Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 902830, 19 pages
doi:10.1155/2011/902830

Research Article

Differential Subordination and Superordination for Srivastava-Attiya Operator

Maisarah Haji Mohd and Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor D. Ehsan, Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

Received 28 March 2011; Accepted 1 June 2011

Academic Editor: Khalil Ezzinbi

Due to the well-known Srivastava-Attiya operator, we investigate here some results relating the p-valent of the operator with differential subordination and subordination. Further, we obtain some interesting results on sandwich-type theorem for the same.

1. Introduction and Motivation

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disc U and let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(U)$ consisting functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$, with $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H} = \mathcal{H}[1,1]$. For two functions f_1 and f_2 analytic in U, the function f_1 is subordinate to f_2, or f_2 superordinate to f_1, written as $f_1 \prec f_2$ if there exists a function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f_1(z) = f_2(w(z))$. In particular, if the function f_2 is univalent in U, then $f_1 \prec f_2$ is equivalent to $f_1(0) = f_2(0)$ and $f_1(U) \subset f_2(U)$.

Let $f, h \in \mathcal{H}(U)$ and $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. If f and $\psi(f(z), zf'(z), z^2 f''(z); z)$ are univalent and f satisfies the second-order differential subordination

$$\psi\Big(f(z), zf'(z), z^2 f''(z); z\Big) \prec h(z), \tag{1.1}$$

then f is called a solution of the differential subordination. The univalent function F is called a dominant if $f \prec F$ for all f satisfying (1.1). Miller and Mocanu discussed many interesting results containing the above mentioned subordination and also many applications of the field of differential subordination in [1]. In that direction, many differential subordination and differential superordination problems for analytic functions defined by means of linear operators were investigated. See [2–11] for related results.

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (z \in U,\ p \in \mathbb{N} = 1, 2, 3, \ldots), \tag{1.2}$$

which are analytic and p-valent in U. For f satisfying (1.2), let the generalized Srivastava-Attiya operator [12] be denoted by

$$J_{s,b} f(z) = G_{s,b} * f(z) \quad \left(b \in \mathbb{C} \setminus \overline{Z}_0 = 0, -1, -2, \ldots\right), \tag{1.3}$$

where

$$G_{s,b} = (1+b)^s \left[\varphi(z, s, b) - b^{-s}\right], \tag{1.4}$$

with

$$\varphi(z, s, b) = \frac{1}{b^s} + \frac{z^p}{(1+b)^s} + \frac{z^{1+p}}{(2+b)^s} + \cdots, \tag{1.5}$$

and the symbol $(*)$ denotes the usual Hadamard product (or convolution). From the equations, we can see that

$$J_{s,b} f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{1+b}{n+1+b}\right)^s a_{n+p} z^{n+p}. \tag{1.6}$$

Note that for $p = 1$ in (1.6), $J_{s,b} f(z)$ coincides with the Srivastava-Attiya operator [13]. Further, observe that for proper choices of s and b, the operator $J_{s,b} f(z)$ coincides with the following:

(i) $J_{0,b} f(z) = f(z)$,
(ii) $J_{1,0} f(z) = A(f)(z)$ [14],
(iii) $J_{1,\gamma} f(z) = \mathcal{J}_\gamma(f)(z)$, $(\gamma > -1)$ [15, 16],
(iv) $J_{\sigma,1} f(z) = I^\sigma(f)(z)$, $(\sigma > 0)$ [17],
(v) $J_{\alpha,\beta} f(z) = P_\beta^\alpha(f)(z)$, $(\alpha \geq 1,\ \beta > 1)$ [18].

Since the above mentioned operator, the generalized Srivastava-Attiya operator, $J_{s,b} f(z)$ reduces to the well-known operators introduced and studied in the literature by suitably specializing the values of s and b and also in view of the several interesting properties and characteristics of well-known differential subordination results, we aim to associate these two motivating findings and obtain certain other related results. Further, we consider the differential superordination problems associated with the same operator. In addition, we also obtain interesting sandwich-type theorems.

The following definitions and theorems were discussed and will be needed to prove our results.

Definition 1.1 (see [1], Definition 2.2b, page 21). Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$ where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} = \infty \right\} \tag{1.7}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$, and $Q(1) \equiv Q_1$.

