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Research Article

Nonlinear Singular BVP of Limit Circle Type and the Presence of Reverse-Ordered Upper and Lower Solutions

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We consider the following class of nonlinear singular differential equation $-(p(x)y'(x))' + q(x)f(x, y(x), p(x)y'(x)) = 0$, $0 < x < 1$ subject to the Neumann boundary condition $y'(0) = y'(1) = 0$. Conditions on $p(x)$ and $q(x)$ ensure that $x = 0$ is a singular point of limit circle type. A simple approximation scheme which is iterative in nature is considered. The initial iterates are upper and lower solutions which can be ordered in one way ($v_0 \le u_0$) or the other ($u_0 \le v_0$).

1. Introduction

The upper and lower solution technique is the most promising technique as far as singular boundary value problems, are concerned [1]. Recently, lot of activities are there regarding upper and lower solutions technique (see [2, 3] and the references therein). To see the application of the similar kind of problems, one should see the references of [3]. In most of the results, upper and lower solutions are well ordered, that is, $u_0 \ge v_0$. As far as reverse-ordered upper and lower solutions are considered, that is, $u_0 \le v_0$, the literature is not that rich. Though references are there for nonsingular boundary value problem, but singular boundary value problems require further exploration. The details of the work done for the nonsingular problem when upper and lower solutions are in reverse order can be seen in [4, 5]. To fill this gap in the present paper, we consider the following singular BVP:

$$\begin{gathered} -\left(p(x)y'(x)\right)' + q(x)f\left(x, y(x), p(x)y'(x)\right) = 0, \quad 0 < x < 1, \\ y'(0) = 0, \qquad y'(1) = 0. \end{gathered} \tag{1.1}$$

(A1) Let $p(x)$ satisfy the following conditions.
(i) $p(0) = 0$ and $p > 0$ in $(0, 1)$.
(ii) $p \in C[0, 1] \cap C^1(0, 1)$.

(A2) Let $q(x)$ satisfy the following conditions.
(i) $q(x) > 0$ in $(0, 1)$ and $q(x) \in C(0, 1]$.
(ii) $\int_0^1 q(x)dx < \infty$.
(iii) $\lim_{x\to 0}(q(x))/(p'(x)) = 0$.
(iv) $\int_0^1 (1/p(x))(\int_0^x q(s)ds)^{1/2} dx < \infty$.

In this paper, we consider a computationally simple iterative scheme defined by

$$-(py_n')' + \lambda q y_n = -qf(x, y_{n-1}, py_{n-1}') + \lambda q y_{n-1}, \quad 0 < x < 1,$$
$$y_n'(0) = 0, \qquad y_n'(1) = 0. \tag{1.2}$$

Starting with upper and lower solutions, we generate monotone sequences. To generate these monotonic sequences, we need the existence of some differential inequalities. To prove these differential inequalities, we analyze the corresponding singular IVP and extract properties of the solutions and their derivative.

We have arranged the paper in four sections. In Section 2, we discuss some elementary results, for example, maximum principles and existence of two differential inequalities. Then using these elementary results, we establish existence results for well-ordered upper and lower solutions in Section 3 and for reverse-ordered upper and lower solutions in Section 4. In Section 5, we conclude this paper with some remarks.

2. Preliminaries

Let $h(x) \in C[0, 1]$, and let $\lambda \in \mathbb{R}_0$ ($\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$), let $A \in \mathbb{R}$ and let $B \in \mathbb{R}$. Now, consider the following class of linear singular problems:

$$-(p(x)y'(x))' + \lambda q(x)y(x) = q(x)h(x), \quad 0 < x < 1, \tag{2.1}$$

$$y'(0) = A, \qquad y'(1) = B. \tag{2.2}$$

The corresponding homogeneous system (eigenvalue problem) is given by

$$-(p(x)y'(x))' + \lambda q(x)y(x) = 0, \quad 0 < x < 1, \tag{2.3}$$

$$y'(0) = 0, \qquad y'(1) = 0. \tag{2.4}$$

The solution of the nonhomogeneous problem (2.1)-(2.2) can be written as follows:

$$w(x) = z_1(x)\left[\int_0^x \frac{q(t)h(t)z_0(t)}{W_p(z_1, z_0)}dt + \frac{A}{z_1'(0)}\right] + z_0(x)\left[\int_x^1 \frac{q(t)h(t)z_1(t)}{W_p(z_1, z_0)}dt + \frac{B}{z_0'(1)}\right], \tag{2.5}$$

where $z_0(x, \lambda)$ is the solution of

$$-\left(p(x)z_0'(x)\right)' + \lambda q(x)z_0(x) = 0, \quad 0 < x < 1, \; z_0(0) = 1, \; z_0'(0) = 0, \tag{2.6}$$

