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The influence of nonnormal noncyclic subgroups on the structure of finite groups

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Abstract. We obtain a complete classification of finite groups in which all noncyclic proper subgroups are nonnormal, and we apply this classification to investigate some structures of finite groups.

Key words: noncyclic subgroup, nonnormal, nonabelian simple group.

1. Introduction

In this paper all groups are assumed to be finite. It is known that a group G is called a Dedekind-group if all subgroups of G are normal in G (see [6, Theorem 5.3.7]), and a group G is said to be a simple group if all nontrivial subgroups of G are nonnormal in G. As generalizations, it is natural to investigate the normality of some particular subgroups. In [1], Buckley characterized groups in which all minimal subgroups are normal, such groups are called PN-groups. Note that a group in which all cyclic subgroups are normal is also a Dedekind-group. For the noncyclic subgroups, [2] and [5] classified all p-groups in which all noncyclic subgroups are normal. And in [3], Kutnar, Marušič and the authors classified noncyclic groups in which all supersolvable noncyclic subgroups are selfnormalizing.

As a further study of the normality of noncyclic subgroups, the main goal of this paper is to classify groups in which all noncyclic proper subgroups are nonnormal. For convenience, we call a group G an NCNN-group if G has at least one noncyclic proper subgroup and all noncyclic proper subgroups of G are nonnormal in G.

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For NCNN-groups, we have the following result, the proof of which is given in Section 2.

Theorem 1.1 *A group G is an NCNN-group if and only if one of the following statements holds*:

(1) *$G/\Phi(G)$ is a nonabelian simple group with $\Phi(G) = Z(G)$ being cyclic, where $\Phi(G)$ is the Frattini subgroup of G and $Z(G)$ is the center of G*;

(2) *$G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 1$ are positive integers and q is the smallest prime divisor of $|G|$ such that $((r-1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.*

Next we will apply Theorem 1.1 to investigate some structures of groups.

Lemma 1.2 ([3, Theorem 1.2]) *Let G be a group having at least one noncyclic proper subgroup. Then all noncyclic proper subgroups of G are selfnormalizing in G if and only if $G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 1$ are positive integers and q is the smallest prime divisor of $|G|$ such that $((r-1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.*

Combining Theorem 1.1 and Lemma 1.2 together, we obtain the following interesting result for noncyclic subgroups.

Theorem 1.3 *Let G be a solvable group having at least one noncyclic proper subgroup. Then all noncyclic proper subgroups of G are nonnormal in G if and only if all noncyclic proper subgroups of G are selfnormalizing in G.*

The alternating group A_5 shows that Theorem 1.3 is not true if G is a nonsolvable group.

Note that all PN-groups are solvable by [1]. Then we can easily get the following theorem by Theorem 1.1.

Theorem 1.4 *Let G be a PN-group having at least one noncyclic proper subgroup. Then all noncyclic proper subgroups of G are nonnormal in G if and only if $G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r \rangle$, where $m > r \geq 1$, $n \geq 2$ are positive integers and q is the smallest prime divisor of $|G|$ such that $((r-1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.*

The following three corollaries are direct consequences of Theorem 1.1.

Corollary 1.5 *Let G be a group having at least one noncyclic proper* *subgroup. Then all noncyclic proper subgroups of G are not subnormal in G if and only if one of the following statements holds*:

(1) $G/\Phi(G)$ *is a nonabelian simple group with* $\Phi(G) = Z(G)$ *being cyclic*;

(2) $G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r \rangle$, *where* $m > r \geq 1$, $n \geq 1$ *are positive integers and* q *is the smallest prime divisor of* $|G|$ *such that* $((r-1)q, m) = 1$ *and* $r^q \equiv 1 \pmod{m}$.

Corollary 1.6 *Let G be a group having at least one noncyclic proper subgroup. Then for any noncyclic proper subgroup H of G we always have that H_G (the largest normal subgroup of G that is contained in H) is cyclic if and only if one of the following statements holds*:

(1) $G/\Phi(G)$ *is a nonabelian simple group with* $\Phi(G) = Z(G)$ *being cyclic*;

(2) $G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r \rangle$, *where* $m > r \geq 1$, $n \geq 1$ *are positive integers and* q *is the smallest prime divisor of* $|G|$ *such that* $((r-1)q, m) = 1$ *and* $r^q \equiv 1 \pmod{m}$.