Definition 1.2 (see [1], Definition 2.3a, page 27). Let Ω be a set in $\mathbb{C}$, $q \in Q$, and let n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(c, d, e; z) \notin \Omega$ whenever $c = q(\zeta)$, $d = k\zeta q'(\zeta)$, and

$$\operatorname{Re}\left\{ \frac{e}{d} + 1 \right\} \geq k \operatorname{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}, \tag{1.8}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq n$. Let $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

Definition 1.3 (see [19], Definition 3, page 817). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(c, d, e; \zeta) \notin \Omega$ *whenever* $c = q(z)$, $d = zq'(z)/m$, and

$$\operatorname{Re}\left\{ \frac{e}{d} + 1 \right\} \geq \frac{1}{m} \operatorname{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}, \tag{1.9}$$

$z \in U$, $\zeta \in \partial U$, and $m \geq n \geq 1$. Let $\Psi'_1[\Omega, q] = \Psi'[\Omega, q]$.

Theorem 1.4 (see [1], Theorem 2.3b, page 28). *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $j(z) \in \mathcal{H}[a, n]$ satisfies*

$$\psi\left(j(z), zj'(z), z^2 j''(z); z \right) \in \Omega, \tag{1.10}$$

then $j(z) \prec q(z)$.

Theorem 1.5 (see [19], Theorem 1, page 818). *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $j \in Q(a)$ and $\psi(j(z), zj'(z), z^2 j''(z); z)$ is univalent in U, then*

$$\Omega \subset \left\{ \psi\left(j(z), zj'(z), z^2 j''(z); z \right) : z \in U \right\} \tag{1.11}$$

implies $q(z) \prec j(z)$.

2. Subordination Results Associated with Generalized Srivastava-Attiya Operator

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mathcal{H}[0,p]$. The class of admissible functions $\Phi_J[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u,v,w;z) \notin \Omega \tag{2.1}$$

whenever

$$\begin{gathered} u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) - [p-(1+b)]q(\zeta)}{1+b} \quad \left(b \in \mathbb{C} \setminus \overline{Z}_0 = 0,-1,-2,\dots, p \in \mathbb{N}\right), \\ \operatorname{Re}\left\{ \frac{(1+b)^2 w - [p-(1+b)]^2 u}{(1+b)v + [p-(1+b)]u} + 2[p-(1+b)] \right\} \geq k \operatorname{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}, \end{gathered} \tag{2.2}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq p$.

Theorem 2.2. *Let $\phi \in \Phi_J[\Omega,q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z\right) : z \in U\right\} \subset \Omega, \tag{2.3}$$

then

$$J_{s+2,b}f(z) \prec q(z) \quad (z \in U). \tag{2.4}$$

Proof. The following relation obtained in [13]

$$zJ'_{s+1,b}f(z) = [p-(1+b)]\,J_{s+1,b}f(z) + (1+b)J_{s,b}f(z) \tag{2.5}$$

is equivalent to

$$J_{s,b}f(z) = \frac{zJ'_{s+1,b}f(z) - [p-(1+b)]\,J_{s+1,b}f(z)}{1+b}, \tag{2.6}$$

and hence

$$\mathrm{J}_{s+1,b}f(z) = \frac{zJ'_{s+2,b}f(z) - [p-(1+b)]\,J_{s+2,b}f(z)}{1+b}. \tag{2.7}$$

Define the analytic function j in U by

$$j(z) = J_{s+2,b}f(z), \tag{2.8}$$

and then we get

$$J_{s+1,b}f(z) = \frac{zj'(z) - [p-(1+b)]j(z)}{1+b}, \qquad J_{s,b}f(z) = \frac{z^2 j''(z) + (1-2[p-(1+b)])zj'(z) + [p-(1+b)]^2 j(z)}{(1+b)^2}. \tag{2.9}$$

Further, let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = c, \quad v = \frac{d - [p-(1+b)]c}{1+b}, \qquad w = \frac{e + (1-2[p-(1+b)])d + [p-(1+b)]^2 c}{(1+b)^2}. \tag{2.10}$$

Let

$$\psi(c,d,e;z) = \phi(u,v,w;z), \tag{2.11}$$

$$\phi(u,v,w;z) = \phi\left(c, \frac{d-[p-(1+b)]c}{1+b}, \frac{e+(1-2[p-(1+b)])d+[p-(1+b)]^2 c}{(1+b)^2}; z\right). \tag{2.12}$$

The proof will make use of Theorem 1.4. Using (2.8) and (2.9), from (2.12) we obtain

$$\psi\left(j(z), zj'(z), z^2 j''(z); z\right) = \phi\left(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z\right). \tag{2.13}$$

