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A note on skew group categories

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Abstract. Let G be a finite group, and $\mathscr{C}$ a G-abelian category. We prove that the skew group category $\mathscr{C}(G)$ is an abelian category under the condition that the order $|G|$ is invertible in $\mathscr{C}$. When the order $|G|$ is not invertible in $\mathscr{C}$, an example is given to show that $\mathscr{C}(G)$ is not an abelian category.

Key words: G-abelian category, skew group category, idempotent completion.

1. Introduction

Let G be a finite group, and $\mathscr{C}$ a preadditive category. By an *action* of G on $\mathscr{C}$ we mean a group homomorphism from G to the group $\mathrm{Aut}(\mathscr{C})$ of autofunctors of $\mathscr{C}$. Recall that $\mathscr{C}$ is called a *G-abelian category* if $\mathscr{C}$ is an abelian category with an action of G on $\mathscr{C}$. Then one can form the orbit category $\mathscr{C}[G]$ of $\mathscr{C}$ with respect to the given action (compare [10], [7], [14], [2], the orbit category $\mathscr{C}[G]$ is called the skew category of $\mathscr{C}$ in [7]). We mention that both works [7] and [2] suggest that the orbit category $\mathscr{C}[G]$ is a more general notion which allow ones to consider the case that the action of G on $\mathscr{C}$ is not free; see [7, Remark 2.9] and [2, Remark 2.2 and Remark 2.14].

In general, the orbit category $\mathscr{C}[G]$ of a G-abelian category $\mathscr{C}$ is not necessarily an abelian category since idempotent morphisms in $\mathscr{C}[G]$ may not split; see [14, page 255]. This leads ones to consider the skew group category $\mathscr{C}(G)$ of $\mathscr{C}$, which is defined as the idempotent completion of the orbit category $\mathscr{C}[G]$; see [14, Section 3]. It is well known that the idempotent completion $\widehat{\mathscr{D}}$ of a category $\mathscr{D}$ is the smallest category containing $\mathscr{D}$ as a full subcategory with split idempotent morphisms; see [12].

The aim of this note is to show that: for a G-abelian category $\mathscr{C}$, the

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skew group category $\mathscr{C}(G)$, which is the smallest category containing the orbit category $\mathscr{C}[G]$ as a full subcategory, has a natural abelian categorial structure, under the condition that the order $|G|$ of the group G is invertible in $\mathscr{C}$ (see page 4 for the definition of the order $|G|$ being invertible in $\mathscr{C}$). The main result is as follows.

Proposition 1.1 *Let $\mathscr{C}$ be a G-abelian category with the order $|G|$ invertible in $\mathscr{C}$. Then the skew group category $\mathscr{C}(G)$ is also an abelian category.*

In Section 2, we recall some basic facts of the orbit categories and the skew group categories.

Section 3 is devoted to the proof of Proposition 1.1.

In Section 4, we present an example where the skew group category $\mathscr{C}(G)$ is not an abelian category with $|G|$ not invertible in $\mathscr{C}$.

Throughout this paper, the composition of two morphisms $f : L \longrightarrow M$ and $g : M \longrightarrow N$ in a category $\mathscr{C}$ is denoted by gf. For a ring A, we denote by $\operatorname{mod} A$ the category of all finitely generated left A-modules. The quivers we consider in this paper are finite. Let α and β be two paths in a quiver Q. We denote by $\beta\,\alpha$ the multiplication of the paths α and β when the terminal vertex of α is the starting vertex of β. For the unexplained notions about quiver algebras and the representation theory of quivers, we refer to [3], [4].

2. Preliminaries

In this section, we recall some basic facts on skew group categories; compare [7], [14], [2].

The notion of the skew group category for a G-preadditive category with a finite group G was introduced in [14] for the study of the representation theory of an artin algebra and its skew group algebra. The skew group category $\mathscr{C}(G)$ for a G-preadditive category $\mathscr{C}$ was defined as the idempotent completion of the category $\mathscr{C}[G]$; see [14, Section 3]. When G is a cyclic group, the category $\mathscr{C}[G]$ is called the orbit category in [10]. For an arbitrary group G and a small preadditive category $\mathscr{C}$ over a commutative ring R, the orbit category $\mathscr{C}[G]$ is called the skew category in [7]. When the action of G on $\mathscr{C}$ is free, two alternative constructions for the orbit category were also introduced in [7]; see [7, Propsition 2.7 and Theorem 2.8]. The notion of the skew group category for a G-preadditive category over a commutative ring R with an arbitrary group G was introduced in [2] as done for the finite group case in [14]. We mention that the skew group category in [2] was defined as the basic category of the idempotent completion of the orbit category; see [2, Definition 3.6].

Throughout this paper, G is a finite group. Let $\mathscr{C}$ be a G-preadditive category, that is, $\mathscr{C}$ is a preadditive category with an action of G on $\mathscr{C}$. In other words, the category $\mathscr{C}$ is equipped with the following data. That is, each element $x \in G$ defines an autofunctor $F_x : \mathscr{C} \longrightarrow \mathscr{C}$. For an object M in $\mathscr{C}$, the action ${}^xM$ of x on M is the value $F_x(M)$ of the functor F_x applying on M. For a morphism $f : M \longrightarrow N$ in $\mathscr{C}$, the action ${}^xf : {}^xM \longrightarrow {}^xN$ of x on f is the value $F_x(f)$ of the functor F_x applying on f. Moreover, the action of G on morphisms is subject to the following rules:

(1) ${}^x(gf) = ({}^xg)({}^xf)$, for $x \in G$, and f, g which can be composed in $\mathscr{C}$.
(2) ${}^{xy}f = {}^x({}^yf)$, for $x, y \in G$.
(3) ${}^1f = f$, for f in $\mathscr{C}$, and 1 the identity element of G.

Following [10], [7], the *orbit category* $\mathscr{C}[G]$ of $\mathscr{C}$ with respect to the given action of G on $\mathscr{C}$ is defined as follows. The objects of $\mathscr{C}[G]$ are the same as $\mathscr{C}$. Each morphism set $\mathrm{Hom}_{\mathscr{C}[G]}(M, N)$ is given by the direct sum $\bigoplus_{x \in G} \mathrm{Hom}_{\mathscr{C}}({}^xM, N)$ of the abelian groups of morphisms. The composition of morphisms is defined in the natural way. More explicitly, let $\overline{f} : L \longrightarrow M$ be a morphism in $\mathscr{C}[G]$, which is given by a family of morphisms $\{f_x : {}^xL \longrightarrow M\}_{x \in G}$ in $\mathscr{C}$, and let $\overline{g} : M \longrightarrow N$ be a morphism in $\mathscr{C}[G]$, which is given by a family of morphisms $\{g_y : {}^yM \longrightarrow N\}_{y \in G}$ in $\mathscr{C}$. The composition $\overline{g} \cdot \overline{f} : L \longrightarrow N$ is the morphism in $\mathscr{C}[G]$, which is given by the family of morphisms $\{\sum_{y \in G} g_y({}^yf_{y^{-1}x}) : {}^xL \longrightarrow N\}_{x \in G}$ in $\mathscr{C}$. It follows immediately that $\mathscr{C}[G]$ is a preadditive category, and it is an additive category provided $\mathscr{C}$ is. If $\mathscr{C}$ is a G-additive category, one can consider a morphism $\overline{f} : L \longrightarrow M$ in $\mathscr{C}[G]$ is given by a morphism $\bigoplus_{x \in G} f_x : \bigoplus_{x \in G} {}^xL \longrightarrow M$ in $\mathscr{C}$.

