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Research Article

The Existence of Positive Solutions for Singular Impulse Periodic Boundary Value Problem

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We obtain new result of the existence of positive solutions of a class of singular impulse periodic boundary value problem via a nonlinear alternative principle of Leray-Schauder. We do not require the monotonicity of functions in paper (Zhang and Wang, 2003). An example is also given to illustrate our result.

1. Introduction

Because of wide interests in physics and engineering, periodic boundary value problems have been investigated by many authors (see [1–19]). In most real problems, only the positive solution is significant.

In this paper, we consider the following periodic boundary value problem (PBVP in short) with impulse effects:

$$\begin{gathered} -u''(t) + Mu(t) = f(t, u(t)), \quad t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad -\Delta u'\big|_{t=t_k} = J_k(u(t_k)), \quad k = 1, 2, \dots l, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{gathered} \tag{1.1}$$

Here, $J = [0, 2\pi]$, $0 < t_1 < t_2 < \cdots < t_l < 2\pi$, $J' = J \setminus \{t_1, t_2, \dots, t_l\}$, $M > 0$, $f \in C(J \times R_+, R^+)$, $I_k \in C(R^+, R)$, $J_k \in C(R^+, R^+)$, $R^+ = [0, +\infty)$, $R_+ = (0, +\infty)$ with $-(1/m)J_k(u) < I_k(u) < (1/m)J_k(u)$, $u \in R^+$, $m = \sqrt{M}$. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, where $u^{(i)}(t_k^+)$ and $u^{(i)}(t_k^-)$, $i = 0, 1$, respectively, denote the right and left limit of $u^{(i)}(t)$ at $t = t_k$.

In [7], Liu applied Krasnoselskii's and Leggett-Williams fixed-point theorem to establish the existence of at least one, two, or three positive solutions to the first-order periodic boundary value problems

$$\begin{gathered}x'(t)+a(t)x(t)=f(t,x(t)),\quad \text{a.e. } t\in[0,T]\setminus\{t_1,\dots,t_p\},\\ \Delta x|_{t=t_k}=I_k(x(t_k)),\quad k=1,\dots,p,\\ x(0)=x(T).\end{gathered}\tag{1.2}$$

Jiang [5] has applied Krasnoselskii's fixed point theorem to establish the existence of positive solutions of problem

$$\begin{gathered}x''(t)+Mx(t)=f(t,x(t)),\quad t\in[0,2\pi],\\ x(0)=x(2\pi),\qquad x'(0)=x'(2\pi).\end{gathered}\tag{1.3}$$

The work [5] proved that periodic boundary value problem (PBVP in short) (1.3) without singularity have at least one positive solutions provided $f(t,x)$ is superlinear or sublinear at $x=0+$ and $x=+\infty$. In [14], Tian et al. researched PBVP (1.1) without singularity. They obtained the existence of multiple positive solutions of PBVP (1.1) by replacing the suplinear condition or sublinear condition of [4] with the following limit inequality condition:

(A_1)

$$\left[2\pi f_0+\sum_{i=1}^{l}J_0(i)\right]\sigma>2\pi M,\qquad \left[2\pi f_\infty+\sum_{i=1}^{l}J_\infty(i)\right]\sigma>2\pi M,\tag{1.4}$$

(A_2)

$$\left[2\pi f^0+\sum_{i=1}^{l}J^0(i)\right]\sigma<2\pi M,\qquad \left[2\pi f^\infty+\sum_{i=1}^{l}J^\infty(i)\right]\sigma<2\pi M.\tag{1.5}$$

Nieto [10] introduced the concept of a weak solution for a damped linear equation with Dirichlet boundary conditions and impulses. These results will allow us in the future to deal with the corresponding nonlinear problems and look for solutions as critical points of weakly lower semicontinuous functionals.

