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Research Article

Positive Solution to a Fractional Boundary Value Problem

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A fractional boundary value problem is considered. By means of Banach contraction principle, Leray-Schauder nonlinear alternative, properties of the Green function, and Guo-Krasnosel'skii fixed point theorem on cone, some results on the existence, uniqueness, and positivity of solutions are obtained.

1. Introduction

Fractional differential equations are a natural generalization of ordinary differential equations. In the last few decades many authors pointed out that differential equations of fractional order are suitable for the metallization of various physical phenomena and that they have numerous applications in viscoelasticity, electrochemistry, control and electromagnetic, and so forth, see [1–4].

This work is devoted to the study of the following fractional boundary value problem (P1):

$$ {}^{c}D_{0^+}^{q} u(t) = f\left(t, u(t), {}^{c}D_{0^+}^{\sigma} u(t)\right), \quad 0 < t < 1, \tag{1.1}$$

$$ u(0) = u''(0) = 0, \qquad u'(1) = \alpha u''(1), \tag{1.2}$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $2 < q < 3$, $1 < \sigma < 2$ and ${}^{c}D_{0^+}^{q}$ denotes the Caputo's fractional derivative. Our results allow the function f to depend on the fractional derivative ${}^cD_{0^+}^{\sigma}u(t)$ which leads to extra difficulties. No contributions exist, as far as we know, concerning the existence of positive solutions of the fractional differential equation (1.1) jointly with the nonlocal condition (1.2).

Our mean objective is to investigate the existence, uniqueness, and existence of positive solutions for the fractional boundary value problem (P1), by using Banach contraction principle, Leray-Schauder nonlinear alternative, properties of the Green function and Guo-Krasnosel'skii fixed point theorem on cone.

The research in this area has grown significantly and many papers appeared on this subject, using techniques of nonlinear analysis, see [5–14].

In [6], El-Shahed considered the following nonlinear fractional boundary value problem

$$\begin{gathered} D_{0^+}^{q}u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \end{gathered} \tag{1.3}$$

where $2 < q \leq 3$ and $D_{0^+}^{q}$ is the Riemann-Liouville fractional derivative. Using the Krasnoselskii's fixed-point theorem on cone, he proved the existence and nonexistence of positive solutions for the above fractional boundary value problem.

Liang and Zhang in [9] studied the existence and uniqueness of positive solutions by the properties of the Green function, the lower and upper solution method and fixed point theorem for the fractional boundary value problem

$$\begin{gathered} D_{0^+}^{q}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\zeta_i), \end{gathered} \tag{1.4}$$

where $2 < q \leq 3$ and $D_{0^+}^{q}$ is the Riemann-Liouville fractional derivative.

In [5] Bai and Lü investigated the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem:

$$\begin{gathered} D_{0^+}^{q}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.5}$$

where $1 < q \leq 1$ and $D_{0^+}^{q}$ is the Riemann-Liouville fractional derivative. Applying fixed-point theorems on cone, they prove some existence and multiplicity results of positive solutions.

This paper is organized as follows, in the Section 2 we cite some definitions and lemmas needed in our proofs. Section 3 treats the existence and uniqueness of solution by using Banach contraction principle, Leray Schauder nonlinear alternative. Section 4 is devoted to prove the existence of positive solutions with the help of Guo-Krasnoselskii Theorem, then we give some examples illustrating the previous results.

2. Preliminaries and Lemmas

In this section, we present some lemmas and definitions from fractional calculus theory which will be needed later.

Definition 2.1. If $g \in C([a,b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds. \tag{2.1}$$

Definition 2.2. Let $\alpha \geq 0$, $n = [\alpha] + 1$. If $f \in AC^n[a,b]$ then the Caputo fractional derivative of order α of f defined by

$${}^cD_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

exists almost everywhere on $[a,b]$ ($[\alpha]$ is the entire part of α).

Lemma 2.3 (see [15]). *Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold: ${}^cD_{0^+}^{\alpha} t^{\beta-1} = (\Gamma(\beta)/\Gamma(\beta-\alpha))t^{\beta-1}$, $\beta > n$ and ${}^cD_{0^+}^{\alpha} t^k = 0$, $k = 0, 1, 2, \ldots, n-1$.*

Lemma 2.4 (see [15]). *For $\alpha > 0$, $g(t) \in C(0,1)$, the homogenous fractional differential equation*

$${}^cD_{a^+}^{\alpha} g(t) = 0 \tag{2.3}$$

has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, \tag{2.4}$$

where, $c_i \in R$, $i = 0, \ldots, n$, and $n = [\alpha] + 1$.

Denote by $L^1([0,1], \mathbb{R})$ the Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|y\|_{L^1} = \int_0^1 |y(t)| dt$.

The following Lemmas gives some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative.

Lemma 2.5 (see [16]). *Let $p, q \geq 0$, $f \in L_1[a,b]$. Then $I_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^{q} I_{0^+}^{p} f(t)$ and ${}^cD_{0^+}^{q} I_{0^+}^{q} f(t) = f(t)$, for all $t \in a, b]$.*

Lemma 2.6 (see [15]). *Let $\beta > \alpha > 0$. Then the formula ${}^cD_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\beta-\alpha} f(t)$, holds almost everywhere on $t \in a, b]$, for $f \in L_1[a,b]$ and it is valid at any point $x \in [a,b]$ if $f \in C[a,b]$.*

Now we start by solving an auxiliary problem.