$z_1(x, \lambda)$ is the solution of

$$-\left(p(x)z_1'(x)\right)' + \lambda q(x)z_1(x) = 0, \quad 0 < x < 1, \; z_1(1) = 1, \; z_1'(1) = 0, \tag{2.7}$$

and $W_p(z_1, z_0) = p(t)(z_1 z_0' - z_1' z_0)$. By replacing x with $1 - x$ in (2.6), it is easy to verify that

$$z_1(x) = z_0(1 - x), \tag{2.8}$$

for both positive and negative values of λ.

Remark 2.1. Existence of $z_0(x)$ and $z_1(x)$ satisfying the IVP (2.6) and IVP (2.7), respectively, is an immediate consequence of the result due to O'Regan [6, Theorem 2.1, page 432].

Remark 2.2. Let $L_q^2(0, 1)$ be a Hilbert space with inner product defined by

$$\langle f, g\rangle = \int_0^1 q(x)f(x)\overline{g(x)}dx. \tag{2.9}$$

From (A2) (iv), it can easily be verified that $x = 0$ is a singular point of limit circle type (see [6, Remark (i) page 434]) in $L_q^2(0, 1)$. Thus, we have pure point spectrum [7, page 125]. It is easy to show that the eigenvalues are real, simple, and negative.

Remark 2.3. Since z_0 and z_1 are two linearly independent solutions of (2.3), the eigenvalues of the eigenvalue problem (2.3)-(2.4) will be the zeros of $z_0'(1, \lambda)$. Since $z_0'(1, \lambda)$ is an analytic function of λ so its zeros will be isolated and they all will be negative. Let them be $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$, where $\lambda_i > 0$ for $i = 0, 1, 2, \ldots$. Now, we have $-\lambda_0$ as the first negative zero of $z_0'(1, \lambda)$ or in other words first negative eigenvalue of (2.3)-(2.4).

Since $z_0(x, \lambda)$ does not change its sign for $-\lambda_0 < \lambda < 0$ and $z_0(0, \lambda) = 1$; therefore, $z_0(x, \lambda) > 0$ for all $x \in [0, 1]$ and for all $-\lambda_0 < \lambda < 0$.

Remark 2.4. Using (2.6), $z_1(x) = z_0(1 - x)$, it is easy to prove that if $\lambda > 0$ then for all $x \in (0, 1]$, $z_0(x) > 1$, and $z_0'(x) > 0$ and for all $x \in [0, 1)$, we have $z_1(x) > 1$ and $z_1'(x) < 0$.

Remark 2.5. Using Remark 2.3, $z_1(x) = z_0(1 - x)$, and the differential equation (2.6), it is easy to prove that if $-\lambda_0 < \lambda < 0$ for all $x \in [0, 1)$, $z_0(x) > 0$ and $z_1'(x) > 0$ and for all $x \in (0, 1]$, we have $z_0'(x) < 0$ and $z_1(x) > 0$.

Remark 2.6. Let $\lambda > 0$ and let $h \in C[0,1]$. If $h \geq 0$ (or $h \leq 0$), then

$$\int_0^x \frac{q(t)h(t)z_0(t)}{W_p(z_1,z_0)}dt, \qquad \int_x^1 \frac{q(t)h(t)z_1(t)}{W_p(z_1,z_0)}dt \tag{2.10}$$

are nonnegative (or nonpositive).

Remark 2.7. Let $-\lambda_0 < \lambda < 0$ and let $h \in C[0,1]$. If $h \geq 0$ (or $h \leq 0$), then

$$\int_0^x \frac{q(t)h(t)z_0(t)}{W_p(z_1,z_0)}dt, \qquad \int_x^1 \frac{q(t)h(t)z_1(t)}{W_p(z_1,z_0)}dt \tag{2.11}$$

are nonpositive (or nonnegative).