We note that the function f involved in above papers [5, 7, 10, 14] does not have singularity. Xiao et al. [16] investigate the multiple positive solutions of singular boundary value problem for second-order impulsive singular differential equations on the halfline, where the function $f(t,u)$ is singular only at $t=0$ and/or $t=1$. Reference [19] studied PBVP (1.3), where the function f has singularity at $x=0$. The authors present the existence of multiple positive solutions via the Krasnoselskii's fixed point theorem under the following conditions.

(A_1') There exist nonnegative valued $\xi(x), \eta(x) \in C((0,\infty))$ and $P(t), Q(t) \in L^1[0,2\pi]$ such that

$$0 \le f(t,x) \le P(t)\xi(x) + Q(t)\eta(x), \quad \text{a.e. } (t,x) \in [0,2\pi] \times (0,\infty),$$
$$\sup_{x\in(0,\infty)} \left\{ \frac{x}{\left(\int_0^{2\pi} P(t)dt\xi(x)/\eta(x) + \int_0^{2\pi} Q(t)dt\right)\eta(\delta_j t)} \right\} > B_j, \tag{1.6}$$

where $\eta(x)$ is nonincreasing and $\xi(x)/\eta(x)$ is nondecreasing on $(0,\infty)$,

(A_2')

$$\liminf_{t\to 0+} \frac{\min\left\{\int_0^{2\pi} f(x,w)dx : \delta_j t \le w \le t\right\}}{t} > \frac{1}{A_j}, \tag{1.7}$$

(A_3')

$$\liminf_{t\to +\infty} \frac{\min\left\{\int_0^{2\pi} f(x,w)dx : \delta_j t \le w \le t\right\}}{t} > \frac{1}{A_j}. \tag{1.8}$$

Here, δ_j, A_j, B_j are some constants.

In this paper, the nonlinear term $f(t,u)$ is singular at $u = 0$, and positive solution of PBVP (1.1) is obtained by a nonlinear alternative principle of Leray-Schauder type in cone. We do not require the monotonicity of functions $\eta, \xi/\eta$ used in [19]. An example is also given to illustrate our result.

This paper is organized as follows. In Section 1, we give a brief overview of recent results on impulsive and periodic boundary value problems. In Section 2, we present some preliminaries such as definitions and lemmas. In Section 3, the existence of one positive solution for PBVP (1.1) will be established by using a nonlinear alternative principle of Leray-Schauder type in cone. An example is given in Section 4.

2. Preliminaries

Consider the space $PC[J,R] = \{u : u$ is a map from J into R such that $u(t)$ is continuous at $t \ne t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, for $k = 1,2,\ldots l.\}$. It is easy to say that $PC[J,R]$ is a Banach space with the norm $\|u\|_{pc} = \sup_{t\in J}|u(t)|$. Let $PC^1[J,R] = \{u \in PC[J,R] : u'(t)$ exists at $t \ne t_k$ and is continuous at $t \ne t_k$, and $u'(t_k^+)$, $u'(t_k^-)$ exist and $u'(t)$ is left continuous at $t = t_k$, for $k = 1,2,\ldots l.\}$ with the norm $\|u\|_{pc^1} = \max\{\|u\|_{pc}, \|u'\|_{pc}\}$. Then, $PC^1[J,R]$ is also a Banach space.

Lemma 2.1 (see [15]). *$u \in PC^1(J,R) \cap C^2(J',R)$ is a solution of PBVP* (1.1) *if and only if $u \in PC(J)$ is a fixed point of the following operator T:*

$$Tu(t) = \int_0^{2\pi} G(t,s)f(s,u(s))ds + \sum_{k=1}^{l} G(t,t_k)J_k(u(t_k)) + \sum_{k=1}^{l} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_k} I_k(u(t_k)), \tag{2.1}$$

where $G(t,s)$ is the Green's function to the following periodic boundary value problem:

$$\begin{gathered} -u'' + Mu = 0, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \\ G(t,s) := \frac{1}{\Gamma}\left\{ \begin{cases} e^{m(t-s)} + e^{m(2\pi-t+s)}, & 0 \le s \le t \le 2\pi, \\ e^{m(s-t)} + e^{m(2\pi-s+t)}, & 0 \le t \le s \le 2\pi, \end{cases} \right. \end{gathered} \tag{2.2}$$

here, $\Gamma = 2m(e^{2m\pi} - 1)$. It is clear that

$$\frac{2e^{m\pi}}{\Gamma} = G(\pi) \le G(t,s) \le G(0) = \frac{e^{2m\pi}+1}{\Gamma}. \tag{2.3}$$

Define

$$K = \left\{ u \in PC[J,R] : u(t) \ge \sigma \|u\|_{pc},\ t \in J \right\}, \tag{2.4}$$

where

$$\sigma = \frac{1}{e^{2m\pi}}. \tag{2.5}$$

The following nonlinear alternative principle of Leray-Schauder type in cone is very important for us.

Lemma 2.2 (see [4]). *Assume that Ω is a relatively open subset of a convex set K in a Banach space $PC[J,R]$. Let $T : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then, either*

(i) *T has a fixed point in $\overline{\Omega}$, or,*

(ii) *there is a $u \in \partial\Omega$ and a $\lambda < 1$ such that $u = \lambda Tu$.*

3. Main Results

In this section, we establish the existence of positive solutions of PBVP (1.1).

Theorem 3.1. *Assume that the following three hypothesis hold:*

(H_1) *there exists nonnegative functions* $\xi(u), \eta(u), \gamma(u) \in C(0,+\infty)$ *and* $p(t), q(t) \in L^1([0,2\pi])$ *such that*

$$f(t,u) \le p(t)\xi(u) + q(t)\eta(u), \quad (t,u) \in [0,2\pi] \times (0,\infty), \tag{3.1}$$

$$\max_{1\le k\le l} J_k(u) \le \gamma(u), \quad (t,u) \in [0,2\pi] \times (0,+\infty), \tag{3.2}$$

(H_2) *there exists a positive number* $r > 0$ *such that*

$$\frac{A}{2}\left\{\max_{x\in[\sigma r,r]} \xi(x) \int_0^{2\pi} p(s)ds + \max_{x\in[\sigma r,r]} \eta(x) \int_0^{2\pi} q(s)ds\right\} + Al\gamma(r) < r, \tag{3.3}$$

(H_3) *for the constant* r *in* (H_2)*, there exists a function* $\Phi_r > 0$ *such that*

$$f(t,u) > \Phi_r(t), \quad (t,u) \in [0,2\pi] \times (0,r], \qquad \int_0^{2\pi} \Phi_r(s)ds > 0. \tag{3.4}$$

Then PBVP (1.1) *has at least one positive periodic solution with* $0 < \|u\| < r$*, where*

$$A = \frac{e^{2m\pi}+1}{m(e^{2m\pi}-1)} = \frac{e^{2\pi\sqrt{M}}+1}{\sqrt{M}\left(e^{2\pi\sqrt{M}}-1\right)}. \tag{3.5}$$

Proof. The existence of positive solutions is proved by using the Leray-Schauder alternative principle given in Lemma 2.2. We divide the rather long proof into six steps.

Step 1. From (3.3), we may choose $n_0 \in \{1,2,\ldots\}$ such that

$$\frac{A}{2}\left\{\max_{x\in[\sigma r,r]} \xi(x) \int_0^{2\pi} p(s)ds + \max_{x\in[\sigma r,r]} \eta(x) \int_0^{2\pi} q(s)ds\right\} + Al\gamma(r) + \frac{1}{n_0} < r. \tag{3.6}$$