However, idempotent morphisms in $\mathscr{C}[G]$ may not split even if idempotent morphisms split in $\mathscr{C}$ as we have mentioned in the introduction. This leads ones to consider the skew group category $\mathscr{C}(G)$. Let us recall the definition of the idempotent completion of a category from [12, Preliminaries]. Some authors call the idempotent completion of a category as Karoubianisation or pseudo-abelian hull of a category; see for instance [5] and [8, Appendix].

Let $\mathscr{D}$ be an arbitrary category. A morphism $f : M \longrightarrow N$ in $\mathscr{D}$ is said to be a *retraction* if there exists a morphism $g : N \longrightarrow M$ such that $fg = 1_N$. In this case, the object N is called a *retract* of M. A full subcategory $\mathscr{C}$ of $\mathscr{D}$ is said to be a *cover* of $\mathscr{D}$ if every object in $\mathscr{D}$ is a retract of some object in $\mathscr{C}$.

An endomorphism $e : M \longrightarrow M$ with $e^2 = e$ is said to be a *split idempotent* provided that there exist two morphisms $\pi : M \longrightarrow L$ and $\iota : L \longrightarrow M$ in $\mathscr{D}$, such that $e = \iota\,\pi$, and $\pi\,\iota = 1_L$. A category is *idempotent complete* if all idempotent morphisms split. It is well known that abelian categories are idempotent complete.

Let $\mathscr{C}$ be a full subcategory of $\mathscr{D}$. $\mathscr{D}$ is called an *idempotent completion* of $\mathscr{C}$ provided that $\mathscr{D}$ is idempotent complete and $\mathscr{C}$ is a cover for $\mathscr{D}$.

There is a well known construction of an idempotent completion $\widehat{\mathscr{C}}$ of a category $\mathscr{C}$; see [9, Chapter 2, Exercise B]. The category $\widehat{\mathscr{C}}$ is defined as follows. The objects are pairs (M, e) with an object X in $\mathscr{C}$ and an idempotent morphism $e : M \longrightarrow M$ in $\mathscr{C}$. A morphism from (M, e_M) to (N, e_N) is a morphism $f : M \longrightarrow N$ in $\mathscr{C}$ such that $f\,e_M = f = e_N\,f$, which can be represented by a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle e_M}\downarrow & \searrow^{f} & \downarrow{\scriptstyle e_N} \\ M & \xrightarrow{f} & N \end{array}$$

in $\mathscr{C}$. We denote by $f : (M, e_M) \longrightarrow (N, e_N)$ the morphism depicted above for simplicity. There is a canonical embedding functor $\mathbb{i} : \mathscr{C} \longrightarrow \widehat{\mathscr{C}}$ assigning M to $(M, 1_M)$, which makes $\mathscr{C}$ as a full subcategory of $\widehat{\mathscr{C}}$. It is an equivalence if and only if $\mathscr{C}$ is idempotent complete. Since idempotent completions are unique up to equivalence of categories, we shall call $\widehat{\mathscr{C}}$ the idempotent completion of $\mathscr{C}$. Moreover, if $\mathscr{C}$ is an additive category, then so is $\widehat{\mathscr{C}}$.

Remark 2.1 (1) We will write an object M in $\widehat{\mathscr{C}}$ to mean the object $(M, 1_M)$ in $\widehat{\mathscr{C}}$, and write a morphism $f : M \longrightarrow N$ in $\widehat{\mathscr{C}}$ to mean the morphism $f : (M, 1_M) \longrightarrow (N, 1_N)$ if the situation is clear.

(2) It is worthy to notice that the identity $1_{(M,e)}$ of an object (M, e) in $\widehat{\mathscr{C}}$ is the endomorphism $e : (M, e) \longrightarrow (M, e)$. And $1_{(M,e)} = 1_M$ if

and only if $(M, e) = (M, 1_M)$. Moreover, if $e' : (M, e) \longrightarrow (M, e)$ is an idempotent morphism, then e' has an epi-mono factorization as $e' = \iota\pi$ with $\pi\iota = 1_{(M,e')}$, where $\pi = e' : (M, e) \longrightarrow (M, e')$, and $\iota = e' : (M, e') \longrightarrow (M, e)$.

Let $\mathscr{A}$ be an idempotent complete category. Then any functor $F : \mathscr{C} \longrightarrow \mathscr{A}$ can be extended (uniquely up to natural equivalence) to a functor $\widehat{F} : \widehat{\mathscr{C}} \longrightarrow \mathscr{A}$. For an object (M, e), set $\widehat{F}(M, e) = \operatorname{Im} F(e)$. For a morphism $f : (M, e_M) \longrightarrow (N, e_N)$ with $e_M = \iota_M \pi_M$ and $e_N = \iota_N \pi_N$, set $\widehat{F}(f) = F(\pi_N)\, F(f)\, F(\iota_M)$. Moreover, any natural transformation $\theta : F \longrightarrow G$ can be extended uniquely to a natural transformation $\widehat{\theta} : \widehat{F} \longrightarrow \widehat{G}$ by means of $\widehat{\theta}_{(M,e)} = G(\pi_M)\, \theta_M\, F(\iota_M)$.

Let $\mathscr{A}$ be an idempotent complete category, and $F : \mathscr{C} \longrightarrow \mathscr{A}$ be a functor. It is well known that F is fully faithful if and only if $\widehat{F} : \widehat{\mathscr{C}} \longrightarrow \mathscr{A}$ is fully faithful. This result implies that $\widehat{\mathscr{C}}$ is the smallest idempotent complete category containing $\mathscr{C}$ as a full subcategory.

By now, we have all the needed ingredients to introduce the following notion, which is due to Reiten and Riedtmann; see [14, Section 3].

Definition 2.2 Let $\mathscr{C}$ be a G-preadditive category and $\mathscr{C}[G]$ the orbit category. Define the *skew group category* of $\mathscr{C}$ as the idempotent completion $\widehat{\mathscr{C}[G]}$ of the orbit category, which we denote by $\mathscr{C}(G)$ instead of $\widehat{\mathscr{C}[G]}$ for simplicity.

Finally, let us recall the constructions of the two functors $F : \mathscr{C} \longrightarrow \mathscr{C}(G)$ and $H : \mathscr{C}(G) \longrightarrow \mathscr{C}$, which will be helpful to investigate the relationship between $\mathscr{C}$ and $\mathscr{C}(G)$.

From now on, unless otherwise stated, we will always assume that $\mathscr{C}$ is a G-abelian category, and the order $|G|$ is invertible in $\mathscr{C}$, that is, any morphism f in any $\operatorname{Hom}_{\mathscr{C}}(M, N)$ is uniquely divisible by $|G|$. For example, if $\mathscr{C}$ is an abelian category over a field k, then $|G|$ is invertible in k.

The additive functor $F : \mathscr{C} \longrightarrow \mathscr{C}[G]$ is given by $F(M) = M$ for an object M in $\mathscr{C}$. If $f : M \longrightarrow N$ is a morphism in $\mathscr{C}$, $F(f) : M \longrightarrow N$ is defined to be the morphism $\overline{f'}$ in $\mathscr{C}[G]$, which is given by the morphism $\bigoplus_{x\in G} f'_x : \bigoplus_{x\in G} {}^{x}M \longrightarrow N$ in $\mathscr{C}$ with $f'_1 = f$ and other component morphisms $f'_x = 0$ for $x \neq 1$. Compositing F with the embedding functor $\mathbb{i} : \mathscr{C}[G] \longrightarrow \mathscr{C}(G)$, then we get a natural functor $\mathscr{C} \longrightarrow \mathscr{C}(G)$, and also denote by F.