Hence (2.3) becomes

$$\psi\left(j(z), zj'(z), z^2 j''(z); z\right) \in \Omega. \tag{2.14}$$

Note that

$$\frac{e}{d} + 1 = \frac{(1+b)^2 w - [p-(1+b)]^2 u}{(1+b)v + [p-(1+b)]u} + 2[p-(1+b)], \tag{2.15}$$

and since the admissibility condition for $\phi \in \Phi_J[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2, hence $\psi \in \Psi_p[\Omega, q]$, and by Theorem 1.4,

$$j(z) \prec q(z), \tag{2.16}$$

or

$$J_{s+2,b}f(z) \prec q(z). \tag{2.17}$$

□

In the case $\phi(u,v,w;z) = v$, we have the following example.

Example 2.3. Let the class of admissible functions $\Phi_{Jv}[\Omega,q]$ consist of those functions ϕ : $\mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$v = \frac{k\zeta q'(\zeta) - [p-(1+b)]q(\zeta)}{1+b} \notin \Omega, \tag{2.18}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq p$ and $\phi \in \Phi_{Jv}[\Omega,q]$. If $f \in \mathcal{A}_p$ satisfies

$$J_{s+1,b}f(z) \subset \Omega, \tag{2.19}$$

then

$$J_{s+2,b}f(z) \prec q(z) \quad (z \in U). \tag{2.20}$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega \in h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class is written as $\Phi_J[h,q]$. The following result follows immediately from Theorem 2.2.

Theorem 2.4. *Let $\phi \in \Phi_J[\Omega,q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) \prec h(z), \tag{2.21}$$

then

$$J_{s+2,b}f(z) \prec q(z). \tag{2.22}$$

The next result occurs when the behavior of q on ∂U is not known.

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$, q be univalent in $U$and $q(0) = 0$. Let $\phi \in \Phi_J[\Omega,q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}_p$ and*

$$\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) \in \Omega, \tag{2.23}$$

then

$$J_{s+2,b}f(z) \prec q(z). \tag{2.24}$$

Proof. From Theorem 2.2, we see that $J_{s+2,b}f(z) \prec q_\rho(z)$ and the proof is complete. □

Theorem 2.6. *Let h and q be univalent in U, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_J[h, q_\rho]$, for some $\rho \in (0,1)$, or*

(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_J[h_\rho, q_\rho]$, for all $\rho_0 \in (0,1)$.*

If $f \in \mathcal{A}_p$ satisfies (2.21), then

$$J_{s+2,b}f(z) \prec q(z). \tag{2.25}$$

Proof. The proof is similar to the one in [1] and therefore is omitted. □

The next results give the best dominant of the differential subordination (2.21).

Theorem 2.7. *Let h be univalent in U. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{2.26}$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

(1) *$q \in Q_0$ and $\phi \in \Phi_J[h, q]$,*

(2) *q is univalent in U and $\phi \in \Phi_J[h, q_\rho]$, for some $\rho \in (0,1)$, or*

(3) *q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_J[h_\rho, q_\rho]$, for all $\rho_0 \in (0,1)$.*

If $f \in \mathcal{A}_p$ satisfies (2.21), then

$$J_{s+2,b}f(z) \prec q(z), \tag{2.27}$$

and q is the best dominant.

Proof. Following the same arguments in [1], we deduce that q is a dominant from Theorem 2.4 and Theorem 2.6. Since q satisfies (2.26), it is also a solution of (2.21) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

Definition 2.8. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{J,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega \tag{2.28}$$

whenever

$$\begin{gathered} u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) - bq(\zeta)}{1+b} \quad \left(b \in \mathbb{C} \setminus \overline{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}\right), \\ \operatorname{Re}\left\{\frac{(1+b)^2 w - b^2 u}{(1+b)v + bu} - 2b\right\} \geq k \operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \end{gathered} \tag{2.29}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq 1$.