Let $N_0 = \{n_0, n_0+1, \ldots\}$. For $n \in N_0$. We consider the family of equations

$$\begin{gathered} -u''(t) + Mu(t) = \lambda f_n(t,u(t)) + \frac{M}{n}, \quad t \in J', \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad -\Delta u'\big|_{t=t_k} = J_k(u(t_k)), \quad k = 1,2,\ldots l, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{gathered} \tag{3.7}$$

where $\lambda \in [0,1]$ and

$$f_n(t,u) = f\left(t, \max\left\{u, \frac{1}{n}\right\}\right), \quad (t,u) \in J \times [0,+\infty). \tag{3.8}$$

For every λ and $n \in N_0$, define an operator as follows:

$$\begin{aligned} T_{\lambda,n}u(t) &= \lambda \int_0^{2\pi} G(t,s) f_n(s,u(s))ds + \sum_{k=1}^{l} G(t,t_k) J_k(u(t_k)) \\ &\quad + \sum_{k=1}^{l} \frac{\partial G(t,s)}{\partial s}\bigg|_{s=t_k} I_k(u(t_k)), \quad u \in K. \end{aligned} \tag{3.9}$$

Then, we may verify that

$$T_{\lambda,n} : K \longrightarrow K \text{ is completely continuous.} \tag{3.10}$$

To find a positive solution of (3.7) is equivalent to solve the following fixed point problem in $PC[J,R]$:

$$u = T_{\lambda,n}u + \frac{1}{n}. \tag{3.11}$$

Let

$$\Omega = \{x \in K : \|x\| < r\}, \tag{3.12}$$

then Ω is a relatively open subset of the convex set K.

Step 2. We claim that any fixed point u of (3.11) for any $\lambda \in [0,1)$ must satisfies $\|u\| \neq r$.

Otherwise, we assume that u is a solution of (3.11) for some $\lambda \in [0,1)$ such that $\|u\| = r$. Note that $f_n(t,u) \geq 0$. $u(t) \geq 1/n$ for all $t \in J$ and $r \geq u(t) \geq (1/n) + \sigma\|u - 1/n\|$. By the choice of n_0, $1/n \leq 1/n_0 < r$. Hence, for all $t \in J$, we get

$$r \geq u(t) \geq \frac{1}{n} + \sigma\left\|u - \frac{1}{n}\right\| \geq \frac{1}{n} + \sigma\left|\|u\| - \frac{1}{n}\right| \geq \frac{1}{n} + \sigma\left(r - \frac{1}{n}\right) > \sigma r. \tag{3.13}$$

From (3.2), we have

$$J_k(u(t_k)) \leq \max_{1 \leq k \leq l} J_k(u(t_k)) \leq \gamma(u(t_k)) \leq \gamma(r). \tag{3.14}$$

Consequently, for any fixed point u of (3.11), by (3.8), (3.13), and (3.14), we have

$$\begin{aligned} u(t) &= \lambda \int_0^{2\pi} G(t,s) f_n(s,u(s))ds + k = 1\sum_{k=1}^{l} G(t,t_k) J_k(u(t_k)) + \sum_{k=1}^{l} \frac{\partial G(t,s)}{\partial s}\bigg|_{s=t_k} I_k(u(t_k)) + \frac{1}{n} \\ &\leq \int_0^{2\pi} G(t,s) f(s,u(s))ds + \sum_{k=1}^{l} G(t,t_k) J_k(u(t_k)) + \sum_{k=1}^{l} \frac{\partial G(t,s)}{\partial s}\bigg|_{s=t_k} I_k(u(t_k)) + \frac{1}{n} \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{2\pi} G(t,s) f(s,u(s)) ds \\
&\quad + \frac{1}{\Gamma} \Bigg\{ \sum_{t_k \leq t} \left[e^{m(t-t_k)} + e^{m(2\pi - t + t_k)} \right] J_k(u(t_k)) \\
&\qquad + \sum_{t_k > t} \left[e^{m(t_k - t)} + e^{m(2\pi - t_k + t)} \right] J_k(u(t_k)) + \sum_{t_k \leq t} \left[-e^{m(t-t_k)} + e^{m(2\pi - t + t_k)} \right] m I_k(u(t_k)) \\
&\qquad + \sum_{t_k > t} \left[e^{m(t_k - t)} - e^{m(2\pi - t_k + t)} \right] m I_k(u(t_k)) \Bigg\} + \frac{1}{n} \\
&= \int_0^{2\pi} G(t,s) f(s,u(s)) ds \\
&\quad + \frac{1}{\Gamma} \Bigg\{ \sum_{t_k \leq t} e^{m(t-t_k)} [J_k(u(t_k)) - m I_k(u(t_k))] \\
&\qquad + \sum_{t_k \leq t} e^{m(2\pi - t + t_k)} [J_k(u(t_k)) + m I_k(u(t_k))] + \sum_{t_k > t} e^{m(t_k - t)} [J_k(u(t_k)) + m I_k(u(t_k))] \\
&\qquad + \sum_{t_k > t} e^{m(2\pi - t_k + t)} [J_k(u(t_k)) - m I_k(u(t_k))] \Bigg\} + \frac{1}{n}.
\end{aligned}
\tag{3.15}
$$