The functor $F : \mathscr{C} \longrightarrow \mathscr{C}[G]$ admits a right adjoint $H : \mathscr{C}[G] \longrightarrow \mathscr{C}$ which is defined as follows. For an object V in $\mathscr{C}[G]$, set $H(V) = \bigoplus_{x\in G} {}^{x}V$. If $\overline{f} : V \longrightarrow W$ is a morphism in $\mathscr{C}[G]$, which is given by a morphism $\bigoplus_{x\in G} f_x : \bigoplus_{x\in G} {}^{x}V \longrightarrow W$ in $\mathscr{C}$, set $H(\overline{f}) : \bigoplus_{x\in G} {}^{x}V \longrightarrow \bigoplus_{y\in G} {}^{y}W$ to be the $|G| \times |G|$ matrix $(f_{y,x})$ of morphisms such that $f_{y,x} : {}^{x}V \longrightarrow {}^{y}W$ is ${}^{y}f_{y^{-1}x}$. Extending H to $\mathscr{C}(G)$, we get a functor $\mathscr{C}(G) \longrightarrow \mathscr{C}$, and also denote by H, which is a right adjoint of the functor $F : \mathscr{C} \longrightarrow \mathscr{C}(G)$.

Put $G = \{x_1 = 1, x_2, \dots, x_n\}$. For the adjoint pair (F, H) on $\mathscr{C}$ and $\mathscr{C}[G]$, the unit $\eta : 1_{\mathscr{C}} \longrightarrow HF$ is defined by $\eta_M : M \longrightarrow HF(M) = {}^{x_1}M \oplus \cdots \oplus {}^{x_n}M$ which sends M to the first coordinate for an object M in $\mathscr{C}$. Moreover, the unit η is a split monomorphism of functors, which has a splitting $\xi : HF \longrightarrow 1_{\mathscr{C}}$ such that $\xi_M : {}^{x_1}M \oplus \cdots \oplus {}^{x_n}M = HF(M) \longrightarrow M$ is the projection to the first summand.

The counit $\overline{\varepsilon} : FH \longrightarrow 1_{\mathscr{C}[G]}$ is defined by $\overline{\varepsilon_V} : {}^{x_1}V \oplus \cdots \oplus {}^{x_n}V = FH(V) \longrightarrow V$ to be the morphism in $\mathscr{C}[G]$ for an object V in $\mathscr{C}[G]$, which is given by the morphism $\bigoplus_{i=1}^{n} {}^{x_i}({}^{x_1}V \oplus \cdots \oplus {}^{x_n}V) \longrightarrow V$ in $\mathscr{C}$, such that the component morphism

$$ {}^{x_i}({}^{x_1}V \oplus \cdots \oplus {}^{x_n}V) \longrightarrow V $$

sends ${}^{x_i}({}^{x_i^{-1}}V)$ to V by the identity for $i = 1, \dots, n$. It is a split epimorphism of functors with a splitting $(1/|G|)\overline{\delta} : 1_{\mathscr{C}[G]} \longrightarrow FH$. The natural transformation $\overline{\delta}$ is defined as follows. For an object V in $\mathscr{C}[G]$, set $\overline{\delta_V} : V \longrightarrow FH(V) = {}^{x_1}V \oplus \cdots \oplus {}^{x_n}V$ to be the morphism in $\mathscr{C}[G]$, which is given by the morphism $\bigoplus_{i=1}^{n} {}^{x_i}V \longrightarrow {}^{x_1}V \oplus \cdots \oplus {}^{x_n}V$ in $\mathscr{C}$, where the component morphism ${}^{x_i}V \longrightarrow {}^{x_1}V \oplus \cdots \oplus {}^{x_n}V$ in $\mathscr{C}$ sends ${}^{x_i}V$ to the ith coordinate for $i = 1, \dots, n$.

Extending this adjunction, we get the split unit $\eta : 1_{\mathscr{C}} \longrightarrow HF$ and the split counit $\overline{\varepsilon} : FH \longrightarrow 1_{\mathscr{C}(G)}$ of the adjoint pair (F, H) on $\mathscr{C}$ and $\mathscr{C}(G)$. Moreover, (H, F) is also an adjoint pair, for details, we refer [14, Section 3, Theorem 3.2].

3. The proof of proposition 1.1

Let $\mathscr{C}$ be a G-abelian category, we will prove Proposition 1.1 under the assumption that $|G|$ is invertible in $\mathscr{C}$. Let us start the proof with the following observation.

Lemma 3.1 *Let* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *be an exact sequence in* $\mathscr{C}$*. Then*

$$0 \longrightarrow FL \xrightarrow{f} FM \xrightarrow{g} FN \longrightarrow 0$$

is also an exact sequence in $\mathscr{C}[G]$*.*

Here, recall that a sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$ in an additive category $\mathscr{D}$ is said to be *left exact* (equivalently, $f : L \longrightarrow M$ is a kernel of $g : M \longrightarrow N$) if for any object X in $\mathscr{D}$, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, L) \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(X,f)} \mathrm{Hom}_{\mathscr{D}}(X, M) \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(X,g)} \mathrm{Hom}_{\mathscr{D}}(X, N)$$

is exact in the category *Ab* of abelian groups.

By duality, a sequence $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is said to be *right exact* (equivalently, $g : M \longrightarrow N$ is a cokernel of $f : L \longrightarrow M$) if for any object Y in $\mathscr{D}$, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}}(N, Y) \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(g,Y)} \mathrm{Hom}_{\mathscr{D}}(M, Y) \xrightarrow{\mathrm{Hom}_{\mathscr{D}}(f,Y)} \mathrm{Hom}_{\mathscr{D}}(L, Y)$$

is exact in *Ab*. A sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is said to be *exact* if it is both left exact and right exact.

Proof. Let X be an object in $\mathscr{C}[G]$, we have the following commutative diagram in *Ab*,

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(HX, L) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(HX,f)} & \mathrm{Hom}_{\mathscr{C}}(HX, M) \\ & \downarrow \wr & & \downarrow \wr \\ 0 \longrightarrow & \mathrm{Hom}_{\mathscr{C}[G]}(X, FL) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}[G]}(X,Ff)} & \mathrm{Hom}_{\mathscr{C}[G]}(X, FM) \end{array}$$

$$\begin{array}{cc} \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(HX,g)} & \mathrm{Hom}_{\mathscr{C}}(HX, N) \\ & \downarrow \wr \\ \xrightarrow{\mathrm{Hom}_{\mathscr{C}[G]}(X,Fg)} & \mathrm{Hom}_{\mathscr{C}[G]}(X, FN)\,, \end{array}$$

where the top row is exact from the left exactness of the sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$, and each column is isomorphic by applying the adjunction of (H, F). Therefore, the bottom row is exact. This gives rise to the left exactness of the sequence $0 \longrightarrow FL \xrightarrow{f} FM \xrightarrow{g} FN$ in $\mathscr{C}[G]$. Dually, the right exactness of $FL \xrightarrow{f} FM \xrightarrow{g} FN \longrightarrow 0$ follows from the right exactness of the sequence $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ by applying the adjunctions of (F, H). □

We need the following result about natural transformations of functors evaluating at split exact sequences, which generalizes the Lemma 20.9 in [1, Chapter 5].

Lemma 3.2 *Let $\mathscr{D}$ and $\mathscr{D}'$ be two additive categories. Let F_1, F_2 and F_3 be three additive functors from $\mathscr{D}$ to $\mathscr{D}'$ with natural transformations $\theta : F_1 \longrightarrow F_2$ and $\theta' : F_2 \longrightarrow F_3$. If $\xi : 0 \longrightarrow X \xrightarrow{i_X} Y \xrightarrow{\pi_Z} Z \longrightarrow 0$ is a split exact sequence in $\mathscr{D}$, then*

$$\zeta_Y : 0 \longrightarrow F_1(Y) \xrightarrow{\theta_Y} F_2(Y) \xrightarrow{\theta'_Y} F_3(Y) \longrightarrow 0$$

is an exact sequence in $\mathscr{D}'$ if and only if both $\zeta_X : 0 \longrightarrow F_1(X) \xrightarrow{\theta_X} F_2(X) \xrightarrow{\theta'_X} F_3(X) \longrightarrow 0$ and $\zeta_Z : 0 \longrightarrow F_1(Z) \xrightarrow{\theta_Z} F_2(Z) \xrightarrow{\theta'_Z} F_3(Z) \longrightarrow 0$ are exact sequences in $\mathscr{D}'$.