Theorem 2.9. *Let $\phi \in \Phi_{J,1}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega, \tag{2.30}$$

then

$$\frac{J_{s+2,b}f(z)}{z^{p-1}} \prec q(z) \quad (z \in U). \tag{2.31}$$

Proof. Define the analytic function j in U by

$$j(z) = \frac{J_{s+2,b}f(z)}{z^{p-1}}. \tag{2.32}$$

Using the relations (2.5) and (2.32), we get

$$\begin{aligned} \frac{J_{s+1,b}f(z)}{z^{p-1}} &= \frac{zj'(z) - bj(z)}{1+b}, \\ \frac{J_{s,b}f(z)}{z^{p-1}} &= \frac{z^2 j''(z) + (2b+1)zj'(z) + b^2 j(z)}{(1+b)^2}. \end{aligned} \tag{2.33}$$

Further, let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{gathered} u = c, \quad v = \frac{d+bc}{1+b}, \\ w = \frac{e + (2b+1)d + b^2 c}{(1+b)^2}. \end{gathered} \tag{2.34}$$

Let

$$\psi(c, d, e; z) = \phi(u, v, w; z) = \phi\left(c, \frac{d+bc}{1+b}, \frac{e + (2b+1)d + b^2 c}{(1+b)^2}; z\right). \tag{2.35}$$

The proof will make use of Theorem 1.4. Using (2.32) and (2.33), from (2.35) we obtain

$$\psi\left(j(z), zj'(z), z^2 j''(z); z\right) = \phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right). \tag{2.36}$$

Hence (2.30) becomes

$$\psi\left(j(z), zj'(z), z^2 j''(z); z\right) \in \Omega. \tag{2.37}$$

Note that

$$\frac{e}{d}+1=\frac{(1+b)^2w-b^2u}{(1+b)v+bu}-2b, \tag{2.38}$$

and since the admissibility condition for $\phi \in \Phi_{J,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2, hence $\psi \in \Psi[\Omega,q]$, and by Theorem 1.4,

$$j(z) \prec q(z), \tag{2.39}$$

or

$$\frac{J_{s+2,b}f(z)}{z^{p-1}} \prec q(z). \tag{2.40}$$

□

In the case $\phi(u,v,w;z)=v-u$, we have the following example.

Example 2.10. Let the class of admissible functions $\Phi_{J_v,1}[\Omega,q]$ consist of those functions ϕ : $\mathbb{C}^3\times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$v-u=\frac{k\zeta q'(\zeta)-pq(\zeta)}{1+b}\notin\Omega, \tag{2.41}$$

$z\in U$, $\zeta\in\partial U\setminus E(q)$, and $k\geq p$ and $\phi\in\Phi_{J_v,1}[\Omega,q]$. If $f\in\mathcal{A}_p$ satisfies

$$\frac{J_{s+1,b}f(z)}{z^{p-1}}-\frac{J_{s,b}f(z)}{z^{p-1}}\subset\Omega \quad (z\in U), \tag{2.42}$$

then

$$\frac{J_{s+2,b}f(z)}{z^{p-1}} \prec q(z) \quad (z\in U). \tag{2.43}$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega\in h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class is written as $\Phi_{J,1}[h,q]$. The following result follows immediately from Theorem 2.9.

Theorem 2.11. *Let $\phi\in\Phi_{J,1}[\Omega,q]$. If $f\in\mathcal{A}_p$ satisfies*

$$\phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}},\frac{J_{s+1,b}f(z)}{z^{p-1}},\frac{J_{s,b}f(z)}{z^{p-1}};z\right)\prec h(z), \tag{2.44}$$

then

$$\frac{J_{s+2,b}f(z)}{z^{p-1}} \prec q(z). \tag{2.45}$$

Definition 2.12. Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{J,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega \tag{2.46}$$

whenever

$$\begin{gathered} u = q(\zeta), \qquad v = q(\zeta) + \frac{k\zeta q'(\zeta)}{(1+b)q(\zeta)} \quad \left(b \in \mathbb{C} \setminus \overline{Z}_0 = 0, -1, -2, \dots, p \in \mathbb{N}\right), \\ \operatorname{Re}\left\{ \frac{(w-u)(1+b)u}{v-u} + (1+b)(w-3u) \right\} \geq k \operatorname{Re}\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}, \end{gathered} \tag{2.47}$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, and $k \geq 1$.