It follows from $-(1/m)J_k(u) < I_k(u) < (1/m)J_k(u)$ that

$$
J_k(u(t_k)) - m I_k(u(t_k)) > 0, \qquad J_k(u(t_k)) + m I_k(u(t_k)) > 0. \tag{3.16}
$$

So, we get from (3.1), (3.2), and (3.3) that

$$
\begin{aligned}
u(t) &\leq \int_0^{2\pi} G(t,s) f(s,u(s)) ds + \frac{2(e^{2m\pi} + 1)}{\Gamma} \sum_{k=1}^{l} J_k(u(t_k)) + \frac{1}{n} \\
&\leq \int_0^{2\pi} G(t,s) [p(s)\xi(u(s)) + q(s)\eta(u(s))] ds + Al\gamma(r) + \frac{1}{n_0} \\
&\leq \frac{A}{2} \left[\int_0^{2\pi} p(s) ds \max_{x \in [\sigma r, r]} \xi(x) + \int_0^{2\pi} q(s) ds \max_{x \in [\sigma r, r]} \xi(x) \right] + Al\gamma(r) + \frac{1}{n_0}.
\end{aligned}
\tag{3.17}
$$

Therefore,

$$
r = \|u\| \leq \frac{A}{2} \left[\int_0^{2\pi} p(s) ds \max_{x \in [\sigma r, r]} \xi(x) + \int_0^{2\pi} q(s) ds \max_{x \in [\sigma r, r]} \xi(x) \right] + Al\gamma(r) + \frac{1}{n_0} < r. \tag{3.18}
$$

This is a contraction, and so the claim is proved.

Step 3. From the above claim and the Leray-Schauder alternative principle, we know that operator (3.9) (with $\lambda = 1$) has a fixed point denoted by u_n in $\overline{\Omega}$. So, (3.7) (with $\lambda = 1$) has a positive solution u_n with

$$\|u_n\| < r, \qquad u_n(t) \geq \frac{1}{n}, \quad t \in J. \tag{3.19}$$

Step 4. We show that $\{u_n\}$ have a uniform positive lower bound; that is, there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$\min_t \{u_n(t)\} \geq \delta. \tag{3.20}$$