Lemma 2.7. *Let $2 < q < 3, 1 < \sigma < 2$ and $y \in C[0,1]$. The unique solution of the fractional boundary value problem*

$$\begin{gathered} {}^cD_{0^+}^q u(t) = y(t), \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u'(1) = \alpha u''(1) \end{gathered} \tag{2.5}$$

is given by

$$u(t) = \frac{1}{\Gamma(q-2)} \int_0^1 G(t,s) y(s) ds, \tag{2.6}$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{q-1}}{q(q-1)} + \dfrac{\alpha t}{(1-s)^{3-q}} - \dfrac{t(1-s)^{q-2}}{q-1}, & s \leq t \\ \dfrac{\alpha t}{(1-s)^{3-q}} - \dfrac{t(1-s)^{q-2}}{q-1}, & t \leq s. \end{cases} \tag{2.7}$$

Proof. Applying Lemmas 2.4 and 2.5 to (2.5) we get

$$u(t) = I_{0^+}^q y(t) + c_1 + c_2 t + c_3 t^2. \tag{2.8}$$

Differentiating both sides of (2.8) and using Lemma 2.6 it yields

$$\begin{gathered} u'(t) = I_{0^+}^{q-1} y(t) + c_2 + c_3 t, \\ u''(t) = I_{0^+}^{q-2} y(t) + c_3. \end{gathered} \tag{2.9}$$

The first condition in (2.5) implies $c_1 = c_3 = 0$, the second one gives $c_2 = \alpha I_{0^+}^{q-2} y(1) - I_{0^+}^{q-1} y(1)$. Substituting c_2 by its value in (2.8), we obtain

$$u(t) = I_{0^+}^q y(t) + t\left(\alpha I_{0^+}^{q-2} y(1) - I_{0^+}^{q-1} y(1)\right) \tag{2.10}$$

that can be written as

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(q-2)} \int_0^t \left[\frac{(t-s)^{q-1}}{q(q-1)} + \frac{\alpha t}{(1-s)^{3-q}} - \frac{t(1-s)^{q-2}}{q-1} \right] y(s) ds \\ & + \frac{1}{\Gamma(q-2)} \int_t^1 \left[\frac{\alpha t}{(1-s)^{3-q}} - \frac{t(1-s)^{q-2}}{q-1} \right] y(s) ds \end{aligned} \tag{2.11}$$

that is equivalent to

$$u(t) = \frac{1}{\Gamma(q-2)} \int_0^1 G(t,s) y(s) ds, \tag{2.12}$$

where G is defined by (2.7). The proof is complete. $\square$

3. Existence and Uniqueness Results

In this section we prove the existence and uniqueness of solutions in the Banach space E of all functions $u \in C^2[0,1]$ into $\mathbb{R}$, with the norm $\|u\| = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|{}^cD_{0^+}^{\sigma} u(t)|$. We know that ${}^cD_{0^+}^{\sigma} u \in C[0,1]$, $1 < \sigma < 2$, see [15]. Denote by $E^+ = \{u \in E,\ u(t) \geq 0,\ t \in [0,1]\}$. Throughout this section, we suppose that $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Define the integral operator $T : E \to E$ by

$$Tu(t) = \frac{1}{\Gamma(q-2)} \int_0^1 G(t,s) f\left(s, u(s),\ {}^cD_{0^+}^{\sigma} u(s)\right) ds. \tag{3.1}$$

Lemma 3.1. *The function $u \in E$ is solution of the fractional boundary value problem (P1) if and only if $Tu(t) = u(t)$, for all $t \in [0,1]$.*

Proof. Let u be solution of (P1) and $v(t) = \int_0^1 G(t,s) f(s, u(s), {}^c D_{0^+}^{\sigma} u(s)) ds$. In view of (2.10) we have

$$\begin{aligned} v(t) &= I_{0^+}^{q} f\left(t, u(t), {}^c D_{0^+}^{\sigma} u(t)\right) \\ &\quad + \alpha t I_{0^+}^{q-2} f\left(1, u(1), {}^c D_{0^+}^{\sigma} u(1)\right) - t I_{0^+}^{q-1} f\left(1, u(1), {}^c D_{0^+}^{\sigma} u(1)\right). \end{aligned} \tag{3.2}$$

With the help of Lemma 2.6 we obtain

$$\begin{aligned} {}^cD_{0^+}^{q} v(t) &= {}^cD_{0^+}^{q} I_{0^+}^{q} f\left(t, u(t), {}^c D_{0^+}^{\sigma} u(t)\right) \\ &\quad + t\alpha I_{0^+}^{q-2} f\left(1, u(1), {}^c D_{0^+}^{\sigma} u(1)\right) - t I_{0^+}^{q-1} f\left(1, u(1), {}^c D_{0^+}^{\sigma} u(1)\right) \\ &= f\left(t, u(t), {}^c D_{0^+}^{\sigma} u(t)\right). \end{aligned} \tag{3.3}$$

It is clear that v satisfies conditions (1.2), then it is a solution for the problem (P1). The proof is complete. $\square$

Theorem 3.2. *Assume that there exist nonnegative functions $g, h \in L^1([0,1], \mathbb{R}_+)$ such that for all $x, y \in \mathbb{R}$ and $t \in [0,1]$, one has*

$$\left| f(t, x, \overline{x}) - f(t, y, \overline{y}) \right| \leq g(t)\left|x - y\right| + h(t)\left|\overline{x} - \overline{y}\right|, \tag{3.4}$$

$$C_g + C_h < 1, \qquad A_g + A_h < (2-\sigma)\Gamma(2-\sigma), \tag{3.5}$$

where

$$C_g = \left\| I_{0^+}^{q-1} g \right\|_{L^1} + |\alpha| I_{0^+}^{q-2} g(1) + I_{0^+}^{q-1} g(1), \qquad A_g = 2 I_{0^+}^{q-1} g(1) + |\alpha| I_{0^+}^{q-2} g(1),$$
$$C_h = \left\| I_{0^+}^{q-1} h \right\|_{L^1} + |\alpha| I_{0^+}^{q-2} h(1) + I_{0^+}^{q-1} h(1), \qquad A_h = 2 I_{0^+}^{q-1} h(1) + |\alpha| I_{0^+}^{q-2} h(1). \tag{3.6}$$

Then the fractional boundary value problem (P1) has a unique solution u in E.

To prove Theorem 3.2, we use the following property of Riemann-Liouville fractional integrals.