Proposition 2.8 (Maximum Principle). *Let $\lambda > 0$. If $A \leq 0$, $B \geq 0$ (or $A \geq 0$, $B \leq 0$) and $h \in C[0,1]$ is such that $h \geq 0$ (or $h \leq 0$), then $w(x) \geq 0$ (or $w(x) \leq 0$), where $w(x)$ is the solution of* (2.1)-(2.2).

Proposition 2.9 (Antimaximum Principle). *Let $-\lambda_0 < \lambda < 0$. If $A \leq 0$, $B \geq 0$ (or $A \geq 0$, $B \leq 0$) and $h \in C[0,1]$ is such that $h \geq 0$ (or $h \leq 0$), then $w(x) \leq 0$ (or $w(x) \geq 0$), where $w(x)$ is the solution of* (2.1)-(2.2).

Now, we derive conditions on λ which will help us to prove the monotonicity of the solutions generated by the iterative scheme (1.2).

Lemma 2.10. *Let M and $N \in \mathbb{R}^+$. If $\lambda > 0$ is such that*

$$\lambda \geq M\left(1 - N\int_0^1 q(x)dx\right)^{-1}, \tag{2.12}$$

then for all $x \in [0,1]$,

$$(M-\lambda)z_0(x) + Np(x)z_0'(x) \leq 0. \tag{2.13}$$

Proof. Integrating (2.6) from 0 to x and using the fact that $z_0'(x) > 0$ in $(0,1]$, we get

$$p(x)z_0'(x) \leq \lambda z_0(x)\int_0^1 q(x)dx. \tag{2.14}$$

Therefore, we get $(M-\lambda)z_0(x) + Np(x)z_0'(x) \leq (M-\lambda)z_0 + N\lambda z_0(x)\int_0^1 q(x)dx$. Hence, (2.13) will hold if $(M-\lambda) + N\lambda\int_0^1 q(x)dx \leq 0$. Hence the result. □

Lemma 2.11. *Let M and $N \in \mathbb{R}^+$. If $-\lambda_0 < \lambda < 0$ is such that $-\left(\int_0^1 (1/p(x)) \int_0^x q(t)dt\,dx\right)^{-1} < \lambda \leq -M$ and*

$$(M+\lambda)\left(1+\lambda\int_0^1 \frac{1}{p(x)}\int_0^x q(t)dt\,dx\right) - N\lambda\int_0^1 q(x)dx \leq 0, \tag{2.15}$$

then for all $x \in [0,1]$,

$$(M+\lambda)z_0(x) - Np(x)z_0'(x) \leq 0. \tag{2.16}$$

Proof. Using (2.6) and Remark 2.5, it can be deduced that $z_0(x)$ and $p(x)z_0'(x)$ are decreasing functions of x for $-\lambda_0 < \lambda < 0$, thus

$$(M+\lambda)z_0(x) - Np(x)z_0'(x) \leq (M+\lambda)z_0(1) - Np(1)z_0'(1). \tag{2.17}$$

Now using (2.6), we get $-p(1)z_0'(1) \leq (-\lambda)\int_0^1 q(x)dx$ and $z_0(1) > 1+\lambda\int_0^1 (1/p(x))\int_0^x q(t)dt\,dx$. This completes the proof. □

Note. In Lemma 2.11, we arrive at the integral $\int_0^1 (1/p(x))\int_0^x q(t)dt\,dx$ which is an improper integral, and it should be convergent. Using the assumption (A2) (iv) and Remark (i) and (ii) at [6, page 434] its convergence can be established.

3. Well-Ordered Upper and Lower Solutions

Let us define upper and lower solutions.

Definition 3.1. A function $u_0 \in C[0,1] \cap C^2(0,1]$ is an upper solution of (1.1) if

$$-(pu_0')' + qf(x,u_0,pu_0') \geq 0, \quad 0<x<1, \; u_0'(0) \leq 0 \leq u_0'(1). \tag{3.1}$$

Definition 3.2. A function $v_0 \in C[0,1] \cap C^2(0,1]$ is a lower solution of (1.1) if

$$-(pv_0')' + qf(x,v_0,pv_0') \leq 0, \quad 0<x<1, \; v_0'(0) \geq 0 \geq v_0'(1). \tag{3.2}$$

Now, for every n, the problem (1.2) has a unique solution y_{n+1} given by (2.5) with $h(x) = -f(x,y_n,py_n') + \lambda y_n$, $A=0$, and $B=0$.