Theorem 2.13. *Let $\phi \in \Phi_{J,2}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{ \phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z \right) : z \in U \right\} \subset \Omega, \tag{2.48}$$

then

$$\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \prec q(z) \quad (z \in U). \tag{2.49}$$

Proof. Define the analytic function j in U by

$$j(z) = \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}. \tag{2.50}$$

Differentiating (2.50) yields

$$\frac{zj'(z)}{j(z)} = \frac{zJ'_{s+2,b}f(z)}{J_{s+2,b}f(z)} - \frac{J'_{s+3,b}f(z)}{J_{s+3,b}f(z)}. \tag{2.51}$$

From the relation (2.5) we get

$$\frac{zJ'_{s+2,b}f(z)}{J_{s+2,b}f(z)} = \left[p - (1+b)\right] + (1+b)j + \frac{zj'(z)}{j(z)}, \tag{2.52}$$

and hence

$$\frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)} = j(z) + \frac{zj'(z)}{(1+b)j(z)}. \tag{2.53}$$

Further computations show that

$$\frac{J_{s,b}f(z)}{J_{s+1,b}f(z)} = j(z) + \frac{[2(1+b)j(z)+1]zj'(z)+z^2j''(z)}{(1+b)^2j(z)^2+(1+b)zj'(z)}. \tag{2.54}$$

Let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{gathered} u = c, \quad v = c + \frac{d}{(1+b)c}, \\ w = c + \frac{[2(b+1)c+1]d+e}{(1+b)^2c^2+(1+b)d}. \end{gathered} \tag{2.55}$$

Let

$$\psi(c,d,e;z) = \phi(u,v,w;z) = \phi\left(c, c+\frac{d}{(1+b)c}, c+\frac{[2(b+1)c+1]d+e}{(1+b)^2c^2+(1+b)d}; z\right). \tag{2.56}$$

The proof will make use of Theorem 1.4. Using (2.50), (2.53) and (2.54), from (2.56) we obtain

$$\psi\left(j(z), zj'(z), z^2j''(z); z\right) = \phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right). \tag{2.57}$$

Hence (2.48) becomes

$$\psi\left(j(z), zj'(z), z^2j''(z); z\right) \in \Omega. \tag{2.58}$$

Note that

$$\frac{e}{d} + 1 = \frac{(w-u)(1+b)u}{v-u} + (1+b)(w-3u), \tag{2.59}$$

and since the admissibility condition for $\phi \in \Phi_{J,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2, hence $\psi \in \Psi[\Omega, q]$ and by Theorem 1.4,

$$j(z) \prec q(z), \tag{2.60}$$

or

$$\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \prec q(z). \tag{2.61}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega \in h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class is written as $\Phi_{J,2}[h, q]$. The following result follows immediately from Theorem 2.13.

Theorem 2.14. *Let $\phi \in \Phi_{J,2}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) \prec h(z), \tag{2.62}$$

then

$$\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \prec q(z). \tag{2.63}$$

3. Superordination Results Associated with Generalized Srivastava-Attiya Operator

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_J[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \notin \Omega \tag{3.1}$$

whenever

$$\begin{gathered} u = q(z), \quad v = \frac{zq'(z) - m[p-(1+b)]q(z)}{m(1+b)} \quad \left(b \in \mathbb{C} \setminus \overline{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}\right), \\ \operatorname{Re}\left\{\frac{(1+b)^2 w - [p-(1+b)]^2 u}{(1+b)v + [p-(1+b)]u} + 2[p-(1+b)]\right\} \geq \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\}, \end{gathered} \tag{3.2}$$

$z \in U$, $\zeta \in \partial U$, and $m \geq p$.

Theorem 3.2. *Let $\phi \in \Phi'_J[\Omega, q]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f \in Q_0$ and*

$$\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) \tag{3.3}$$

is univalent in U, then

$$\Omega \subset \{\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) : z \in U\} \tag{3.4}$$

implies that

$$q(z) \prec J_{s+2,b}f(z). \tag{3.5}$$

Proof. From (2.13) and (3.4), we have

$$\Omega \subset \left\{\psi\left(j(z), zj'(z), z^2 j''(z); z\right) : z \in U\right\}. \tag{3.6}$$

From (2.10), we see that the admissibility condition for $\phi \in \Phi'_J[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'_p[\Omega, q]$, and by Theorem 1.5, $q(z) \prec j(z)$ or

$$q(z) \prec J_{s+2,b} f(z). \tag{3.7}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega \in h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class is written as $\Phi'_J[h, q]$. The next result follows immediately from Theorem 3.2.

Theorem 3.3. *Let h be analytic in U and $\phi \in \Phi'_J[\Omega, q]$. If $f \in \mathcal{A}_p$, $J_{s+2,b} f(z) \in Q_0$ and*

$$\phi(J_{s+2,b} f(z), J_{s+1,b} f(z), J_{s,b} f(z); z) \tag{3.8}$$

is univalent in U, then

$$h(z) \prec \phi(J_{s+2,b} f(z), J_{s+1,b} f(z), J_{s,b} f(z); z), \tag{3.9}$$

and then

$$q(z) \prec J_{s+2,b} f(z). \tag{3.10}$$

Theorems 3.2 and 3.3 can only be used to obtain subordinants for differential superordination of the form (3.4) and (3.9). The following theorems prove the existence of the best subordinant of (3.9) for certain ϕ.