Lemma 3.3. *Let $q > 0$, $f \in L_1([a,b],\mathbb{R}_+)$. Then, for all $t \in a,b]$ we have*

$$I_{0^+}^{q+1} f(t) \leq \left\| I_{0^+}^{q} f \right\|_{L^1}. \tag{3.7}$$

Proof. Let $f \in L_1([a,b],\mathbb{R}_+)$, then

$$\begin{aligned} \left\| I_{0^+}^{q} f \right\|_{L^1} &= \int_0^1 I_{0^+}^{q} f(r) dr \geq \frac{1}{\Gamma(q)} \int_a^t \int_a^r \frac{f(s)}{(r-s)^{1-q}} ds\, dr \\ &= \frac{1}{\Gamma(q)} \int_a^t \left(\int_s^t \frac{f(s)}{(r-s)^{1-q}} dr \right) ds = I_{0^+}^{q+1} f(t). \end{aligned} \tag{3.8}$$

□

Now we prove Theorem 3.2.

Proof. We transform the fractional boundary value problem to a fixed point problem. By Lemma 3.1, the fractional boundary value problem (P1) has a solution if and only if the operator T has a fixed point in E. Now we will prove that T is a contraction. Let $u, v \in E$, in view of (2.10) we get

$$\begin{aligned} Tu(t) - Tv(t) &= \frac{1}{\Gamma(q-2)} \int_0^1 G(t,s) \left(f\left(s, u(s), {}^c D_{0^+}^{\sigma} u(s)\right) - f\left(s, v(s), {}^c D_{0^+}^{\sigma} v(s)\right) \right) ds \\ &= I_{0^+}^{q} \left(f\left(t, u(t), {}^c D_{0^+}^{\sigma} u(t)\right) - f\left(s, v(t), {}^c D_{0^+}^{\sigma} v(t)\right) \right) \\ &\quad + t\alpha \Big[I_{0^+}^{q-2} \left(f\left(1, u(1), {}^c D_{0^+}^{\sigma} u(1)\right) - f\left(1, v(1), {}^c D_{0^+}^{\sigma} v(1)\right) \right) \\ &\qquad - t I_{0^+}^{q-1} \left(f\left(1, u(1), {}^c D_{0^+}^{\sigma} u(1)\right) - f\left(1, v(1), {}^c D_{0^+}^{\sigma} v(1)\right) \right) \Big], \end{aligned} \tag{3.9}$$

with the help of (3.4) we obtain

$$\begin{aligned} |Tu(t) - Tv(t)| &\leq \max |u(t) - v(t)| \left(I_{0^+}^{q} g(t) + |\alpha| I_{0^+}^{q-2} g(1) + I_{0^+}^{q-1} g(1) \right) \\ &\quad + \max \left| {}^c D_{0^+}^{\sigma} u(t) - {}^c D_{0^+}^{\sigma} v(t) \right| \left(I_{0^+}^{q} h(t) + |\alpha| I_{0^+}^{q-2} h(1) + I_{0^+}^{q-1} h(1) \right). \end{aligned} \tag{3.10}$$

Lemma 3.3 implies

$$|Tu(t)-Tv(t)| \le \|u-v\|\Big[\Big\|I_{0^+}^{q-1}g\Big\|_{L^1}+|\alpha|I_{0^+}^{q-2}g(1)+I_{0^+}^{q-1}g(1) \\ +\Big\|I_{0^+}^{q-1}h\Big\|_{L^1}+|\alpha|I_{0^+}^{q-2}h(1)+I_{0^+}^{q-1}h(1)\Big]=\|u-v\|\left(C_g+C_h\right). \tag{3.11}$$

In view of (3.5) it yields

$$|Tu-Tv|<\|u-v\|. \tag{3.12}$$

On the other hand we have

$${}^cD_{0^+}^{\sigma}Tu-{}^cD_{0^+}^{\sigma}Tv=\frac{1}{\Gamma(2-\sigma)}\int_0^t\frac{(Tu)'(s)-(Tv)'(s)}{(t-s)^{\sigma-1}}ds, \tag{3.13}$$

where

$$(Tu)'(t)=\frac{1}{\Gamma(q-2)}\int_0^1 G_1(t,s)f\left(s,u(s),{}^cD_{0^+}^{\sigma}u(s)\right)ds,$$
$$G_1(t,s)=\frac{\partial G(t,s)}{\partial t}=\begin{cases}\dfrac{(t-s)^{q-2}}{q}+\dfrac{\alpha}{(1-s)^{3-q}}-\dfrac{(1-s)^{q-2}}{q-1}, & 0\le s\le t\le 1\\ \dfrac{\alpha}{(1-s)^{3-q}}-\dfrac{(1-s)^{q-2}}{q-1}, & 0\le t\le s\le 1.\end{cases} \tag{3.14}$$

Therefore

$${}^cD_{0^+}^{\sigma}Tu-{}^cD_{0^+}^{\sigma}Tv=\frac{1}{\Gamma(q-2)\Gamma(2-\sigma)}\int_0^t\int_0^1(t-s)^{-\sigma+1}G_1(s,r) \\ \times\left(f\left(r,u(r),{}^cD_{0^+}^{\sigma}u(r)\right)-f\left(r,v(r),{}^cD_{0^+}^{\sigma}v(r)\right)\right)dr\,ds. \tag{3.15}$$

Applying hypothesis (3.4) we get

$$\left|{}^cD_{0^+}^{\sigma}Tu-{}^cD_{0^+}^{\sigma}Tv\right|\le\frac{\max|u-v|}{\Gamma(q-2)\Gamma(2-\sigma)}\int_0^t\int_0^1(t-s)^{-\sigma+1}|G_1(s,r)|g(r)dr\,ds \\ +\frac{\max\left|{}^cD_{0^+}^{\sigma}u-{}^cD_{0^+}^{\sigma}v\right|}{\Gamma(q-2)\Gamma(2-\sigma)}\int_0^t\int_0^1(t-s)^{-\sigma+1}|G_1(s,r)|h(r)dr\,ds. \tag{3.16}$$