In this section, we show that for the proposed scheme (1.2) a good choice of λ is possible so that the solutions generated by the approximation scheme converge monotonically to solutions of (1.1). We require a number of results.

Lemma 3.3. *Let $\lambda > 0$. If u_n is an upper solution of* (1.1) *and u_{n+1} is defined by* (1.2)*, then $u_{n+1} \leq u_n$.*

Proof. Let $w_n = u_{n+1} - u_n$, then

$$-(pw_n')' + \lambda q w_n = (pu_n')' - qf(x, u_n, pu_n') \le 0, \qquad w_n'(0) \ge 0, \qquad w_n'(1) \le 0, \tag{3.3}$$

and using Proposition 2.8, we have $u_{n+1} \le u_n$. □

Proposition 3.4. *Assume that*

($H1$) *there exist upper solution* (u_0) *and lower solution* (v_0) *in* $C[0,1] \cap C^2(0,1]$ *such that* $v_0 \le u_0$ *for all* $x \in [0,1]$,

($H2$) *the function* $f : D \to \mathbb{R}$ *is continuous on*

$$D := \{(x, y, py') \in [0,1] \times R \times R : v_0 \le y \le u_0\}, \tag{3.4}$$

($H3$) *there exists* $M \ge 0$ *such that for all* $(x, \tau, pv'), (x, \sigma, pv') \in D$,

$$f(x, \tau, pv') - f(x, \sigma, pv') \ge M(\tau - \sigma), \quad (\tau \le \sigma), \tag{3.5}$$

($H4$) *there exist* $N \ge 0$ *such that for all* $(x, u, pv_1')(x, u, pv_2') \in D$,

$$|f(x, u, pv_1') - f(x, u, pv_2')| \le N|pv_2' - pv_1'|. \tag{3.6}$$

Let $\lambda > 0$ *be such that* $\lambda \ge M(1 - N\int_0^1 q(x)dx)^{-1}$. *Then the functions* u_{n+1} *defined recursively by* (1.2) *are such that, for all* $n \in \mathbb{N}$,

(i) u_n *is an upper solution of* (1.1).

(ii) $u_{n+1} \le u_n$.

Proof. We prove the claims by the principle of mathematical induction. Since u_0 is an upper solution and by Lemma 3.3 $u_0 \ge u_1$; therefore, both the claims are true for $n = 0$.

Further, let the claims be true for $n-1$, that is, u_{n-1} is an upper solution and $u_{n-1} \ge u_n$. Now, we are required to prove that u_n is an upper solution and $u_{n+1} \le u_n$. To prove this, let $w = u_n - u_{n-1}$, then we have

$$-(pu_n')' + qf(x, u_n, pu_n') \ge p[(M - \lambda)w - N(\text{sign } w')pw']. \tag{3.7}$$

Thus, to prove that u_n is an upper solution, we are required to prove that

$$(M - \lambda)w - N(\text{sign } w')pw' \ge 0. \tag{3.8}$$

Now, since w satisfies

$$-(pw')' + \lambda q w = (pu_{n-1}')' - qf(x, u_{n-1}, pu_{n-1}') \le 0, \quad w^{'(0)} \ge 0, \ w'(1) \le 0, \tag{3.9}$$

from Proposition 2.8, we have $w \le 0$ for $\lambda > 0$. Now, putting the value of w from (2.5) in (3.8), and in view of $h = (pu'_{n-1})' - qf(x, u_{n-1}, pu'_{n-1}) \le 0$, we deduce that to prove (3.8) it is sufficient to prove that

$$(M - \lambda)z_0 - N(\operatorname{sign} w')pz_0' \le 0,$$
$$(M - \lambda)z_1 - N(\operatorname{sign} w')pz_1' \le 0\ , \tag{3.10}$$

for all $x \in [0,1]$. Since $z_1 = z_0(1 - x)$, using Remark 2.6, the above inequalities will be true if for all $x \in [0,1]$ we have

$$(M - \lambda)z_0(x) + Np(x)z_0'(x) \le 0. \tag{3.11}$$

Which is true (Lemma 2.10). Therefore, (3.8) holds, and hence u_n is an upper solution.