Theorem 3.4. *Let h be analytic in U and $\phi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\Big(q(z), zq'(z), z^2 q''(z); z\Big) = h(z) \tag{3.11}$$

has a solution $q \in Q_0$. If $\phi \in \Phi'_J[\Omega, q]$, $f \in \mathcal{A}_p$, $J_{s+2,b} f(z) \in Q_0$, and

$$\phi(J_{s+2,b} f(z), J_{s+1,b} f(z), J_{s,b} f(z); z) \tag{3.12}$$

is univalent in U, then

$$h(z) \prec \phi(J_{s+2,b} f(z), J_{s+1,b} f(z), J_{s,b} f(z); z) \tag{3.13}$$

implies that

$$q(z) \prec J_{s+2,b} f(z), \tag{3.14}$$

and $q(z)$ is the best subordinant.

Proof. The result can be obtained by similar proof of Theorem 2.7. □

The next result, the sandwich-type theorem follows from Theorems 2.4 and 3.3.

Corollary 3.5. *Let h_1 and q_1 be analytic in U, and let h_2 be univalent function in U, $q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_J[h_2, q_2] \cap \Phi'_J[h_1, q_1]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f(z) \in \mathcal{H}[0,p] \cap Q_0$, and*

$$\phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) \tag{3.15}$$

is univalent in U, then

$$h_1(z) \prec \phi(J_{s+2,b}f(z), J_{s+1,b}f(z), J_{s,b}f(z); z) \prec h_2(z) \tag{3.16}$$

implies that

$$q_1(z) \prec J_{s+2,b}f(z) \prec q_2(z). \tag{3.17}$$

Definition 3.6. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $zq\prime(z) \neq 0$. The class of admissible functions $\Phi'_{J,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega \tag{3.18}$$

whenever

$$\begin{gathered} u = q(z), \quad v = \frac{zq'(z) - mbq(z)}{m(1+b)} \quad \left(b \in \mathbb{C} \setminus \overline{\mathbb{Z}}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}\right), \\ \operatorname{Re}\left\{\frac{(1+b)^2 w - b^2 u}{(1+b)v + bu} - 2b\right\} \geq \frac{1}{m} \operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\}, \end{gathered} \tag{3.19}$$

$z \in U$, $\zeta \in \partial U$, and $m \geq 1$.

The following result is associated with Theorem 2.9.

Theorem 3.7. *Let $\phi \in \Phi'_{H,1}[\Omega, q]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f(z)/z^{p-1} \in Q_0$, and*

$$\phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) \tag{3.20}$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) : z \in U\right\} \tag{3.21}$$

implies that

$$q(z) \prec \frac{J_{s+2,b}f(z)}{z^{p-1}}. \tag{3.22}$$

Proof. From (2.36) and (3.21), we have

$$\Omega \subset \left\{\phi\left(j(z), zj'(z), z^2 j''(z); z\right) : z \in U\right\}. \tag{3.23}$$

From (2.34), we see that the admissibility condition for $\phi \in \Phi'_{J,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.5, $q(z) \prec j(z)$ or

$$q(z) \prec \frac{J_{s+2,b}f(z)}{z^{p-1}}. \tag{3.24}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega \in h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class is written as $\Phi'_{J,1}[h, q]$. The next result follows immediately from Theorem 3.7.

Theorem 3.8. *Let $q \in \mathcal{H}_0$, and let h be analytic on U, and let $\phi \in \Phi'_{J,1}[\Omega, q]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f(z)/z^{p-1} \in Q_0$, and*

$$\phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) \tag{3.25}$$

is univalent in U, then

$$h(z) \prec \phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) \tag{3.26}$$

implies that

$$q(z) \prec \frac{J_{s+2,b}f(z)}{z^{p-1}}. \tag{3.27}$$

Combining Theorems 2.11 and 3.8, we obtain the following sandwich-type theorem.