Let us estimate the term $\int_0^1(\partial G(s,r)/\partial s)g(r)dr$. We have

$$\int_0^1|G_1(s,r)|g(r)dr\le\Gamma(q-2)\left(\frac{2(q-2)I_{0^+}^{q-1}g(1)}{q-1}+|\alpha|I_{0^+}^{q-2}g(1)\right) \\ =\Gamma(q-2)\left(2I_{0^+}^{q-1}g(1)+|\alpha|I_{0^+}^{q-2}g(1)\right)=\Gamma(q-2)A_g. \tag{3.17}$$

Consequently (3.16) becomes

$$\left|{}^{c}D_{0^+}^{\sigma}Tu - {}^{c}D_{0^+}^{\sigma}Tv\right| \leq \|u-v\| \frac{1}{(2-\sigma)\Gamma(2-\sigma)}\left(A_g + A_h\right). \tag{3.18}$$

With the help of hypothesis (3.5) it yields

$$\left|{}^{c}D_{0^+}^{\sigma}Tu - {}^{c}D_{0^+}^{\sigma}Tv\right| \leq \|u-v\|. \tag{3.19}$$

Taking into account (3.12)–(3.19) we obtain

$$\|Tu - Tv\| < \|u-v\|, \tag{3.20}$$

from here, the contraction principle ensures the uniqueness of solution for the fractional boundary value problem (P1). This finishes the proof. □

Now we give an existence result for the fractional boundary value problem (P1).

Theorem 3.4. *Assume that $f(t,0,0) \neq 0$ and there exist nonnegative functions $k, h, g \in L^1([0,1], \mathbb{R}_+)$, $\phi, \varphi \in C(\mathbb{R}_+, \mathbb{R}_+^*)$ nondecreasing on $\mathbb{R}_+$ and $r > 0$, such that*

$$\left|f(t,x,\overline{x})\right| \leq k(t)\varphi(|x|) + h(t)\phi(|\overline{x}|) + g(t), \quad \text{a.e. } (t,x) \in [0,1]\times\mathbb{R}. \tag{3.21}$$

$$(\varphi(r) + \phi(r) + 1)\left(\frac{C_1}{\Gamma(q-2)} + \frac{C_2}{(2-\sigma)\Gamma(2-\sigma)}\right) < r, \tag{3.22}$$

where $C_1 = \max\{C_k, C_h, C_g\}$, $C_2 = \max\{A_k, A_h, A_g\}$, C_h and C_g are defined as in Theorem 3.2 and

$$\begin{aligned} C_k &= \left\|I_{0^+}^{q-1}k\right\|_{L^1} + |\alpha| I_{0^+}^{q-2}k(1) + I_{0^+}^{q-1}k(1), \\ A_k &= 2I_{0^+}^{q-1}k(1) + |\alpha| I_{0^+}^{q-2}k(1). \end{aligned} \tag{3.23}$$

Then the fractional boundary value problem (P1) has at least one nontrivial solution $u^* \in E$.

To prove this Theorem, we apply Leray-Schauder nonlinear alternative.

Lemma 3.5 (see [17]). *Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Proof. First let us prove that T is completely continuous. It is clear that T is continuous since f and G are continuous. Let $B_r = \{u \in E, \|u\| \leq r\}$ be a bounded subset in E. We shall prove that $T(B_r)$ is relatively compact.

(i) For $u \in B_r$ and using (3.21) we get

$$|Tu(t)| \leq \frac{1}{\Gamma(q-2)} \int_0^1 |G(t,s)| \left[k(s)\psi(|u(s)|) + h(s)\phi\left(\left|{}^cD_{0^+}^{\sigma}u(s)\right|\right) + g(s)\right] ds. \tag{3.24}$$

Since ψ and ϕ are nondecreasing then (3.24) implies

$$\begin{aligned} |Tu(t)| &\leq \frac{1}{\Gamma(q-2)} \int_0^1 |G(t,s)| \left[k(s)\psi(\|u\|) + h(s)\phi(\|u\|) + g(s)\right] ds \\ &\leq \frac{1}{\Gamma(q-2)} \int_0^1 |G(t,s)| \left[k(s)\psi(r) + h(s)\phi(r) + g(s)\right] ds, \end{aligned} \tag{3.25}$$

using similar techniques as to get (3.12) it yields

$$\begin{aligned} |Tu(t)| \leq \frac{1}{\Gamma(q-2)} &\Big[\psi(r)\Big(\left\|I_{0^+}^{q-1}k\right\|_{L^1} + |\alpha| I_{0^+}^{q-2}k(1) + I_{0^+}^{q-1}k(1)\Big) \\ &\quad + \phi(r)\Big(\left\|I_{0^+}^{q-1}h\right\|_{L^1} + |\alpha| I_{0^+}^{q-2}h(1) + I_{0^+}^{q-1}h(1)\Big) \\ &\quad + \Big(\left\|I_{0^+}^{q-1}g\right\|_{L^1} + |\alpha| I_{0^+}^{q-2}g(1) + I_{0^+}^{q-1}g(1)\Big)\Big] \\ = \frac{1}{\Gamma(q-2)} &\left(C_k\psi(r) + C_h\phi(r) + C_g\right). \end{aligned} \tag{3.26}$$

Hence

$$|Tu(t)| \leq \frac{C_1}{\Gamma(q-2)} \left[\psi(r) + \phi(r) + 1\right]. \tag{3.27}$$

Moreover, we have

$$\left|(Tu(t))'\right| \leq \frac{1}{\Gamma(q-2)} \left[\psi(r)\int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| k(s)ds + \phi(r)\int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| h(s) + \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| g(s)ds\right]. \tag{3.28}$$

Using (3.17) we obtain

$$\left|{}^cD_{0^+}^{\sigma}Tu\right| \leq \frac{C_2}{(2-\sigma)\Gamma(2-\sigma)} \left(\psi(r) + \phi(r) + 1\right). \tag{3.29}$$

From (3.27) and (3.29) we get

$$\|Tu\| = \left(\psi(r) + \phi(r) + 1\right)\left(\frac{C_1}{\Gamma(q-2)} + \frac{C_2}{(2-\sigma)\Gamma(2-\sigma)}\right), \tag{3.30}$$

then $T(B_r)$ is uniformly bounded.