Now applying Lemma 3.3, we deduce that $u_{n+1} \le u_n$. This completes the proof. □

Similarly, we can prove the following two results (Lemma 3.5, Proposition 3.6) for lower solutions.

Lemma 3.5. *Let $\lambda > 0$. If v_n is a lower solution of* (1.1) *and v_{n+1} is defined by* (1.2)*, then $v_n \le v_{n+1}$.*

Proposition 3.6. *Assume that* $(H1)$, $(H2)$, $(H3)$, *and* $(H4)$ *hold, and let $\lambda > 0$ be such that $\lambda \ge M(1 - N\int_0^1 q(x)dx)^{-1}$. Then the functions v_{n+1} defined recursively by* (1.2) *are such that for all $n \in \mathbb{N}$,*

(i) *v_n is a lower solution of* (1.1).

(ii) *$v_n \le v_{n+1}$.*

In the next result, we prove that upper solution u_n is larger than lower solution v_n for all n.

Proposition 3.7. *Assume that* $(H1)$, $(H2)$, $(H3)$, *and* $(H4)$ *hold, and let $\lambda > 0$ such that $\lambda \ge M(1 - N\int_0^1 q(x)dx)^{-1}$ and for all $x \in [0,1]$*

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \ge 0. \tag{3.12}$$

Then for all $n \in \mathbb{N}$, the functions u_n and v_n defined recursively by (1.2) *satisfy $v_n \le u_n$.*

Proof. We define a function

$$h_i(x) = f(x, v_i pv_i') - f(x, u_i, pu_i') + \lambda(u_i - v_i), \quad i \in \mathbb{N}. \tag{3.13}$$

It is easy to see that for all $i \in \mathbb{N}$, $w_i = u_i - v_i$ satisfies the following differential equation:

$$-(pw_i')' + \lambda q w_i = q\{f(x, v_{i-1}, pv_{i-1}') - f(x, u_{i-1}, pu_{i-1}') + \lambda(u_{i-1} - v_{i-1})\} = qh_{i-1}. \tag{3.14}$$

Now to prove this proposition again, we use the principle of mathematical induction. For $i = 1$, we have $h_0 \ge 0$, and w_1 is the solution of (2.1)-(2.2) with $A = 0$ and $B = 0$. Using Proposition 2.8, we deduce that $w_1 \ge 0$, that is, $u_1 \ge v_1$.

Now, let $n \geq 2$, let $h_{n-2} \geq 0$, and let $u_{n-1} \geq v_{n-1}$, then we are required to prove that $h_{n-1} \geq 0$ and $u_n \geq v_n$. First, we show that for all $x \in [0,1]$ the function h_{n-1} is nonnegative. Indeed, we have

$$\begin{aligned} h_{n-1} &= f\left(x, v_{n-1}, pv_{n-1}'\right) - f\left(x, u_{n-1},\ pu_{n-1}'\right) + \lambda(u_{n-1} - v_{n-1}) \\ &\geq -\left[(M-\lambda)w_{n-1} + N\left(\operatorname{sign} w_{n-1}'\right)pw_{n-1}'\right]. \end{aligned} \tag{3.15}$$

Here w_{n-1} is a solution of (2.1) with $h(x) = h_{n-2} \geq 0$, $A = 0$, and $B = 0$. Arguments similar to Proposition 3.4 can be used to prove that $h_{n-1} \geq 0$. Now, we have $h_{n-1} \geq 0$, $w_n'(0) = 0$, and $w_n'(1) = 0$, thus from Proposition 2.8, we deduce that $w_n \geq 0$, that is, $u_n \geq v_n$. □

Lemma 3.8. *If $f(x,u,pu')$ satisfies $(H1)$, $(H2)$, and*

$(H5)$ *for all $(x,u,pu') \in D$, $|f(x,u,pu')| \leq \varphi(|pu'|)$, where $\varphi : [0,\infty) \to (0,\infty)$ is continuous and satisfies.*

$$\int_0^\infty \frac{ds}{\varphi(s)} > \int_0^1 q(x)dx, \tag{3.16}$$

then there exists $R_0 > 0$ such that any solution of

$$-(pu')' + qf(x,u,pu') \geq 0, \quad 0 < x < 1,\ u'(0) = 0 = u'(1) \tag{3.17}$$

with $u \in [v_0, u_0]$, for all $x \in [0,1]$, satisfies $\|pu'\|_\infty < R_0$.