Corollary 1.7 *Let G be an NCNN-group. If G is solvable, then G is supersolvable.*

Note that a group having at most three conjugacy classes of noncyclic proper subgroups is solvable by [4]. Combining Corollary 1.7 and [4] together, we have the following corollary.

Corollary 1.8 *Let G be an NCNN-group. If G has at most three conjugacy classes of noncyclic proper subgroups, then G is supersolvable.*

2. Proof of Theorem 1.1

Proof. (1) For the necessity part.

(i) Suppose that G is nonsolvable. By [6, Exercise 10.5.7], we have that all maximal subgroups of G are noncyclic. If G is a nonabelian simple group, then G clearly satisfies the hypothesis. Next we assume that G is not a nonabelian simple group. Let N be a maximal nontrivial normal subgroup of G. By the hypothesis, we have that N is cyclic. Then G/N must be a nonabelian simple group. We claim that

$$N \leq \Phi(G).$$

Otherwise, assume $N \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $N \nleq M$. Then $G = NM$. It is obvious that $N \cap M \trianglelefteq M$. Moreover, $N \cap M \trianglelefteq N$ since N is cyclic. So $N \cap M \trianglelefteq G$. We have $G/(N \cap M) = N/(N \cap M) \rtimes M/(N \cap M)$. Let $\bar{G} = G/(N \cap M)$, $\bar{N} = N/(N \cap M)$ and $\bar{M} = M/(N \cap M)$. It is obvious that $\bar{M} \cong G/N$ is a nonabelian simple group. By N/C-theorem, $\bar{G}/C_{\bar{G}}(\bar{N}) = N_{\bar{G}}(\bar{N})/C_{\bar{G}}(\bar{N}) \lesssim \mathrm{Aut}(\bar{N})$. Since $\bar{N}$ is cyclic, we have that $\mathrm{Aut}(\bar{N})$ is abelian. However, since $\bar{G}/C_{\bar{G}}(\bar{N}) \cong (\bar{G}/\bar{N})/(C_{\bar{G}}(\bar{N})/\bar{N})$ and $\bar{G}/\bar{N} \cong \bar{M}$ is a nonabelian simple group, it follows that $C_{\bar{G}}(\bar{N}) = \bar{G}$. That is, $\bar{N} \leq Z(\bar{G})$. Thus $\bar{M} \trianglelefteq \bar{G}$. It implies that $M \trianglelefteq G$. By the hypothesis, we have that M is cyclic. Then it is easy to see that G is solvable, a contradiction. Hence $N \leq \Phi(G)$.

Since G/N is a nonabelian simple group, it follows that $N = \Phi(G)$. So $G/\Phi(G)$ is a nonabelian simple group, where $\Phi(G)$ is cyclic. Moreover, we can easily get $\Phi(G) = Z(G)$ by N/C-theorem.

(ii) Suppose that G is solvable. If G is nilpotent, then all maximal subgroups of G are normal in G. By the hypothesis, we have that all maximal subgroups of G are cyclic, this contradicts that G has at least one noncyclic proper subgroup.

Thus G is nonnilpotent. Since G is solvable, one has that G has a maximal subgroup L such that $L \trianglelefteq G$. By the hypothesis, we have that L is cyclic. Assume $G/L \cong \mathbb{Z}_e$, where e is a prime divisor of $|G|$. Let $E \in Syl_e(G)$. Then $G = LE$. Let K be a e'-Hall subgroup of L. It is obvious that $K \trianglelefteq G$ since L is cyclic. Thus $G = K \rtimes E$. We claim that

$$E \text{ is cyclic.}$$

Otherwise, assume that E is noncyclic. Let E_1 and E_2 be two distinct maximal subgroups of E. It is easy to see that $K \rtimes E_1$ and $K \rtimes E_2$ are normal in $K \rtimes E = G$. By the hypothesis, we have that $K \rtimes E_1$ and $K \rtimes E_2$ are cyclic. It follows that $E_1 \leq C_G(K)$ and $E_2 \leq C_G(K)$. So $E = E_1E_2 \leq C_G(K)$. It implies that G is nilpotent, a contradiction. Hence E is cyclic.