Corollary 3.9. *Let h_1 and q_1 be analytic in U, let h_2 be univalent function in U, $q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$, and $\phi \in \Phi_{J,1}[h_2, q_2] \cap \Phi'_{J,1}[h_1, q_1]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f(z)/z^{p-1} \in \mathcal{H}_0 \cap Q_0$, and*

$$\phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) \tag{3.28}$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{J_{s+2,b}f(z)}{z^{p-1}}, \frac{J_{s+1,b}f(z)}{z^{p-1}}, \frac{J_{s,b}f(z)}{z^{p-1}}; z\right) \prec h_2(z) \tag{3.29}$$

implies that

$$q_1(z) \prec \frac{J_{s+2,b}f(z)}{z^{p-1}} \prec q_2(z). \tag{3.30}$$

Definition 3.10. Let Ω be a set in $\mathbb{C}$ and $q(z) \neq 0$, $zq'(z) \neq 0$, and $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{J,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \notin \Omega \tag{3.31}$$

whenever

$$\begin{gathered} u = q(z), \qquad v = q(z) + \frac{zq'(z)}{m(1+b)q(z)} \quad \left(b \in \mathbb{C} \setminus \overline{Z}_0 = 0, -1, -2, \ldots, p \in \mathbb{N}\right), \\ \operatorname{Re}\left\{\frac{(w-u)(1+b)u}{v-u} + (1+b)(w-3u)\right\} \geq \frac{1}{m} \operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\}, \end{gathered} \tag{3.32}$$

$z \in U$, $\zeta \in \partial U$, and $m \geq 1$.

The following result is associated with Theorem 2.13.

Theorem 3.11. *Let $\phi \in \Phi'_{J,2}[\Omega, q]$. If $f \in \mathcal{A}_p$, $J_{s+2,b}f(z)/J_{s+3,b}f(z) \in Q_1$, and*

$$\phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) \tag{3.33}$$

is univalent in U, then

$$\Omega \subset \left\{\phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) : z \in U\right\} \tag{3.34}$$

implies that

$$q(z) \prec \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}. \tag{3.35}$$

Proof. From (2.57) and (3.34), we have

$$\Omega \subset \left\{\phi\left(j(z), zj'(z), z^2 j''(z); z\right) : z \in U\right\}. \tag{3.36}$$

From (2.55), we see that the admissibility condition for $\phi \in \Phi'_{J,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.5, $q(z) \prec j(z)$ or

$$q(z) \prec \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}. \tag{3.37}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega \in h(U)$ for some conformal mapping $h(z)$ of U onto Ω and the class is written as $\Phi'_{J,2}[h, q]$. The next result follows immediately from Theorem 3.11 as in the previous section.

Theorem 3.12. *Let $q \in \mathscr{H}$, let h be analytic in U, and let $\phi \in \Phi'_{J,2}[\Omega, q]$. If $f \in \mathscr{A}_p$, $J_{s+2,b}f(z)/J_{s+3,b}f(z) \in Q_1$ and*

$$\phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) \tag{3.38}$$

is univalent in U, then

$$h(z) \prec \phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) \tag{3.39}$$

implies that

$$q(z) \prec \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}. \tag{3.40}$$

Combining Theorems 2.14 and 3.12, we obtain the following sandwich-type theorem.

Corollary 3.13. *Let h_1 and q_1 be analytic in U, let h_2 be univalent function in U, $q_2 \in Q_0$ with $q_1(0) = q_2(0) = 0$, and $\phi \in \Phi_{J,2}[h_2, q_2] \cap \Phi'_{J,2}[h_1, q_1]$. If $f \in \mathscr{A}_p$, $J_{s+2,b}f(z)/J_{s+3,b}f(z) \in \mathscr{H} \cap Q_1$, and*

$$\phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) \tag{3.41}$$

is univalent in U, then

$$h_1(z) \prec \phi\left(\frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)}, \frac{J_{s+1,b}f(z)}{J_{s+2,b}f(z)}, \frac{J_{s,b}f(z)}{J_{s+1,b}f(z)}; z\right) \prec h_2(z) \tag{3.42}$$

implies that

$$q_1(z) \prec \frac{J_{s+2,b}f(z)}{J_{s+3,b}f(z)} \prec q(z). \tag{3.43}$$

Other work related to certain operators concerning the subordination and superordination can be found in [20–25].

Acknowledgment

The work presented here was partially supported by UKM-ST-FRGS-0244-2010.

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, NY, USA, 2000.

[2] R. Aghalary, R. M. Ali, S. B. Joshi, and V. Ravichandran, "Inequalities for analytic functions defined by certain linear operators," *International Journal of Mathematical Sciences*, vol. 4, no. 2, pp. 267–274, 2005.