Proof. Consider an interval $[x, x_0] \subset [0,1]$ such that

$$\forall s \in [x, x_0), \quad u'(s) < 0, \qquad u'(x_0) = 0. \tag{3.18}$$

Now using $(H5)$, we have

$$(pu')' \leq q\varphi(|pu'|), \tag{3.19}$$

and after integrating it from x to x_0 and using $(H5)$, we have

$$-pu' \leq R_0. \tag{3.20}$$

Similarly for the interval $[x_0, x]$, we have

$$pu' \leq R_0. \tag{3.21}$$

Thus

$$\|pu'\|_\infty \leq R_0. \tag{3.22}$$

In the same way, we can prove the following result for lower solutions. □

Lemma 3.9. *If $f(x, v, pv')$ satisfies $(H1)$, $(H2)$, and $(H5)$, then there exists $R_0 > 0$ such that any solution of*

$$-(pv')' + qf(x, v, pv') \leq 0, \quad 0 < x < 1, \ v'(0) = 0 = v'(1) \tag{3.23}$$

with $v \in [v_0, u_0]$, for all $x \in [0,1]$, satisfies $\|pv'\|_\infty < R_0$.

Now we are in a situation to prove our final result for the case when upper and lower solutions are well ordered.

Theorem 3.10. *Assume $(H1)$, $(H2)$, $(H3)$, $(H4)$, and $(H5)$ are true. Let $\lambda > 0$ be such that*

$$\lambda \geq M\left(1 - N\int_0^1 q(x)dx\right)^{-1}, \tag{3.24}$$

and for all $x \in [0,1]$,

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \geq 0. \tag{3.25}$$

Then the sequences $\{u_n\}$ and $\{v_n\}$ defined by (1.2) converge monotonically to solutions $\tilde{u}(x)$ and $\tilde{v}(x)$ of (1.1). Any solution $z(x)$ of (1.1) in D satisfies

$$\tilde{v}(x) \leq z(x) \leq \tilde{u}(x). \tag{3.26}$$

Proof. Using Lemma 3.3 to Lemma 3.9 and Proposition 3.4 to Proposition 3.7, we deduce that the sequences $\{u_n\}$ and $\{v_n\}$ are monotonic ($u_0 \geq u_1 \geq u_2 \cdots \geq u_n \geq v_n \cdots \geq v_2 \geq v_1 \geq v_0$) and are bounded by v_0 and u_0 in $C[0,1]$, and by Dini's theorem, they converge uniformly to $\tilde{u}$ and $\tilde{v}$ (say). We can also deduce that the sequences $\{pu_n'\}$ and $\{pv_n'\}$ are uniformly bounded and equicontinuous in $C[0,1]$, and by Arzela-Ascoli theorem, there exists uniformly convergent subsequences $\{pu_{n_k}'\}$ and $\{pv_{n_k}'\}$ in $C[0,1]$. It is easy to observe that $u_n \to \tilde{u}$ and $v_n \to \tilde{v}$ imply $pu_n' \to p\tilde{u}'$ and $p\tilde{v}_n' \to p\tilde{v}'$.

Solution of (1.2) is given by (2.5) where $h(x) = -f(x, y_{n-1}, py_{n-1}') + \lambda y_{n-1}$. Since the sequences are uniformly convergent taking limit as $n \to \infty$, we get $\tilde{u}$ and $\tilde{v}$ as the solutions of the nonlinear boundary value problem (1.1). Any solution $z(x)$ in D plays the role of u_0. Hence, $z(x) \geq \tilde{v}(x)$. Similarly $z(x) \leq \tilde{u}(x)$. □

Remark 3.11. When the source function is derivative independent, that is, $N = 0$, in this case we can choose $\lambda = M$.