Thus G is a group in which all Sylow subgroups are cyclic. By [6, Theorem 10.1.10], we have $G = \langle a, b \mid a^m = b^s = 1, b^{-1}ab = a^r \rangle$, where m and s are positive integers such that $((r-1)s, m) = 1$ and $r^s \equiv 1 \pmod{m}$. We claim that

$$s \text{ is a prime-power.}$$

Otherwise, assume that t_1 and t_2 are two distinct prime divisors of s. Then $\langle b^{t_1}\rangle$ and $\langle b^{t_2}\rangle$ are two distinct maximal subgroups of $\langle b\rangle$. It is easy to see that $\langle a\rangle \rtimes \langle b^{t_1}\rangle$ and $\langle a\rangle \rtimes \langle b^{t_2}\rangle$ are normal in $\langle a\rangle \rtimes \langle b\rangle = G$. By the hypothesis, we have that $\langle a\rangle \rtimes \langle b^{t_1}\rangle$ and $\langle a\rangle \rtimes \langle b^{t_2}\rangle$ are cyclic. Then $\langle b^{t_1}\rangle \leq C_G(\langle a\rangle)$ and $\langle b^{t_2}\rangle \leq C_G(\langle a\rangle)$. Thus $\langle b\rangle = \langle b^{t_1}\rangle\langle b^{t_2}\rangle \leq C_G(\langle a\rangle)$. It follows that G is cyclic, a contradiction. So s is a prime-power.

Assume $s = q^n$, where q is a prime and $n \geq 1$. Since $\langle a\rangle \rtimes \langle b^q\rangle$ is normal in $\langle a\rangle \rtimes \langle b\rangle = G$. By the hypothesis, we have that $\langle a\rangle \rtimes \langle b^q\rangle$ is cyclic. Thus $r^q \equiv 1 \pmod{m}$.

Next we claim that

$$q \text{ is the smallest prime divisor of } |G|.$$

Otherwise, let f be the smallest prime divisor of $|G|$ and $f \neq q$. Let $F \in Syl_f(G)$. By above argument, F is cyclic. Then G is f-nilpotent by [6, Theorem 10.1.9]. That is, there exists a normal subgroup T of G such that $G = T \rtimes F$. By the hypothesis, T is cyclic. Since $q \neq f$, we have $\langle b\rangle \leq T$. Thus $\langle b\rangle \trianglelefteq G$. It follows that G is cyclic, a contradiction. So q is the smallest prime divisor of $|G|$.

(2) For the sufficiency part.

If $G/\Phi(G)$ is a nonabelian simple group with $\Phi(G) = Z(G)$ being cyclic, it is easy to show that all noncyclic proper subgroups of G are nonnormal in G.

Next assume $G = \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r\rangle$, where $m \geq 1$, $n \geq 1$ are positive integers and q is the smallest prime divisor of $|G|$ such that $((r-1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$. Let R be a noncyclic proper subgroup of G. By the definition of G, it is easy to show that $R = \langle a^i\rangle \rtimes \langle b^x\rangle$ for some $x \in G$ and some positive integer i such that $\langle a^i\rangle < \langle a\rangle$. If $R \trianglelefteq G$, then $G/R = \langle a\rangle \rtimes \langle b^x\rangle/\langle a^i\rangle \rtimes \langle b^x\rangle$ is cyclic. It follows that $G' \leq R$. Since $b^{-1}ab = a^r$, one has $[a, b] = a^{-1}b^{-1}ab = a^{1-r}$. Thus $a^{1-r} \in R$. It follows that $\langle a^{r-1}\rangle \leq R$. Since $(r-1, m) = 1$, we have $\langle a^{r-1}\rangle = \langle a\rangle$. Then $\langle a\rangle \leq R$, this contradicts that $\langle a^i\rangle < \langle a\rangle$. Hence all noncyclic proper subgroups of G are nonnormal in G. □

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