Proof. Let π_X be a splitting of i_X, and i_Z a splitting of π_Z such that $i_X\pi_X + i_Z\pi_Z = 1_Y$. Since ξ is split exact in $\mathscr{D}$, then each $0 \longrightarrow F_j(X) \longrightarrow F_j(Y) \longrightarrow F_j(Z) \longrightarrow 0$ is split exact in $\mathscr{D}'$ for $j = 1, 2, 3$. Note that, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1(Y) & \xrightarrow{\theta_Y} & F_2(Y) & \xrightarrow{\theta'_Y} & F_3(Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \binom{F_1(\pi_X)}{F_1(\pi_Z)}} & & \downarrow{\scriptstyle \binom{F_2(\pi_X)}{F_2(\pi_Z)}} & & \downarrow{\scriptstyle \binom{F_3(\pi_X)}{F_3(\pi_Z)}} & & \\
0 & \longrightarrow & F_1(X) \oplus F_1(Z) & \xrightarrow[\left(\begin{smallmatrix}\theta_X & 0 \\ 0 & \theta_Z\end{smallmatrix}\right)]{} & F_2(X) \oplus F_2(Z) & \xrightarrow[\left(\begin{smallmatrix}\theta'_X & 0 \\ 0 & \theta'_Z\end{smallmatrix}\right)]{} & F_3(X) \oplus F_3(Z) & \longrightarrow & 0,
\end{array}$$

where each column is an isomorphism with an inverse $(F_j(i_X)\ F_j(i_Z))$ for $j = 1, 2, 3$. Therefore, the top row is an exact sequence in $\mathscr{D}'$ if and only if so is the bottom row. Denote by $\zeta_X \oplus \zeta_Z$ the bottom row. For any object

E in $\mathscr{D}'$, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_1(X) \oplus F_1(Z)) \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_2(X) \oplus F_2(Z)) \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_3(X) \oplus F_3(Z))$$

is exact in *Ab*, if and only if both sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_1(X)) \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_2(X)) \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_3(X))$$

and

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_1(Z)) \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_2(Z)) \longrightarrow \mathrm{Hom}_{\mathscr{D}'}(E, F_3(Z))$$

are exact in *Ab* by the fact that the Hom functor $\mathrm{Hom}_{\mathscr{D}'}(E, -)$ commutes with the direct sums. That is, we have that the sequence $\zeta_X \oplus \zeta_Z$ is left exact in $\mathscr{D}'$ if and only if both sequences ζ_X and ζ_Z are left exact in $\mathscr{D}'$. Dually, we have the similar conclusion about the right exactness of the sequence $\zeta_X \oplus \zeta_Z$. Thus the result immediately follows. □

Corollary 3.3 *Let* $\zeta : 0 \longrightarrow U \xrightarrow{\overline{f}} V \xrightarrow{\overline{g}} W \longrightarrow 0$ *be an exact sequence in* $\mathscr{C}[G]$. *Then the sequence* ζ *is also an exact sequence in* $\mathscr{C}(G)$.

Proof. It is equivalent to show that ζ is both left exact and right exact in $\mathscr{C}(G)$. We first show ζ is right exact. To this end, let (X, e_X) be an object in $\mathscr{C}(G)$, we have to show that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{C}(G)}((W, 1_W), (X, e_X)) \longrightarrow \mathrm{Hom}_{\mathscr{C}(G)}((V, 1_V), (X, e_X)) \longrightarrow \mathrm{Hom}_{\mathscr{C}(G)}((U, 1_U), (X, e_X))$$

is exact in *Ab*. Since $\mathscr{C}[G]$ is a cover of $\mathscr{C}(G)$, then there exists an object $(X', e_{X'})$ such that $Y = (X, e_X) \oplus (X', e_{X'})$ is an object in $\mathscr{C}[G]$. That is, we have a split exact sequence $0 \longrightarrow (X, e_X) \longrightarrow (Y, 1_Y) \longrightarrow (X', e_{X'}) \longrightarrow 0$ in $\mathscr{C}(G)$. Set $F_1 = \mathrm{Hom}_{\mathscr{C}(G)}((W, 1_W), -)$, $F_2 = \mathrm{Hom}_{\mathscr{C}(G)}((V, 1_V), -)$, and $F_3 = \mathrm{Hom}_{\mathscr{C}(G)}((U, 1_U), -)$. Observe that the sequence $F_1((Y, 1_Y)) \longrightarrow F_2((Y, 1_Y)) \longrightarrow F_3((Y, 1_Y)) \longrightarrow 0$ is just the sequence

$$\delta : 0 \longrightarrow \mathrm{Hom}_{\mathscr{C}[G]}(W,Y) \xrightarrow{\mathrm{Hom}_{\mathscr{C}[G]}(\overline{g},Y)} \mathrm{Hom}_{\mathscr{C}[G]}(V,Y) \xrightarrow{\mathrm{Hom}_{\mathscr{C}[G]}(\overline{f},Y)} \mathrm{Hom}_{\mathscr{C}[G]}(U,Y)$$

by the fact that $\mathscr{C}[G]$ is a full subcategory of $\mathscr{C}(G)$. It follows that the sequence δ is exact in Ab from the assumption that ζ is exact in $\mathscr{C}[G]$. Now, we are in the setting of Lemma 3.2. The right exactness of the sequence ζ in $\mathscr{C}(G)$ immediately follows. The left exactness of of the sequence ζ in $\mathscr{C}(G)$ can be obtained by duality. □

Lemma 3.4 *If every morphism $\overline{f} : V \longrightarrow W$ in $\mathscr{C}[G]$ has a kernel (resp. a cokernel) in $\mathscr{C}(G)$, then every morphism in $\mathscr{C}(G)$ has a kernel (resp. a cokernel).*

Proof. Let $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$ be a morphism in $\mathscr{C}(G)$. Let $\overline{k} : (K, \overline{e_K}) \longrightarrow (V, \overline{1_V})$ be a kernel of the morphism $\overline{f} : V \longrightarrow W$ in $\mathscr{C}(G)$ by the assumption. Then there exists a unique morphism $\overline{e_K}' : (K, \overline{e_K}) \longrightarrow (K, \overline{e_K})$ such that the following diagram

$$\begin{array}{ccccc}
(K,\overline{e_K}) & \xrightarrow{\overline{k}} & (V,\overline{1_V}) & \xrightarrow{\overline{f}} & (W,\overline{1_W}) \\
\Big\downarrow{\scriptstyle \overline{e_K}'} & & \Big\downarrow{\scriptstyle \overline{e_V}} & & \Big\downarrow{\scriptstyle \overline{e_W}} \\
(K,\overline{e_K}) & \xrightarrow{\overline{k}} & (V,\overline{1_V}) & \xrightarrow{\overline{f}} & (W,\overline{1_W})
\end{array}$$

commutes in $\mathscr{C}(G)$. Therefore, we have $\overline{k} \cdot \overline{e_K}' \cdot \overline{e_K}' = \overline{e_V} \cdot \overline{e_V} \cdot \overline{k} = \overline{e_V} \cdot \overline{k} = \overline{k} \cdot \overline{e_K}'$. This immediately yields that $\overline{e_K}'$ is an idempotent on $(K, \overline{e_K})$ by the fact that $\overline{k}$ is monic in $\mathscr{C}(G)$, where $\overline{e_K}' \cdot \overline{e_K} = \overline{e_K}' = \overline{e_K} \cdot \overline{e_K}'$.