4. Upper and Lower Solutions in Reverse Order

In this section, we consider the case when the upper and lower solutions are in reverse order, that is,

$$u_0(x) \le v_0(x). \tag{4.1}$$

For this, we require opposite one-sided Lipschitz condition, and we assume that

($F1$) there exists upper solution (u_0) and lower solution (v_0) in $C[0,1] \cap C^2(0,1]$ such that $u_0 \le v_0$ for all $x \in [0,1]$,

($F2$) the function $f : D_0 \to \mathbb{R}$ is continuous on

$$D_0 := \{(x, y, py') \in [0,1] \times R \times R : u_0 \le y \le v_0\}, \tag{4.2}$$

($F3$) there exists $M \ge 0$ such that for all $(x, \tilde{\tau}, pv'), (x, \tilde{\sigma}, pv') \in D_0$,

$$f(x, \tilde{\sigma}, pv') - f(x, \tilde{\tau}, pv') \ge -M(\tilde{\sigma} - \tilde{\tau}), \quad (\tilde{\tau} \le \tilde{\sigma}), \tag{4.3}$$

($F4$) there exist $N \ge 0$ such that for all $(x, u, pv_1')(x, u, pv_2') \in D_0$,

$$\left|f(x, u, pv_1') - f(x, u, pv_2')\right| \le N\left|pv_2' - pv_1'\right|. \tag{4.4}$$

Here again we define the approximation scheme by (1.2) and use the Antimaximum principle. We make a good choice of λ so that the sequences thus generated converge to the solution of the nonlinear problem. Similar to Section 3, we require the following lemmas and propositions.

Lemma 4.1. *Let $-\lambda_0 < \lambda < 0$. If u_n is an upper solution of* (1.1) *and u_{n+1} is defined by* (1.2)*, then $u_{n+1} \ge u_n$.*

Proof. Let $w_n = u_{n+1} - u_n$, then

$$\begin{gathered} -(pw_n')' + \lambda q w_n = (pu_n')' - qf(x, u_n, pu_n') \le 0, \\ w_n'(0) \ge 0, \qquad w_n'(1) \le 0, \end{gathered} \tag{4.5}$$

and using Proposition 2.8, we have $u_{n+1} \ge u_n$. □

Proposition 4.2. *Assume that* ($F1$), ($F2$), ($F3$), *and* ($F4$) *hold. Let $-\lambda_0 < \lambda < 0$ be such that $-\left(\int_0^1 (1/p(x)) \int_0^x q(t)dt\,dx\right)^{-1} < \lambda \le -M$ and $(M + \lambda)(1 + \lambda \int_0^1 (1/p(x)) \int_0^x q(t)dt\,dx) - N\lambda \int_0^1 q(x)dx \le 0$. Then the functions u_{n+1} defined recursively by* (1.2) *are such that, for all $n \in \mathbb{N}$,*

(i) *u_n is an upper solution of* (1.1);

(ii) $u_{n+1} \ge u_n$.

Proof. Using Remarks 2.5 and 2.7, Lemmas 2.11 and 4.1, and on the lines of the proof of Proposition 3.4, this proposition can be deduced easily.

In the same way, we can prove the following results for the lower solutions. □

Lemma 4.3. *Let* $-\lambda_0 < \lambda < 0$. *If* v_n *is a lower solution of* (1.1) *and* v_{n+1} *is defined by* (1.2), *then* $v_n \geq v_{n+1}$.

Proposition 4.4. *Assume that* $(F1)$, $(F2)$, $(F3)$, *and* $(F4)$ *hold. Let* $-\lambda_0 < \lambda < 0$ *be such that* $-(\int_0^1 (1/p(x)) \int_0^x q(t)dt\,dx)^{-1} < \lambda \leq -M$ *and* $(M+\lambda)(1+\lambda \int_0^1 (1/p(x)) \int_0^x q(t)dt\,dx) - N\lambda \int_0^1 q(x)dx \leq 0$. *Then the functions* v_{n+1} *defined recursively by* (1.2) *are such that, for all* $n \in \mathbb{N}$,

(i) v_n *is a lower solution of* (1.1);

(ii) $v_n \geq v_{n+1}$.

In the next result, we prove that lower solution v_n is larger than upper solution u_n for all n.