In fact, from (3.4), (3.8), (3.16), and (3.19), we get

$$\begin{aligned}
u_n(t) &= \int_0^{2\pi} G(t,s) f_n(s, u_n(s)) ds + \sum_{k=1}^{l} G(t,t_k) J_k(u_n(t_k)) \\
&\quad + \sum_{k=1}^{l} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_k} I_k(u_n(t_k)) + \frac{1}{n} \\
&= \int_0^{2\pi} G(t,s) f(s, u_n(s)) ds + \sum_{k=1}^{l} G(t,t_k) J_k(u_n(t_k)) \\
&\quad + \sum_{k=1}^{l} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_k} I_k(u_n(t_k)) + \frac{1}{n} \\
&\geq \int_0^{2\pi} G(t,s) \Phi_r(s) ds \\
&\quad + \frac{1}{\Gamma} \Bigg\{ \sum_{t_k \leq t} e^{m(t-t_k)} \left[J_k(u_n(t_k)) - m I_k(u_n(t_k)) \right] \\
&\qquad + \sum_{t_k \leq t} e^{m(2\pi - t + t_k)} \left[J_k(u_n(t_k)) + m I_k(u_n(t_k)) \right] \\
&\qquad + \sum_{t_k > t} e^{m(t_k - t)} \left[J_k(u_n(t_k)) + m I_k(u_n(t_k)) \right] \\
&\qquad + \sum_{t_k > t} e^{m(2\pi - t_k + t)} \left[J_k(u_n(t_k)) - m I_k(u_n(t_k)) \right] \Bigg\} + \frac{1}{n} \\
&\geq \int_0^{2\pi} G(t,s) \Phi_r(s) ds \\
&\geq \frac{2e^{m\pi}}{\Gamma} \int_0^{2\pi} \Phi_r(s) ds := \delta > 0.
\end{aligned} \tag{3.21}$$

Step 5. We prove that

$$\|u_n'\| < H, \quad n \geq n_0 \tag{3.22}$$

for some constant $H > 0$. Equations (3.19) and (3.20) tell us that $\delta \leq u_n(t) \leq r$, so we may let

$$M_1 = \max_{t\in J,\, u\in[\delta,r]} f(t,u), \qquad M_2 = \max_{t,\, s\in J}\left|G_t'(t,s)\right|, \qquad M_3 = \max_{u\in[\delta,r]} \sum_{k=1}^{l} J_k(u). \tag{3.23}$$

Then,

$$\begin{aligned}
\|u_n'\| &= \sup_{t\in J}\left|u_n'(t)\right| \\
&= \sup_{t\in J}\Bigg|\int_0^{2\pi} G_t'(t,s) f(s,u_n(s))ds + \sum_{k=1}^{l} G_t'(t,t_k) J_k(u_n(t_k)) \\
&\qquad + \sum_{k=1}^{l} \frac{\partial}{\partial t}\left(\frac{\partial G(t,s)}{\partial s}\bigg|_{s=t_k}\right) I_k(u_n(t_k))\Bigg| \\
&= \sup_{t\in J}\Bigg|\int_0^{2\pi} G_t'(t,s) f(s,u_n(s))ds \\
&\qquad + \frac{m}{\Gamma}\Bigg\{\sum_{t_k\leq t} e^{m(t-t_k)}\left[J_k(u_n(t_k)) - mI_k(u_n(t_k))\right] \\
&\qquad\qquad - \sum_{t_k\leq t} e^{m(2\pi-t+t_k)}\left[J_k(u_n(t_k)) + mI_k(u_n(t_k))\right] \\
&\qquad\qquad - \sum_{t_k>t} e^{m(t_k-t)}\left[J_k(u_n(t_k)) + mI_k(u_n(t_k))\right] \\
&\qquad\qquad + \sum_{t_k>t} e^{m(2\pi-t_k+t)}\left[J_k(u_n(t_k)) - mI_k(u_n(t_k))\right]\Bigg\}\Bigg| \\
&\leq \sup_{t\in J}\int_0^{2\pi}\left|G_t'(t,s)\right| f(s,u_n(s))ds + \frac{2m\left(e^{2m\pi}+1\right)}{\Gamma}\sum_{k=1}^{l} J_k(u_n(t_k)) \\
&\leq 2\pi M_1 M_2 + \frac{2m\left(e^{2m\pi}+1\right)}{\Gamma} M_3 := H.
\end{aligned} \tag{3.24}$$