We now claim that $\overline{k} \cdot \overline{e_K}' : (K, \overline{e_K}') \longrightarrow (V, \overline{e_V})$ is a kernel of $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$ in $\mathscr{C}(G)$. In fact, let $\overline{h} : (X, \overline{e_X}) \longrightarrow (V, \overline{e_V})$ be a morphism in $\mathscr{C}(G)$ such that $\overline{f} \cdot \overline{h} = 0$. Then $(X, \overline{e_X}) \xrightarrow{\overline{h}} (V, \overline{e_V}) \xrightarrow{\overline{e_V}} (V, \overline{1_V}) \xrightarrow{\overline{f}} (W, \overline{1_W}) = 0$. Since $\overline{k} : (K, \overline{e_K}) \longrightarrow (V, \overline{1_V})$ is a kernel of the morphism $\overline{f} : V \longrightarrow W$ in $\mathscr{C}(G)$, then there exists a unique morphism $\overline{\alpha} : (X, \overline{e_X}) \longrightarrow (K, \overline{e_K})$ such that $\overline{e_V} \cdot \overline{h} = \overline{k} \cdot \overline{\alpha}$.

Let $\overline{\beta} : (X, \overline{e_X}) \longrightarrow (K, \overline{e_K}') = (X, \overline{e_X}) \xrightarrow{\overline{\alpha}} (K, \overline{e_K}) \xrightarrow{\overline{e_K}'} (K, \overline{e_K}')$, then we have that $\overline{h} = (\overline{k} \cdot \overline{e_K}') \cdot \overline{\beta}$. That is, the morphism $\overline{h} : (X, \overline{e_X}) \longrightarrow (V, \overline{e_V})$

in $\mathscr{C}(G)$ with $\overline{f} \cdot \overline{h} = 0$ can factor through $\overline{k} \cdot \overline{e_K}' : (K, \overline{e_K}') \longrightarrow (V, \overline{e_V})$.

Finally, we have to show that the factorization of $\overline{h}$ through $\overline{k} \cdot \overline{e_K}'$ is unique. Assume that $\overline{\gamma} : (X, \overline{e_X}) \longrightarrow (K, \overline{e_K}')$ is a morphism in $\mathscr{C}(G)$ such that $\overline{h} = (\overline{k} \cdot \overline{e_K}') \cdot \overline{\gamma}$. It is easy to verify that $(X, \overline{e_X}) \xrightarrow{\overline{\gamma}} (K, \overline{e_K}') \xrightarrow{\overline{e_K}'} (K, \overline{e_K})$ satisfying that $\overline{e_V} \cdot \overline{h} = \overline{k} \cdot \overline{\alpha} = \overline{k} \cdot (\overline{e_K}' \cdot \overline{\gamma})$. This yields that $\overline{e_K}' \cdot \overline{\gamma} = \overline{\alpha}$ since $\overline{k} : (K, \overline{e_K}) \longrightarrow (V, \overline{1_V})$ is a kernel of the morphism $\overline{f} : V \longrightarrow W$ in $\mathscr{C}(G)$. Immediately, we can conclude that $\overline{\gamma} = \overline{\beta}$ by composing with the split epimorphism $\overline{e_K}' : (K, \overline{e_K}) \longrightarrow (K, \overline{e_K}')$ on $\overline{e_K}' \cdot \overline{\gamma} = \overline{\alpha}$. Hence, $\overline{k} \cdot \overline{e_K}' : (K, \overline{e_K}') \longrightarrow (V, \overline{e_V})$ is a kernel of $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$ in $\mathscr{C}(G)$.

Dually, by switching from $\mathscr{C}$ to the opposite category $\mathscr{C}^{op}$, the statement about cokernels immediately follows. □

Lemma 3.5 *Any morphism $\overline{f} : V \longrightarrow W$ in $\mathscr{C}[G]$ has a kernel and a cokernel in $\mathscr{C}(G)$.*

Proof. First, we know that the morphism $H(\overline{f}) : H(V) \longrightarrow H(W)$ has a kernel $k : K \longrightarrow H(V)$ in $\mathscr{C}$. This gives rise to that $F(k) : F(K) \longrightarrow FH(V)$ is a kernel of $FH(\overline{f}) : FH(V) \longrightarrow FH(W)$ in $\mathscr{C}[G]$ by Lemma 3.1. Hence, it follows that $F(k) : F(K) \longrightarrow FH(V)$ is a kernel of $FH(\overline{f}) : FH(V) \longrightarrow FH(W)$ in $\mathscr{C}(G)$ by Corollary 3.3. Note that the counit $\overline{\varepsilon} : FH \longrightarrow 1_{\mathscr{C}[G]}$ of (F, H) is a split epimorphism of functors with a splitting $(1/|G|)\overline{\delta} : 1_{\mathscr{C}[G]} \longrightarrow FH$. Consider the following commutative diagram

$$\begin{array}{ccccc}
F(K) & \xrightarrow{F(k)} & FH(V) & \xrightarrow{FH(\overline{f})} & FH(W) \\
\vdots & & \downarrow{\scriptstyle \overline{\varepsilon}_V} & & \downarrow{\scriptstyle \overline{\varepsilon}_W} \\
\vdots\ {\scriptstyle \overline{e_{F(K)}}} & & V & \xrightarrow{\overline{f}} & W \\
\vdots & & \downarrow{\scriptstyle \frac{1}{|G|}\overline{\delta}_V} & & \downarrow{\scriptstyle \frac{1}{|G|}\overline{\delta}_W} \\
F(K) & \xrightarrow{F(k)} & FH(V) & \xrightarrow{FH(\overline{f})} & FH(W)
\end{array}$$

in $\mathscr{C}[G]$. Immediately, we have $\overline{e_{FH(V)}} = (1/|G|)\overline{\delta}_V \cdot \overline{\varepsilon}_V$ and $\overline{e_{FH(W)}} = (1/|G|)\overline{\delta}_W \cdot \overline{\varepsilon}_W$ are idempotent morphisms in $\mathscr{C}[G]$. This gives rise to an idempotent morphism $\overline{e_{F(K)}} : F(K) \longrightarrow F(K)$ in $\mathscr{C}[G]$. Since $\mathscr{C}(G)$ is idempotent complete, then the idempotent morphism $\overline{e_{F(K)}} : F(K) \longrightarrow F(K)$ is the composition of a split epimorphism $\overline{\pi} = \overline{e_{F(K)}} : (F(K), \overline{1_{F(K)}}) \longrightarrow$

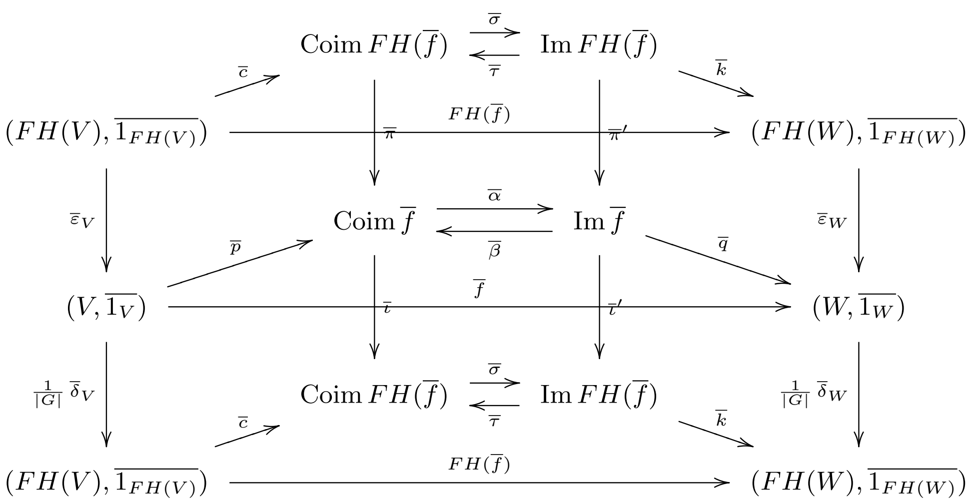
$(F(K), \overline{e_{F(K)}})$ and a split monomorphism $\overline{\iota} = \overline{e_{F(K)}} : (F(K), \overline{e_{F(K)}}) \longrightarrow (F(K), \overline{1_{F(K)}})$ in $\mathscr{C}(G)$.