Proposition 4.5. *Assume that* $(F1)$, $(F2)$, $(F3)$, *and* $(F4)$ *hold. Let* $-\lambda_0 < \lambda < 0$ *be such that* $-(\int_0^1 (1/p(x)) \int_0^x q(t)dt\,dx)^{-1} < \lambda \leq -M$ *and*

$$(M+\lambda)\left(1+\lambda\int_0^1 \frac{1}{p(x)}\int_0^x q(t)dt\,dx\right) - N\lambda\int_0^1 q(x)dx \leq 0, \tag{4.6}$$

and for all $x \in [0,1]$,

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \geq 0. \tag{4.7}$$

Then, for all $n \in \mathbb{N}$, *the functions* u_n *and* v_n *defined recursively by* (1.2) *satisfy* $v_n \geq u_n$.

Now similar to Lemmas 3.8 and 3.9, we state the following two results. These results establish a bound on $p(x)u'(x)$ and $p(x)v'(x)$.

Lemma 4.6. *If* $f(x,u,pu')$ *satisfies* $(F1)$, $(F2)$, *and*

$(F5)$ *for all* $(x,u,pu') \in D_0$, $|f(x,u,pu')| \leq \varphi(|pu'|)$, *where* $\varphi : [0,\infty) \to (0,\infty)$ *is continuous and satisfies*

$$\int_0^\infty \frac{ds}{\varphi(s)} > \int_0^1 q(x)dx, \tag{4.8}$$

then there exists $R_0 > 0$ *such that any solution of*

$$-(pu')' + qf(x,u,pu') \geq 0, \quad 0 < x < 1,\ u'(0) = 0 = u'(1) \tag{4.9}$$

with $u \in [u_0, v_0]$, *for all* $x \in [0,1]$, *satisfies* $\|pu'\|_\infty < R_0$.

Lemma 4.7. *If $f(x, v, pv')$ satisfies* $(F1)$, $(F2)$, *and* $(F5)$, *then there exists $R_0 > 0$ such that any solution of*

$$-(pv')' + qf(x, v, pv') \le 0, \quad 0 < x < 1, \; v'(0) = 0 = v'(1) \tag{4.10}$$

with $v \in [u_0, v_0]$, for all $x \in [0, 1]$, satisfies $\|pv'\|_\infty < R_0$.

Finally we arrive at the theorem similar to Theorem 3.10.

Theorem 4.8. *Assume* $(F1)$, $(F2)$, $(F3)$, $(F4)$, *and* $(F5)$ *are true. Let $-\lambda_0 < \lambda < 0$ be such that $-\left(\int_0^1 (1/p(x)) \int_0^x q(t)dt\, dx\right)^{-1} < \lambda \le -M$ and $(M + \lambda)(1 + \lambda \int_0^1 (1/p(x)) \int_0^x q(t)dt\, dx) - N\lambda \int_0^1 q(x)dx \le 0$, and for all $x \in [0, 1]$,*

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \ge 0. \tag{4.11}$$

Then the sequences $\{u_n\}$ and $\{v_n\}$ defined by (1.2) *converge monotonically to solutions $\tilde{u}(x)$ and $\tilde{v}(x)$ of* (1.1). *Any solution $z(x)$ of* (1.1) *in D_0 satisfies*

$$\tilde{u}(x) \le z(x) \le \tilde{v}(x). \tag{4.12}$$

Proof. Using Lemma 4.1 to Lemma 4.7 and Proposition 4.2 to Proposition 4.5, we deduce that

$$u_0 \le u_1 \le u_2 \le \cdots \le u_n \le v_n \cdots \le v_1 \le v_0. \tag{4.13}$$

Now similar to the proof of Theorem 3.10, the result of this theorem can be deduced. □

Remark 4.9. When the source function is derivative independent, that is, $N = 0$, in this case we can choose $\lambda = -M$.

5. Conclusion

We establish some existence results under quite general conditions on $p(x)$, $q(x)$, and $f(x, y, py')$. We prove some fundamental differential inequalities which enables us to prove the monotonicity of the sequences $\{u_n\}$ and $\{v_n\}$. For this we have analyzed the singular differential equation $-(py')' + \lambda qy = 0$ and derived properties of the solutions and their derivatives. This work generalizes our previous work [3]. Lot of exploration is still left. For example, one can consider different type of boundary conditions, and one can also try to remove the Lipschitz condition.

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