Step 6. Now, we pass the solution u_n of the truncation equation (3.7) (with $\lambda = 1$) to that of the original equation (1.1). The fact that $\|u_n\| < r$ and (3.22) show that $\{u_n\}_{n\in N_0}$ is a bounded and equi-continuous family on $[0,2\pi]$. Then, the Arzela-Ascoli Theorem guarantees that $\{u_n\}_{n\in N_0}$ has a subsequence $\{u_{n_j}\}_{j\in N}$, converging uniformly on $[0,2\pi]$. From the fact $\|u_n\| < r$ and

(3.20), u satisfies $\delta \le u(t) \le r$ for all $t \in J$. Moreover, u_{n_j} also satisfies the following integral equation:

$$\begin{aligned} u_{n_j}(t) = {} & \int_0^{2\pi} G(t,s) f\left(s, u_{n_j}(s)\right) ds \\ & + \sum_{k=1}^{l} G(t,t_k) J_k\left(u_{n_j}(t_k)\right) + \sum_{k=1}^{l} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_k} I_k\left(u_{n_j}(t_k)\right) + \frac{1}{n_j}. \end{aligned} \tag{3.25}$$

Let $j \to +\infty$, and we get

$$\begin{aligned} u(t) = {} & \int_0^{2\pi} G(t,s) f(s, u(s)) ds \\ & + \sum_{k=1}^{l} G(t,t_k) J_k(u(t_k)) + \sum_{k=1}^{l} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_k} I_k(u(t_k)), \end{aligned} \tag{3.26}$$

where the uniform continuity of $f(t,u)$ on $J \times [\delta, r]$ is used. Therefore, u is a positive solution of PBVP (1.1). This ends the proof. □

4. An Example

Consider the following impulsive PBVP:

$$\begin{gathered} -u''(t) + Mu(t) = t^2\left(1 + \frac{|\sin u|}{u^{3/2}}\right) + t(1 + |\cos u|), \quad t \in J', \\ \Delta u|_{t=t_k} = \frac{\min\{c_1, c_2, \ldots, c_l\}}{2\sqrt{M}} u(t_k), \quad -\Delta u'\big|_{t=t_k} = c_k u(t_k), \quad k = 1, 2, \ldots, l, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{gathered} \tag{4.1}$$

where $c_k > 0$ are constants. Then, PBVP (4.1) has at least one positive solution u with $0 < \|u\| < 1$.

To see this, we will apply Theorem 3.1.

Let

$$f(t,u) = t^2\left(1 + \frac{|\sin u|}{u^{3/2}}\right) + t(1 + |\cos u|), \tag{4.2}$$

then $f(t,u)$ has a repulsive singularity at $u = 0$

$$\lim_{u \to 0^+} f(t,u) = +\infty, \quad \text{uniformaly in } t. \tag{4.3}$$

Denote

$$p(t) = t^2, \quad q(t) = t, \quad \xi(u) = 1 + \frac{|\sin u|}{u^{3/2}}, \quad \eta(u) = 1 + |\cos u|,$$
$$\gamma(u) = \max\{c_1, c_2, \dots, c_l\}u, \tag{4.4}$$
$$r = 1, \quad \Phi_r(t) = t + t^2.$$

Then, it is easy to say that (3.1), (3.2), and (3.3) hold. From (3.5), we know

$$\lim_{M\to+\infty} A = \lim_{M\to+\infty} \frac{e^{2\pi\sqrt{M}} + 1}{\sqrt{M}\left(e^{2\pi\sqrt{M}} - 1\right)} = 0. \tag{4.5}$$

So, we may choose M large enough to guarantee that (3.3) holds. Then, the result follows from Theorem 3.1.

Remark 4.1. Functions ξ, η in example (4.1) do not have the monotonicity required as in [19]. So, the results of [19] cannot be applied to PBVP (4.1).

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