Now, we claim that $\overline{\varepsilon}_V \cdot F(k) \cdot \overline{\iota} : (F(K), \overline{e_{F(K)}}) \longrightarrow (V, 1_V)$ is a kernel of $\overline{f} : V \longrightarrow W$ in $\mathscr{C}(G)$. In fact, let $\overline{h} : (Y, \overline{e_Y}) \longrightarrow (V, 1_V)$ be a morphism in $\mathscr{C}(G)$ with $\overline{f} \cdot \overline{h} = 0$. Then we have that $FH(\overline{f}) \cdot (1/|G|)\overline{\delta}_V \cdot \overline{h} = (1/|G|)\overline{\delta}_W \cdot \overline{f} \cdot \overline{h} = 0$. Since $F(k) : F(K) \longrightarrow FH(V)$ is a kernel of $FH(\overline{f}) : FH(V) \longrightarrow FH(W)$ in $\mathscr{C}(G)$, then there exists a morphism $\overline{\alpha} : (Y, \overline{e_Y}) \longrightarrow (F(K), \overline{1_{F(K)}})$ such that $(1/|G|)\overline{\delta}_V \cdot \overline{h} = F(k) \cdot \overline{\alpha}$. Define $\overline{\beta} : (Y, \overline{e_Y}) \longrightarrow (F(K), \overline{e_{F(K)}})$ to be the composition of $\overline{\alpha} : (Y, \overline{e_Y}) \longrightarrow (F(K), \overline{1_{F(K)}})$ and $\overline{\pi} : (F(K), \overline{1_{F(K)}}) \longrightarrow (F(K), \overline{e_{F(K)}})$. It follows that $\overline{h} = (\overline{\varepsilon}_V \cdot F(k) \cdot \overline{\iota}) \cdot \overline{\beta}$ by a direct verification. Suppose that there is a morphism $\overline{\gamma} : (Y, \overline{e_Y}) \longrightarrow (F(K), \overline{e_{F(K)}})$ in $\mathscr{C}(G)$ such that $\overline{h} = (\overline{\varepsilon}_V \cdot F(k) \cdot \overline{\iota}) \cdot \overline{\gamma}$. Then the composed morphism $\overline{\iota} \cdot \overline{\gamma} : (Y, \overline{e_Y}) \longrightarrow (F(K), \overline{1_{F(K)}})$ is just the morphism $\overline{\alpha} : (Y, \overline{e_Y}) \longrightarrow (F(K), \overline{1_{F(K)}})$ such that $(1/|G|)\overline{\delta}_V \cdot \overline{h} = F(k) \cdot \overline{\alpha}$. This yields that $\overline{\gamma} = \overline{\pi} \cdot \overline{\iota} \cdot \overline{\gamma} = \overline{\pi} \cdot \overline{\alpha} = \overline{\beta}$. We have shown that the factorization of $\overline{h}$ through $\overline{\varepsilon}_V \cdot F(k) \cdot \overline{\iota}$ in $\mathscr{C}(G)$ is unique. Hence, $\overline{\varepsilon}_V \cdot F(k) \cdot \overline{\iota} : (F(K), \overline{e_{F(K)}}) \longrightarrow (V, 1_V)$ is a kernel of $\overline{f} : V \longrightarrow W$ in $\mathscr{C}(G)$. The existence of a cokernel can be proved by duality. □

Lemma 3.6 *Let $\overline{f} : V \longrightarrow W$ be a morphism in $\mathscr{C}[G]$. Then* $\operatorname{Coim} \overline{f}$ *and* $\operatorname{Im} \overline{f}$ *exist in $\mathscr{C}(G)$. Moreover, The canonical factorization $(V, \overline{1_V}) \longrightarrow \operatorname{Coim} \overline{f} \longrightarrow \operatorname{Im} \overline{f} \longrightarrow (W, \overline{1_W})$ of $\overline{f}$ in $\mathscr{C}(G)$ induces an isomorphism of* $\operatorname{Coim} \overline{f}$ *and* $\operatorname{Im} \overline{f}$.

Here, for a morphism $f : X \longrightarrow Y$ in an additive category $\mathscr{D}$ with a kernel $k : K \longrightarrow X$ and a cokernel $c : Y \longrightarrow C$. If $k : K \longrightarrow X$ has a cokernel in $\mathscr{D}$, define the *coimage* $\operatorname{Coim} f$ of f as $\operatorname{Coker} k$. If $c : Y \longrightarrow C$ has a kernel, define the *image* $\operatorname{Im} f$ of f as $\operatorname{Ker} c$. In this case, the morphism $f : X \longrightarrow Y$ has a natural *canonical factorization* $X \longrightarrow \operatorname{Coim} f \overset{\delta}{\longrightarrow} \operatorname{Im} f \longrightarrow Y$, where the morphism $\delta : \operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is called the *induced morphism* of the canonical factorization.

Proof. Since $\overline{f} : V \longrightarrow W$ has a kernel and a cokernel in $\mathscr{C}(G)$ by Lemma 3.5. Then we can form the $\operatorname{Coim} \overline{f}$ and the $\operatorname{Im} \overline{f}$ of the morphism $\overline{f} : V \longrightarrow W$ in $\mathscr{C}(G)$ by carrying a similar proof of Lemma 3.5. That is, we have the following commutative diagram



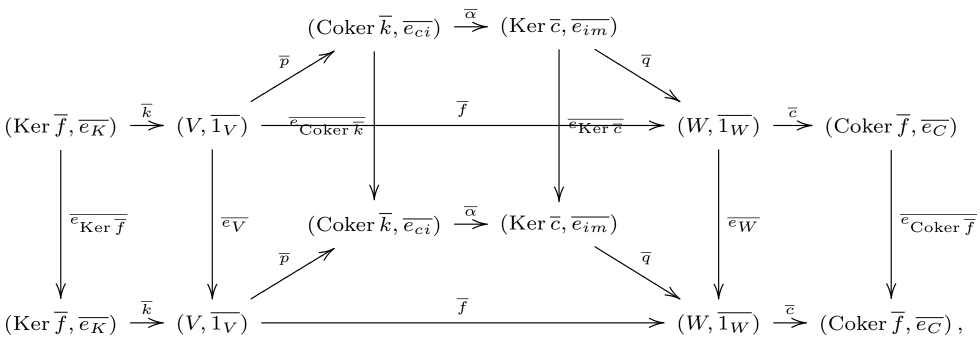
in $\mathscr{C}(G)$, where $\overline{p} = \overline{\pi} \cdot \overline{c} \cdot (1/|G|)\overline{\delta}_V$ and $\overline{q} = \overline{\varepsilon}_W \cdot \overline{k} \cdot \overline{\iota}'$; $\overline{\pi} \cdot \overline{\iota} = \overline{1_{\mathrm{Coim}\,\overline{f}}}$ and $\overline{\pi}' \cdot \overline{\iota}' = \overline{1_{\mathrm{Im}\,\overline{f}}}$; the morphism $\overline{\sigma}$ is the isomorphism induced by the canonical factorization of $FH(\overline{f})$ with an inverse $\overline{\tau}$.