(ii) $T(B_r)$ is equicontinuous. Indeed for all $t_1,\ t_2 \in [0,1]$, $t_1 < t_2$, $u \in B_r$, let $C = \max(|f(t, u(t), {}^cD_{0^+}^{\sigma} u(t))|, 0 \le t \le 1, \|u\| < r)$, therefore

$$|Tu(t_1) - Tu(t_2)| \le \frac{C}{\Gamma(q-2)}\left(\int_0^{t_1} |G(t_1, s) - G(t_2, s)| ds + \int_{t_1}^{t_2} |G(t_1, s) - G(t_2, s)| ds + \int_{t_2}^{1} |G(t_1, s) - G(t_2, s)| ds\right), \tag{3.31}$$

that implies

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| \le \frac{C}{\Gamma(q-2)} \int_0^{t_1} & \frac{(t_2 - s)^{q-1} - (t_1 - s)^{q-1}}{q(q-1)} \\ &+ (t_2 - t_1)\left(\frac{|\alpha|}{(1-s)^{3-q}} - \frac{(1-s)^{q-2}}{q-1}\right) ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{q(q-1)} \\ &+ (t_2 - t_1)\left(\frac{|\alpha|}{(1-s)^{3-q}} - \frac{(1-s)^{q-2}}{q-1}\right) ds \\ &+ \int_{t_2}^{1} (t_1 - t_2)\left(\frac{|\alpha|}{(1-s)^{3-q}} - \frac{(1-s)^{q-2}}{q-1}\right) ds. \end{aligned} \tag{3.32}$$

Let us consider the function $\Phi(x) = x^{q-1} - (q-1)x$, we see that Φ is decreasing on $[0,1]$, consequently $(t_2 - s)^{q-1} - (t_1 - s)^{q-1} \le (q-1)(t_2 - t_1)$, from which we deduce

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| \le \frac{C}{\Gamma(q-2)}\Bigg[(t_2 - t_1) \int_0^{t_1} & \frac{1}{q(q-1)} + \left|\frac{\alpha}{(1-s)^{3-q}} - \frac{(1-s)^{q-2}}{q-1}\right| ds \\ &+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{q(q-1)} + (t_2 - t_1)\left|\frac{|\alpha|}{(1-s)^{3-q}} - \frac{(1-s)^{q-2}}{q-1}\right| ds \\ &+ (t_1 - t_2) \int_{t_2}^{1} \left|\frac{|\alpha|}{(1-s)^{3-q}} - \frac{(1-s)^{q-2}}{q-1}\right| ds \Bigg]. \end{aligned} \tag{3.33}$$

Some computations give

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| \le{}& \frac{C(t_2 - t_1)}{\Gamma(q-2)}\left(\frac{1}{q(q-1)} + \frac{3|\alpha|}{(q-2)} + \frac{3}{q-1}\right) \\ &+ \frac{C}{\Gamma(q-2)} \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{q(q-1)} ds. \end{aligned} \tag{3.34}$$

On the other hand we have

$$\left|{}^cD_{0^+}^{\sigma}Tu(t_1) -{}^c D_{0^+}^{\sigma}Tu(t_2)\right| \leq +\frac{1}{\Gamma(2-\sigma)}\int_0^{t_1}\left((t_1-s)^{-\sigma+1}-(t_2-s)^{-\sigma+1}\right)\left|(Tu(s))'\right|ds$$
$$+\frac{1}{\Gamma(2-\sigma)}\int_{t_1}^{t_2}(t_2-s)^{-\sigma+1}\left|(Tu(s))'\right|ds. \tag{3.35}$$

Using (3.17) and (3.28) it yields

$$\left|(Tu(t))'\right| \leq \left[\psi(r)+\phi(r)+1\right]C_2, \tag{3.36}$$

then

$$\left|{}^cD_{0^+}^{\sigma}Tu(t_1) -{}^c D_{0^+}^{\sigma}Tu(t_2)\right| \leq \frac{\left[\psi(r)+\phi(r)+1\right]C_2}{(2-\sigma)\Gamma(2-\sigma)}\left[2(t_2-t_1)^{2-\sigma}+t_2^{2-\sigma}-t_1^{2-\sigma}\right], \tag{3.37}$$

when $t_1 \to t_2$, in (3.34) and (3.37) then $|Tu(t_1)-Tu(t_2)|$ and $|{}^cD_{0^+}^{\sigma}Tu(t_1) -{}^c D_{0^+}^{\sigma}Tu(t_2)|$ tend to 0. Consequently $T(B_r)$ is equicontinuous. From Arzelá-Ascoli Theorem we deduce that T is completely continuous operator.

Now we apply Leray Schauder nonlinear alternative to prove that T has at least one nontrivial solution in E.

Letting $\Omega = \{u \in E : \|u\| < r\}$, for any $u \in \partial\Omega$, such that $u = \lambda Tu$, $0 < \lambda < 1$, we get, with the help of (3.27),

$$|u(t)| = \lambda|Tu(t)| \leq |Tu(t)| \leq \frac{C_1}{\Gamma(q-2)}\left[\psi(r)+\phi(r)+1\right]. \tag{3.38}$$

Taking into account (3.29) we obtain

$$\left|{}^cD_{0^+}^{\sigma}u(t)\right| \leq \frac{C_2}{(2-\sigma)\Gamma(2-\sigma)}\left(\psi(r)+\phi(r)+1\right). \tag{3.39}$$

From (3.38), (3.39), and (3.22) we deduce that

$$\|u\| \leq \left(\psi(r)+\phi(r)+1\right)\left(\frac{C_1}{\Gamma(q-2)}+\frac{C_2}{(2-\sigma)\Gamma(2-\sigma)}\right) < r, \tag{3.40}$$

this contradicts the fact that $u \in \partial\Omega$. Lemma 3.5 allows us to conclude that the operator T has a fixed point $u^* \in \overline{\Omega}$ and then the fractional boundary value problem (P1) has a nontrivial solution $u^* \in E$. The proof is complete. □

4. Existence of Positive Solutions

In this section we investigate the positivity of solution for the fractional boundary value problem (P1), for this we make the following hypotheses.

