{\gamma_x}{\lambda}, \tag{4.1}$$

$$\gamma_t = \frac{1}{2\lambda}\left(\gamma + \lambda u_x - 2\lambda u\gamma\right)_x = \frac{\gamma_x}{2\lambda} + \frac{u_{xx}}{2} - \partial(\gamma u). \tag{4.2}$$

Equation (4.1) is an analogue of the Miura transformation of KdV theory. We can repeat, purely formally, the procedure for obtaining the "modified" CH (mCH) equation [5] in this case. However, it is clear that since μCH contains nonlocal term, one can expect that the "modified" equation also will have nonlocal terms.

Denote by A the operator $A = \mu - \partial^2$, $A(u) = m = \mu(u) - u_{xx}$. The operators A^{-1} and ∂ commute and $\mu(u) = \mu(Au)$.

We have

$$u = A^{-1}m, \quad u_x = A^{-1}m_x, \quad u_{xx} = A^{-1}m_{xx}, \tag{4.3}$$

in which m is determined by (4.1). Then, the second equation (4.2) takes the form

$$\gamma_t = \frac{\gamma_x}{2\lambda} + \frac{A^{-1}m_{xx}}{2} - \gamma A^{-1}m_x - \gamma_x A^{-1}m, \tag{4.4}$$

or

$$A\gamma_t = \frac{A\gamma_x}{2\lambda} + \frac{m_{xx}}{2} - A\partial\left(\gamma A^{-1}m\right). \tag{4.5}$$

Formally, this equation can be named as a "modified" μCH equation. One can simplify further this equation using (4.1) or even to present it as a system as in [5], it remains nonlocal and, hence, it is of no immediate advantage.

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