Define $\overline{\alpha} = \overline{\pi}' \cdot \overline{\sigma} \cdot \overline{\iota}$ and $\overline{\beta} = \overline{\pi} \cdot \overline{\tau} \cdot \overline{\iota}'$, then $\overline{f} = \overline{q} \cdot \overline{\alpha} \cdot \overline{p}$ which means that $\overline{\alpha}$ is the induced morphism of the canonical factorization of $\overline{f}$ in $\mathscr{C}(G)$. Moreover, It follows that $\overline{\sigma}$ and $\overline{\beta}$ are inverses of each other from a direct verification. Hence, the canonical factorization of $\overline{f}$ in $\mathscr{C}(G)$ induces an isomorphism $\overline{\alpha} : \mathrm{Coim}\,\overline{f} \longrightarrow \mathrm{Im}\,\overline{f}$. □

Proof of Proposition 1.1. To prove that $\mathscr{C}(G)$ is an abelian category, we have to show that, any morphism $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$ in $\mathscr{C}(G)$ has a kernel and a cokernel, and the canonical factorization induces an isomorphism of Coim $\overline{f}$ and Im $\overline{f}$.

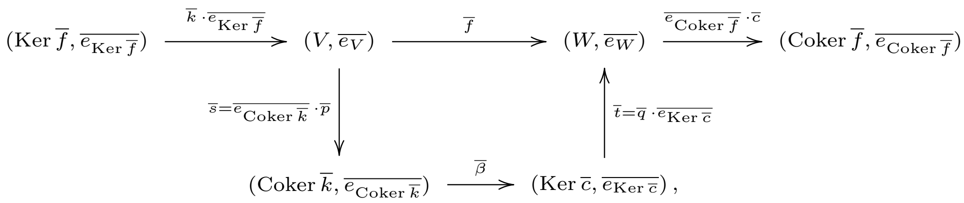
By Lemma 3.5 and Lemma 3.4, we can conclude that any morphism in $\mathscr{C}(G)$ has a kernel and a cokernel. Therefore, the coimage and the image of the morphism $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$ exist in $\mathscr{C}(G)$.

Now, we show that for any morphism $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$, whose canonical factorization induces an isomorphism of the coimage and image. To this end, consider the following commutative diagram with the canonical factorization of $\overline{f} : (V, \overline{1_V}) \longrightarrow (W, \overline{1_W})$ in $\mathscr{C}(G)$:



where $\overline{\alpha}$ is the isomorphism from Lemma 3.6.

Denote by ι the split monomorphism $\overline{e_{\mathrm{Coker}\,\overline{k}}} : (\mathrm{Coker}\,\overline{k}, \overline{e_{\mathrm{Coker}\,\overline{k}}}) \longrightarrow (\mathrm{Coker}\,\overline{k}, \overline{e_{ci}})$ and by π' the split epimorphism $\overline{e_{\mathrm{Ker}\,\overline{c}}} : (\mathrm{Ker}\,\overline{c}, \overline{e_{im}}) \longrightarrow (\mathrm{Ker}\,\overline{c}, \overline{e_{\mathrm{Ker}\,\overline{c}}})$. Set $\overline{\beta} : (\mathrm{Coker}\,\overline{k}, \overline{e_{\mathrm{Coker}\,\overline{k}}}) \longrightarrow (\mathrm{Ker}\,\overline{c}, \overline{e_{\mathrm{Ker}\,\overline{c}}})$ be the composed morphism $\pi' \cdot \overline{\alpha} \cdot \iota$. By Lemma 3.4 and carrying a similar procedure of the proof for Lemma 3.6, we have the canonical factorization of $\overline{f} : (V, \overline{e_V}) \longrightarrow (W, \overline{e_W})$ in $\mathscr{C}(G)$ as follows



where $\overline{\beta}$ is an isomorphism in $\mathscr{C}(G)$ since $\overline{\alpha}$ is an isomorphism. We have completed the proof of Proposition 1.1. □

Corollary 3.7 *Let $\mathscr{C}$ be a G-abelian category with the order $|G| = n$ invertible in $\mathscr{C}$. Then $F : \mathscr{C} \longrightarrow \mathscr{C}(G)$ and $H : \mathscr{C}(G) \longrightarrow \mathscr{C}$ are exact functors between abelian categories.*

Proof. Combine Lemma 3.1 and Corollary 3.3, we know that F is an exact functor. Note that F is a left adjoint and also a right adjoint of H, it follows that the additive functor H preserves kernels and cokernels from [11, Chapter V, Section 5, Theorem 1] and its dual version. Hence, H is also an exact fucntor. □

Corollary 3.8 *Let $\mathscr{C}$ be a Hom-finite abelian k-category, G a finite group* *acting on $\mathscr{C}$ with the order $|G|$ invertible in $\mathscr{C}$. Then $\mathscr{C}(G)$ is also a Hom-finite abelian k-category, and hence a Krull-Schmidt abelian category.*

Proof. Let $(V, \overline{e_V})$ and $(W, \overline{e_W})$ be two objects in $\mathscr{C}(G)$. $\mathrm{Hom}_{\mathscr{C}(G)}((V, \overline{e_V}), (W, \overline{e_W}))$ is the subvector space of $\mathrm{Hom}_{\mathscr{C}[G]}(V, W)$ which consists of morphisms $\overline{f} : V \longrightarrow W$ subject to $\overline{f} \cdot \overline{e_V} = \overline{f} = \overline{e_W} \cdot \overline{f}$. Since $\mathrm{Hom}_{\mathscr{C}[G]}(V, W)$ is a finite direct sum of $\bigoplus_{x \in G} \mathrm{Hom}_{\mathscr{C}}({}^xM, N)$ of finite dimensional vector spaces by the assumption that $\mathscr{C}$ is a Hom-finite k-category, then $\mathscr{C}(G)$ is also a Hom-finite k-category. The abelianness of $\mathscr{C}(G)$ follows from Proposition 1.1 since $\mathscr{C}$ is an abelian category.

It is well known that a Hom-finite abelian category is Krull-Schmidt; see [15, p. 52] also [6, Appendix, Remark A.2]. Then we have completed the proof of the corollary. □

Remark 3.9 Let G be a finite group, and $\mathscr{C}$ a preadditive category. An action of G on $\mathscr{C}$ is called to be *free* if ${}^xM = M$ for an object M in $\mathscr{C}$, then $x = 1$. Let $\mathscr{C}$ be a free G-preadditive category over a commutative ring R, that is, $\mathscr{C}$ is a preadditive category over a commutative ring R with a free action of G on $\mathscr{C}$. In this case, the orbit category $\mathscr{C}[G]$ is equivalent to the quotient category $\mathscr{C}/G$ defined in [7, Definition 2.1]; see [7, Theorem 2.8]. We are interested in the following question which is presented by the referee: if the action of a group on an abelian category is free, is it true that the orbit category is abelian? We thank the referee for this question.

4. Example of a non-abelian category $\mathscr{C}(G)$

Let $\mathscr{C}$ be a G-abelian category, we have proven that the skew group category $\mathscr{C}(G)$ is an abelian category under the assumption that the order $|G|$ is invertible in $\mathscr{C}$. This condition seems to be a usual assumption, which can be traced back to the study of relationships between the module categories of an artin algebra A and the skew group algebra AG in [14]. However, one might ask that whether the skew group category $\mathscr{C}(G)$ an abelian category or not when the order $|G|$ is not invertible in $\mathscr{C}$?