(H1) $f(t,u,v) = a(t)f_1(u,v)$ where $a \in C((0,1),(0,\infty))$ and $f_1 \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$.

(H2) $0 < \int_0^1 G(s,s)a(s)ds < \infty$.

Now we give the properties of the Green function.

Lemma 4.1. *Let $G(t,s)$ be the function defined by* (2.7). *If $\alpha \geq 1$ then $G(t,s)$ has the following properties:*

(i) $G(t,s) \in C([0,1] \times [0,1])$, $G(t,s) > 0$, *for all* t, $s \in]0,1[$.

(ii) *If* $t, s \in (\tau, 1)$, $\tau > 0$, *then*

$$0 < \tau G(s,s) \leq G(t,s) \leq \frac{2}{\tau} G(s,s). \tag{4.1}$$

Proof. (i) It is obvious that $G(t,s) \in C([0,1] \times [0,1])$, moreover, we have

$$\begin{aligned} \frac{\alpha t}{(1-s)^{3-q}} - \frac{t(1-s)^{q-2}}{q-1} &= \frac{t}{q-1(1-s)^{3-q}}\big[(q-1)\alpha - (1-s)\big] \\ &\geq \frac{t}{q-1(1-s)^{3-q}}[\alpha - 1 + s], \end{aligned} \tag{4.2}$$

which is positive if $\alpha \geq 1$. Hence $G(t,s)$ is nonnegative for all $t, s \in]0,1[$.

(ii) Let $t, s \in (\tau, 1)$, it is easy to see that $G(s,s) \neq 0$, then we have

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \frac{(t-s)^{q-1}(1-s)^{3-q}}{qs\big[(q-1)\alpha - (1-s)\big]} + \frac{t}{s} \\ &\leq \frac{1+(1-s)^2}{s} \leq \frac{2}{\tau}, \quad 0 < \tau \leq s \leq t < 1, \\ \frac{G(t,s)}{G(s,s)} &= \frac{t}{s} \leq \frac{2}{\tau}, \quad 0 < \tau \leq t \leq s \leq 1. \end{aligned} \tag{4.3}$$

Now we look for lower bounds of $G(t,s)$

$$\frac{G(t,s)}{G(s,s)} \geq \frac{t}{s} \geq \frac{\tau}{s} \geq \tau, \quad 0 < \tau \leq s \leq t < 1, \ 0 < \tau \leq t \leq s \leq 1. \tag{4.4}$$

Finally, since $G(s,s)$ is nonnegative we obtain $0 < \tau G(s,s) \leq G(t,s) \leq (2/\tau)G(s,s)$. □

We recall the definition of positive of solution.

Definition 4.2. A function $u(t)$ is called positive solution of the fractional boundary value problem (P1) if $u(t) \geq 0$, for all $t \in [0,1]$.

Lemma 4.3. *If $u \in E^+$ and $\alpha \geq 1$, then the solution of the fractional boundary value problem (P1) is positive and satisfies*

$$\min_{t\in(\tau,1)} \left(u(t) + {}^c D_{0^+}^{\sigma} u(t)\right) \geq \frac{\tau^2}{2}\|u\|. \tag{4.5}$$

Proof. First let us remark that under the assumptions on u and f, the function ${}^cD_{0^+}^{\sigma}u$ is nonnegative. From Lemma 3.1 we have

$$u(t) = \frac{1}{\Gamma(q-2)}\int_0^1 G(t,s)a(s)f_1\left(u(s), {}^c D_{0^+}^{\sigma}u(s)\right)ds. \tag{4.6}$$

Applying the right-hand side of inequality (4.1) we get

$$u(t) \leq \frac{2}{\tau\Gamma(q-2)}\int_0^1 G(s,s)a(s)f_1\left(u(s), {}^c D_{0^+}^{\sigma}u(s)\right)ds. \tag{4.7}$$

Moreover, (4.1) gives

$$\begin{aligned} {}^cD_{0^+}^{\sigma}u(t) &= \frac{1}{\Gamma(q-2)\Gamma(2-\sigma)}\int_0^t\int_0^1 (t-s)^{-\sigma+1}G_1(s,r) \\ &\quad\times a(r)f_1\left(u(r), {}^cD_{0^+}^{\sigma}u(r)\right)ds\,dr \\ &\leq \frac{1}{\tau(2-\sigma)\Gamma(q-2)\Gamma(2-\sigma)}\int_0^1 G_1(r,r)a(r)f_1\left(u(r), {}^c D_{0^+}^{\sigma}u(r)\right)dr. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) yields

$$\|u\| \leq \frac{2}{\tau\Gamma(q-2)}\int_0^1\left[G(s,s) + \frac{G_1(s,s)}{(2-\sigma)\Gamma(2-\sigma)}\right]a(s)f_1\left(u(s), {}^c D_{0^+}^{\sigma}u(s)\right)ds, \tag{4.9}$$

hence

$$\int_0^1\left[G(s,s) + \frac{G_1(s,s)}{(2-\sigma)\Gamma(2-\sigma)}\right]a(s)f_1\left(u(s), {}^c D_{0^+}^{\sigma}u(s)\right)ds \geq \frac{\tau\Gamma(q-2)}{2}\|u\|. \tag{4.10}$$

In view of the-left hand side of (4.1), we obtain for all $t \in (\tau, 1)$

$$u(t) \geq \frac{\tau}{\Gamma(q-2)}\int_0^1 G(s,s)a(s)f_1\left(u(s), {}^c D_{0^+}^{\sigma}u(s)\right)ds, \tag{4.11}$$

on the other hand we have

$$ {}^{c}D_{0^+}^{\sigma}u(t) \geq \frac{\tau^{2-\sigma}}{(2-\sigma)\Gamma(q-2)\Gamma(2-\sigma)} \int_0^1 G_1(r,r)a(r)f_1\left(u(r),{}^{c}D_{0^+}^{\sigma}u(r)\right)dr. \tag{4.12} $$