[3] R. M. Ali, R. Chandrashekar, S. K. Lee, V. Ravichandran, and A. Swaminathan, "Differential sandwich theorem for multivalent analytic fucntions associated with the Dziok-Srivastava operator," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 27, no. 3, 2011.

[4] R. M. Ali, R. Chandrashekar, S. K. Lee, V. Ravichandran, and A. Swaminathan, "Differential sandwich theorem for multivalent meromorphic fucntions associated with the Liu-Srivastava operator," *Kyungpook Mathematical Journal*, vol. 51, no. 2, pp. 217–232, 2011.

[5] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Differential subordination and superordination of analytic functions defined by the multiplier transformation," *Mathematical Inequalities & Applications*, vol. 12, no. 1, pp. 123–139, 2009.

[6] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Subordination and superordination on Schwarzian derivatives," *Journal of Inequalities and Applications*, Article ID 712328, 18 pages, 2008.

[7] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "On subordination and superordination of the multiplier transformation for meromorphic functions," *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, vol. 33, no. 2, pp. 311–324, 2010.

[8] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions," *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, vol. 31, no. 2, pp. 193–207, 2008.

[9] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Differential subordination and superordination of analytic functions defined by the Dziok-Srivastava linear operator," *Journal of the Franklin Institute*, vol. 347, no. 9, pp. 1762–1781, 2010.

[10] M. K. Aouf, H. M. Hossen, and A. Y. Lashin, "An application of certain integral operators," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 475–481, 2000.

[11] Y. C. Kim and H. M. Srivastava, "Inequalities involving certain families of integral and convolution operators," *Mathematical Inequalities & Applications*, vol. 7, no. 2, pp. 227–234, 2004.

[12] J. L. Liu, "Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator," *Integral Transforms and Special Functions*, vol. 19, no. 11-12, pp. 893–901, 2008.

[13] H. M. Srivastava and A. A. Attiya, "An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination," *Integral Transforms and Special Functions*, vol. 18, no. 3-4, pp. 207–216, 2007.

[14] J. W. Alexander, "Functions which map the interior of the unit circle upon simple regions," *Annals of Mathematics. Second Series*, vol. 17, no. 1, pp. 12–22, 1915.

[15] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.

[16] R. J. Libera, "Some classes of regular univalent functions," *Proceedings of the American Mathematical Society*, vol. 16, pp. 755–758, 1965.

[17] I. B. Jung, Y. C. Kim, and H. M. Srivastava, "The Hardy space of analytic functions associated with certain one-parameter families of integral operators," *Journal of Mathematical Analysis and Applications*, vol. 176, no. 1, pp. 138–147, 1993.

[18] J. Patel and P. Sahoo, "Some applications of differential subordination to certain one-parameter families of integral operators," *Indian Journal of Pure and Applied Mathematics*, vol. 35, no. 10, pp. 1167–1177, 2004.

[19] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables. Theory and Application*, vol. 48, no. 10, pp. 815–826, 2003.

[20] O. Al-Refai and M. Darus, "An extension to the Owa-Srivastava fractional operator with applications to parabolic starlike and uniformly convex functions," *International Journal of Differential Equations*, Article ID 597292, 18 pages, 2009.

[21] O. Al-Refai and M. Darus, "Main differential sandwich theorem with some applications," *Lobachevskii Journal of Mathematics*, vol. 30, no. 1, pp. 1–11, 2009.

[22] R. G. Xiang, Z. G. Wang, and M. Darus, "A family of integral operators preserving subordination and superordination," *Bulletin of the Malaysian Mathematical Sciences Society. Second Series*, vol. 33, no. 1, pp. 121–131, 2010.

[23] R. W. Ibrahim and M. Darus, "Subordination and superordination for functions based on Dziok-Srivastava linear operator," *Bulletin of Mathematical Analysis and Applications*, vol. 2, no. 3, pp. 15–26, 2010.

[24] M. Darus, I. Faisal, and M. A. M. Nasr, "Differential subordination results for some classes of the family zeta associate with linear operator," *Acta Universitatis Sapientiae-Mathematica*, vol. 2, no. 2, pp. 184–194, 2010.

[25] R. Ibrahim and M. Darus, "Differential subordination results for new classes of the family $\in (\Phi, \Upsilon)$," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 1, article 8, p. 9, 2009.