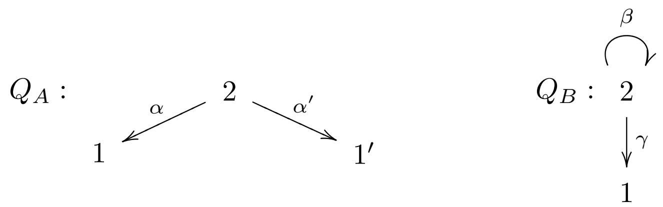
In this section, we consider a finite dimensional k-algebra A of the Dynkin type $\mathbb{A}_3$ with an action of a cyclic group G of order 2 on the quiver Q_A, where k is an algebraically closed field with the characteristic $\mathrm{char}\, k = 2$. The condition $\mathrm{char}\, k = 2$ means that the order $|G|$ is not invertible in $\mathrm{mod}\, A$. Then, we show that the skew group category $(\mathrm{mod}\, A)(G)$ of the finitely generated module category $\operatorname{mod} A$ is not an abelian category.

Before giving the example, let us recall the definition of skew group algebras; see [14, Introduction] and [13, Chapter 1, 1.4]. Let k be a field, and A a finite dimensional k-algebra with a finite group G acting on A. The skew group algebra AG is defined as a free left module $\bigoplus_{x \in G} Ax$ with the basis G, and the multiplication is defined by

$$(ax)(by) = a\, {}^{x}b\, xy$$

for $a, b \in A$ and $x, y \in G$. There is a natural algebra monomorphism $i : A \longrightarrow AG$ by $i(a) = a\, 1_G$ with 1_G the identity of G. Then we have the tensor functor $F = - \bigotimes_A AG : \operatorname{mod} A \longrightarrow \operatorname{mod} AG$, which admits the restriction functor $H : \operatorname{mod} AG \longrightarrow \operatorname{mod} A$ both as a right adjoint and as a left adjoint, we refer to [14, Section 3] for more details.

Example 4.1 Let k be an algebraically closed field with the characteristic $\operatorname{char} k = 2$, A a finite dimensional k-algebra given by the Dynkin quiver Q_A.



Let $G = \{1, x\}$ be a cyclic group of order 2 with a generator x, which acts on A by $x(e_1) = e_{1'}$, $x(e_{1'}) = e_1$, $x(\alpha) = \alpha'$, $x(\alpha') = \alpha$ and $x(e_2) = e_2$. Then the skew group algebra AG is Morita equivalent to a basic finite dimensional k-algebra B given by quiver Q_B with the relation $\beta^2 = 0$. Put $\mathscr{C} = \operatorname{mod} A$, we can form the skew group category $\mathscr{C}(G)$, which is a non-abelian full subcategory of $\operatorname{mod} AG$.

Proof. For computing the quiver Q_B, we refer the reader to [14, Section 2], [2, Example 9.1] and [4, Chapter II, Section 3]. The Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$ is as follows:

$$
\begin{array}{ccccccccc}
P(1') = \begin{smallmatrix} & 0 & \\ 0 & & 1 \end{smallmatrix} & & & & I(1) = \begin{smallmatrix} & 1 & \\ 1 & & 0 \end{smallmatrix} & & \\
& \searrow & & \nearrow & & \searrow & \\
& & P(2) = \begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix} & & & & I(2) = \begin{smallmatrix} & 1 & \\ 0 & & 0 \end{smallmatrix}. \\
& \nearrow & & \searrow & & \nearrow & \\
P(1) = \begin{smallmatrix} & 0 & \\ 1 & & 0 \end{smallmatrix} & & & & I(1') = \begin{smallmatrix} & 1 & \\ 0 & & 1 \end{smallmatrix} & &
\end{array}
$$

The Auslander-Reiten quiver $\Gamma(\operatorname{mod} B)$ is given by the following:

$$
\begin{array}{ccccccccc}
P(1) = \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} & & & & I(1) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} & & & & \\
& \searrow & & \nearrow & & \searrow & & & \\
& & P(2) = \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} & & & & I(2) = \begin{smallmatrix} 2 \\ 0 \end{smallmatrix} & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
N = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & & & & M = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} & & & & S(2) = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}, \\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & S(2) = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} & & & & N = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} & &
\end{array}
$$

where we identify the two copies of $S(2)$ as one vertex, and also identify the two copies of N as one vertex in $\Gamma(\operatorname{mod} B)$.

It is easy to verify that $F(P(1)) = F(P(1'))$, $F(P(2))$, and $F(I(1)) = F(I(1'))$, $F(I(2))$ are the projective modules $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$, $\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$, and the injective modules $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$, $\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}$ in $\operatorname{mod} AG$ respectively, under Morita equivalence.

For the tensor functor $F = -\bigotimes_A AG : \operatorname{mod} A \longrightarrow \operatorname{mod} AG$, denote by $\operatorname{Im} F$ the full subcategory of $\operatorname{mod} AG$ consisting of modules that is isomorphic to $F(M)$ for some module M in $\operatorname{mod} A$. Then the orbit category $\mathscr{C}[G]$ is equivalent to $\operatorname{Im} F$ as we have mentioned in the proof of Remark 3.9; see also [14, p. 255]. Note that, all the modules in $\operatorname{mod} A$ are either projective or injective. Since F preserves projective and injective modules, then $\mathscr{C}[G] = \mathscr{C}(G)$ is the full subcategory of $\operatorname{mod} AG$ consisting of all finitely generated projective modules, injective modules, and the finite direct sums of projective and injective modules.

Now, if we suppose that $\mathscr{C}(G)$ is an abelian category. Consider the minimal projective resolution $0 \longrightarrow P(1) \xrightarrow{f} P(2) \longrightarrow M \longrightarrow 0$ of the B-module M, that is,

$$
\begin{array}{ccccc}
\overset{0}{\circlearrowright} & & \overset{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\circlearrowright} & & \overset{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\circlearrowright} \\
0 & \xrightarrow[0]{} & k\oplus k & \xrightarrow[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)]{} & k\oplus k \\
{\scriptstyle 0}\big\downarrow & & {\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\big\downarrow & & \big\downarrow{\scriptstyle (1\ 0)} \\
k & \xrightarrow[\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)]{} & k\oplus k & \xrightarrow[(1\ 0)]{} & k
\end{array}
$$

where the left hand square is the B-module monomorphism $f : P(1) \longrightarrow P(2)$. Since the skew group category $\mathscr{C}(G)$ is a full subcategory of $\operatorname{mod} B$, then $f : P(1) \longrightarrow P(2)$ is a monomorphism in $\mathscr{C}(G)$. But it is not an epimorphism by the fact that $P(1)$ is not isomorphism to $P(2)$ in $\mathscr{C}(G)$. Let Q be the object in $\mathscr{C}(G)$ such that $0 \longrightarrow P(1) \xrightarrow{f} P(2) \longrightarrow Q \longrightarrow 0$ is an exact sequence in $\mathscr{C}(G)$. Then we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P(1) & \xrightarrow{f} & P(2) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & P(1) & \xrightarrow{f} & P(2) & \longrightarrow & Q & &
\end{array}
$$

in $\operatorname{mod} B$. From the Auslander-Reiten quiver $\Gamma(\operatorname{mod} B)$, it follows that $\dim \operatorname{Hom}_B(M, I(2)) = 1$, $\dim \operatorname{Hom}_B(M, I(1)) = 0$, $\dim \operatorname{Hom}_B(M, P(1)) = 0$ and $\dim \operatorname{Hom}_B(P(2), P(1)) = 0$. This implies that the object Q must contain the B-module $I(2)$ as a direct summand and not contain the B-module $P(1)$ as a direct summand. Thus, we get a contradiction to the fact that $\dim P(2) = \dim P(1) + \dim Q$ by applying the exact functor $H : \mathscr{C}(G) \longrightarrow \mathscr{C}$ (see Corollary 3.7) on the exact sequence $0 \longrightarrow P(1) \xrightarrow{f} P(2) \longrightarrow Q \longrightarrow 0$ in $\mathscr{C}(G)$. Therefore, we can conclude that $\mathscr{C}[G] = \mathscr{C}(G)$ is not an abelian category. $\square$

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