From (4.11) and (4.12) we get

$$ \begin{aligned} \min_{t\in(\tau,1)}\left(u(t)+{}^{c}D_{0^+}^{\sigma}u(t)\right) \geq \frac{\tau}{\Gamma(q-2)} \int_0^1 &\left[G(s,s)+\frac{G_1(s,s)}{(2-\sigma)\Gamma(2-\sigma)}\right] \\ &\times a(s)f_1\left(u(s),{}^{c}D_{0^+}^{\sigma}u(s)\right)ds, \end{aligned} \tag{4.13} $$

with the help of (4.10) we deduce

$$ \min_{t\in(\tau,1)}\left(u(t)+{}^{c}D_{0^+}^{\sigma}u(t)\right) \geq \frac{\tau^2}{2}\|u\|. \tag{4.14} $$

The proof is complete. □

Define the quantities A_0 and A_∞ by

$$ A_0 = \lim_{(|u|+|v|)\to 0}\frac{f_1(u,v)}{|u|+|v|}, \qquad A_\infty = \lim_{(|u|+|v|)\to \infty}\frac{f_1(u,v)}{|u|+|v|}. \tag{4.15} $$

The case $A_0 = 0$ and $A_\infty = \infty$ is called superlinear case and the case $A_0 = \infty$ and $A_\infty = 0$ is called sublinear case.

The main result of this section is as follows.

Theorem 4.4. *Under the assumption of Lemma 4.3, the fractional boundary value problem (P1) has at least one positive solution in the both cases superlinear as well as sublinear.*

To prove Theorem 4.4 we apply the well-known Guo-Krasnosel'skii fixed point theorem on cone.

Theorem 4.5 (see [18]). *Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let*

$$ \mathcal{A} : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \longrightarrow K \tag{4.16} $$

be a completely continuous operator such that

(i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Proof. To prove Theorem 4.4 we define the cone K by

$$K = \left\{ u \in E^+, \min_{t\in(\tau,1)} \left(u(t) +^c D_{0^+}^{\sigma} u(t)\right) \geq \frac{\tau^2}{2}\|u\| \right\}. \tag{4.17}$$

It is easy to check that K is a nonempty closed and convex subset of E, hence it is a cone. Using Lemma 4.3 we see that $TK \subset K$. From the prove of Theorem 3.4, we know that T is completely continuous in E.

Let us prove the superlinear case. First, since $A_0 = 0$, for any $\varepsilon > 0$, there exists $R_1 > 0$, such that

$$f_1(u,v) \leq \varepsilon(|u| + |v|) \tag{4.18}$$

for $0 < |u| + |v| \leq R_1$. Letting $\Omega_1 = \{u \in E, \|u\| < R_1\}$, for any $u \in K \cap \partial\Omega_1$, it yields

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q-2)} \int_0^1 G(t,s)a(s)f_1\left(u(s),^c D_{0^+}^{\sigma} u(s)\right)ds \\ &\leq \frac{2\varepsilon\|u\|}{\tau\Gamma(q-2)} \int_0^1 G(s,s)a(s)ds. \end{aligned} \tag{4.19}$$

Moreover, we have

$$\begin{aligned} {}^cD_{0^+}^{\sigma} Tu(t) &= \frac{1}{\Gamma(q-2)\Gamma(2-\sigma)} \int_0^t \int_0^1 (t-s)^{-\sigma+1} G_1(s,r) \\ &\quad \times a(r) f_1\left(u(r),^c D_{0^+}^{\sigma} u(r)\right) ds\, dr \\ &\leq \frac{1}{\tau(2-\sigma)\Gamma(q-2)\Gamma(2-\sigma)} \int_0^1 G_1(r,r)a(r)\left(|u(r)| + \left|{}^cD_{0^+}^{\sigma} u(r)\right|\right)dr \\ &\leq \frac{\varepsilon\|u\|}{\tau(2-\sigma)\Gamma(q-2)\Gamma(2-\sigma)} \int_0^1 G_1(s,s)a(s)ds. \end{aligned} \tag{4.20}$$

From (4.19) and (4.20) we conclude

$$\|Tu\| \leq \frac{2\varepsilon\|u\|}{\tau\Gamma(q-2)} \int_0^1 \left[G(s,s) + \frac{G_1(s,s)}{(2-\sigma)\Gamma(2-\sigma)}\right] a(s)ds. \tag{4.21}$$

In view of hypothesis (H2), one can choose ε such that

$$\varepsilon \leq \frac{\tau\Gamma(q-2)}{2\int_0^1 [G(s,s) + (G_1(s,s)/(2-\sigma)\Gamma(2-\sigma))]a(s)ds}. \tag{4.22}$$

The inequalities (4.21) and (4.22) imply that $\|Tu\| \leq \|u\|$, for all $u \in K \cap \partial\Omega_1$. Second, in view of $A_\infty = \infty$, then for any $M > 0$, there exists $R_2 > 0$, such that $f_1(u,v) \geq M(|u| + |v|)$ for $|u| + |v| \geq R_2$. Let $R = \max\{2R_1, (2R_2/\tau^2)\}$ and denote by Ω_2 the open set $\{u \in E / \|u\| < R\}$. If $u \in K \cap \partial\Omega_2$ then

$$\min_{t\in(\tau,1)} \left(u(t) +^c D_{0^+}^{\sigma} u(t)\right) \geq \frac{\tau^2}{2}\|u\| = \frac{\tau^2}{2} R \geq R_2. \tag{4.23}$$

Using the left-hand side of (4.1) and Lemma 4.3, we obtain

$$\begin{aligned} Tu(t) &\geq \frac{\tau M}{\Gamma(q-2)} \int_0^1 G(s,s)a(s)\left(|u(s)| + \left|{}^c D_{0^+}^{\sigma} u(s)\right|\right) ds \\ &\geq \frac{\tau^3 M\|u\|}{2\Gamma(q-2)} \int_0^1 G(s,s)a(s)ds. \end{aligned} \tag{4.24}$$

Moreover, we get with the help of (4.12)

$${}^c D_{0^+}^{\sigma} Tu(t) \geq \frac{\tau^{4-\sigma} M\|u\|}{2(2-\sigma)\Gamma(q-2)\Gamma(2-\sigma)} \int_0^1 G_1(s,s)a(s)ds. \tag{4.25}$$

In view of (4.26) and (4.24) we can write

$$\begin{aligned} Tu(t) +^c D_{0^+}^{\sigma} Tu(t) &\geq \frac{\tau^3 M\|u\|}{2\Gamma(q-2)} \int_0^1 \left[G(s,s) + \frac{\tau^{1-\sigma} G_1(s,s)}{(2-\sigma)\Gamma(2-\sigma)}\right] a(s)ds \\ &\geq \frac{\tau^3 M\|u\|}{2\Gamma(q-2)} \int_0^1 \left[G(s,s) + \frac{G_1(s,s)}{(2-\sigma)\Gamma(2-\sigma)}\right] a(s)ds. \end{aligned} \tag{4.26}$$

Let us choose M such that

$$M \geq \frac{2\Gamma(q-2)}{\tau^3 \int_0^1 [G(s,s) + (G_1(s,s)/(2-\sigma)\Gamma(2-\sigma))] a(s)ds}, \tag{4.27}$$

then we get $Tu(t) +^c D_{0^+}^{\sigma} Tu(t) \geq \|u\|$. Hence,

$$\|Tu\| \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_2. \tag{4.28}$$

The first part of Theorem 4.5 implies that T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $R_2 \leq \|u\| \leq R$. To prove the sublinear case we apply similar techniques. The proof is complete. □

In order to illustrate our results, we give the following examples.

Example 4.6. The fractional boundary value problem

$$
\begin{gathered}
{}^cD_{0^+}^{5/2}u = \left(\frac{t-1}{10}\right)^3 u + t^2 D_{0^+}^{5/4}u + \ln t, \quad 0<t<1, \\
u(0)=u''(0)=0, \qquad u'(1)=\frac{-1}{2}u''(1)
\end{gathered}
\tag{4.29}
$$

has a unique solution in E.

Proof. In this case we have $f(t,x,y)=((t-1)/10)^3x+t^2y+\ln t$, $2<q=5/2<3$, $\sigma=(5/4)<2$, $\alpha=-1/2$ and

$$
\left|f(t,x,\overline{x})-f(t,y,\overline{y})\right| \le \left(\frac{1-t}{10}\right)^3|x-\overline{x}|+t^2\left|y-\overline{y}\right|,
\tag{4.30}
$$

then $g(t)=((1-t)/10)^3$ and $h(t)=t^2$. Some calculus give

$$
\begin{gathered}
\left\|I_{0^+}^{q-1}g\right\|_{L^1}=0.17846\times10^{-3}, \qquad I_{0^+}^{q-1}g(1)=0.48001\times10^{-3}, \\
I_{0^+}^{q-2}g(1)=016120\times10^{-3}, \qquad C_g=7.3907\times10^{-4}, \qquad A_g=1.0406\times10^{-3}, \\
\left\|I_{0^+}^{q-1}h\right\|_{L^1}=0.038210, \qquad I_{0^+}^{q-1}h(1)=0.17194, \\
I_{0^+}^{q-2}h(1)=0.6018, \qquad C_h=0.49747, \qquad A_h=0.42448, \\
C_g+C_h=0.49821<1, \\
A_g+A_h=0.42552<(2-\sigma)\Gamma(2-\sigma)=0.91906.
\end{gathered}
\tag{4.31}
$$

Thus Theorem 3.2 implies that fractional boundary value problem (4.29) has a unique in E. □

Example 4.7. The fractional boundary value problem

$$
\begin{gathered}
{}^cD_{0^+}^{7/3}u=(1-t)^2\left(\frac{u^4}{100(1+u^2)}+\frac{\ln\left(1+\left({}^cD_{0^+}^{6/5}u\right)^2\right)+1}{9}\right)=0, \quad 0<t<1, \\
u(0)=u''(0)=0, \qquad u'(1)=\frac{3}{2}u''(1)
\end{gathered}
\tag{4.32}
$$

has at least one nontrivial solution in E.

Proof. We apply Theorem 3.4 to prove that the fractional boundary value problem (4.32) has at least one nontrivial solution. We have $q = 7/3$, $\sigma = 6/5$, $\alpha = 3/2$, and

$$\begin{aligned} \left|f(t,x,\overline{x})\right| &= \frac{(1-t)^2 x^4}{100(1+x^2)} + (1-t)^2 \frac{\ln\left(1+\overline{x^2}\right)}{9} + (1-t)^2 \\ &\le \frac{|x|^2}{100}(1-t)^2 + (1-t)^2 \frac{\ln\left(1+\overline{x^2}\right)}{9} + (1-t)^2 \\ &\le k(t)\psi(|x|) + h(t)\phi(|\overline{x}|) + g(t), \end{aligned} \tag{4.33}$$

where $k(t) = h(t) = g(t) = (1-t)^2$, $\psi(x) = (x/10)^2$, $\phi(\overline{x}) = \ln(1+\overline{x^2})/9$, $f(t,0,0) \neq 0$. Let us find r such that (3.22) holds, for this we have

$$\begin{gathered} \left\| I_{0^+}^{q-1} g \right\|_{L^1} = 0.19382, \qquad I_{0^+}^{q-2} g(1) = 0.15998, \qquad I_{0^+}^{q-1} g(1) = 0.33595, \\ C_g = C_1 = 0.76974, \qquad A_g = C_2 = 0.91187. \end{gathered} \tag{4.34}$$

We see that (3.22) is equivalent to $1.2664(((r^2)/100) + (\ln(1+r^2)/9) + 1) - r$ which is negative for $r = 6$. □

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