Finite groups with some nonnormal subgroups
of non-prime-power order

Jiangtao SHI, Cui ZHANG and Dengfeng LIANG

Abstract. We prove that a group G with exactly three classes of nonnormal proper subgroups of the same non-prime-power order is nonsolvable if and only if $G \cong A_5$, and a group G with exactly four classes of nonnormal proper subgroups of the same non-prime-power order is nonsolvable if and only if $G \cong PSL(2, 7)$ or $PSL(2, 8)$. Moreover, we prove that any group G with at most nine classes of nonnormal nontrivial subgroups of the same order is always solvable except for $G \cong A_5$, $PSL_2(7)$ or $SL_2(5)$.

Key words: nonsolvable group, class of the same order, nonnormal, non-prime-power order.

1. Introduction

In this paper all groups are considered to be finite. In [1], Gallian and Moulton gave a complete classification of minimal non-prime-power order groups. As a direct corollary of [2, Theorems 4.2 and 4.4], any group G with at most three conjugacy classes of proper subgroups of non-prime-power order is always solvable except for $G \cong A_5$. Note that any two conjugate subgroups must have the same order but any two subgroups of the same order might not be conjugate. As a direct corollary of [3, Theorems 3.1 and 3.2], the following result holds:

Theorem 1.1 *Any group with at most two classes of proper subgroups of the same non-prime-power order is always solvable, and a group G with exactly three classes of proper subgroups of the same non-prime-power order is nonsolvable if and only if $G \cong A_5$.*

In [4], we extended the above results:

Theorem 1.2 ([4])   *A group G with exactly four conjugacy classes of proper subgroups of non-prime-power order is nonsolvable if and only if $G \cong PSL_2(8)$, and a group G with exactly four classes of proper subgroups of the same non-prime-power order is nonsolvable if and only if $G \cong PSL_2(7)$ or $PSL_2(8)$.*

In order to further extend the above results, we only consider the non-normal proper subgroups of non-prime-power order. [5, Lemma 3.1] shows that any group with at most two classes of nonnormal nonnilpotent proper subgroups of the same order is always solvable. As a direct corollary of [5, Lemma 3.1], the following result holds:

Theorem 1.3   *Any group with at most two classes of nonnormal proper subgroups of the same non-prime-power order is always solvable.*

In this paper, our main results are as follows, the proofs of which are given in Section 2 and Section 3 respectively.

Theorem 1.4   *Suppose that G is a nonsolvable group with exactly three classes of nonnormal proper subgroups of the same non-prime-power order, then $G \cong A_5$.*

Theorem 1.5   *Suppose that G is a nonsolvable group with exactly four classes of nonnormal proper subgroups of the same non-prime-power order, then $G \cong PSL_2(7)$ or $PSL_2(8)$.*

By considering the number of classes of nonnilpotent proper subgroups of the same order and the number of classes of nontrivial nilpotent subgroups of the same order in [3], and by considering the number of classes of proper subgroups of the same non-prime-power order and the number of classes of nontrivial subgroups of the same prime-power order in [4], we all obtained the following result.

Theorem 1.6 ([3], [4])   *Let G be a group.*

(1) *Suppose that G has at most six classes of nontrivial subgroups of the same order, then G is solvable.*
(2) *Suppose that G is a nonsolvable group with exactly seven classes of nontrivial subgroups of the same order, then $G \cong A_5$.*
(3) *Suppose that G has exactly eight classes of nontrivial subgroups of the same order, then G is solvable.*

(4) *Suppose that G is a nonsolvable group with exactly nine classes of nontrivial subgroups of the same order, then $G \cong PSL_2(7)$.*

As an extension of Theorem 1.6, only consider the number of classes of nonnormal nontrivial subgroups of the same order, we have the following result, the proof of which is given in Section 4.

Theorem 1.7 *Let G be a group.*

(1) *Suppose that G has at most six classes of nonnormal nontrivial subgroups of the same order, then G is solvable.*
(2) *Suppose that G is a nonsolvable group with exactly seven classes of nonnormal nontrivial subgroups of the same order, then $G \cong A_5$.*
(3) *Suppose that G has exactly eight classes of nonnormal nontrivial subgroups of the same order, then G is solvable.*
(4) *Suppose that G is a nonsolvable group with exactly nine classes of nonnormal nontrivial subgroups of the same order, then $G \cong PSL_2(7)$ or $SL_2(5)$.*

2. Proof of Theorem 1.4

Lemma 2.1 *Suppose that G is a nonsolvable group with exactly three classes of nonnormal nonnilpotent proper subgroups of the same order, then $G \cong A_5$ or $SL_2(5)$.*

Proof. Let H be a minimal nonsolvable subgroup of G. Then $H/\Phi(H)$ is a minimal nonabelian simple group. It is easy to see that H has no normal nonnilpotent proper subgroups. By the hypothesis and [3, Theorem 3.1], H has exactly three classes of nonnilpotent proper subgroups of the same order. Therefore, by [3, Theorem 3.2], one has $H \cong A_5$ or $SL_2(5)$.

We claim that $G = H$.

Otherwise, assume that $H < G$. By the hypothesis, one must have $H \trianglelefteq G$. It follows that $\Phi(H) \trianglelefteq G$ since $\Phi(H)$ char H. Consider the quotient group $G/\Phi(H)$. Let $N/\Phi(H)$ be a subgroup of $G/\Phi(H)$ such that $H/\Phi(H)$ is a maximal subgroup of $N/\Phi(H)$. Let $H/\Phi(H) = \bar{H}$ and $N/\Phi(H) = \bar{N}$. Obviously, $\bar{N}/\bar{H}$ is a cyclic group of prime order. Assume that $\bar{N}/\bar{H} \cong \mathbb{Z}_p$ for some prime p. Obviously, $C_{\bar{N}}(\bar{H}) \cap \bar{H} \trianglelefteq \bar{H}$. Since $\bar{H} \cong A_5$ is a nonabelian simple group, we have $C_{\bar{N}}(\bar{H}) \cap \bar{H} = 1$. By N/C-theorem, $\bar{N}/C_{\bar{N}}(\bar{H}) = N_{\bar{N}}(\bar{H})/C_{\bar{N}}(\bar{H}) \lesssim \operatorname{Aut}(\bar{H}) = \operatorname{Aut}(A_5) = S_5$.

(a) Assume that $C_{\bar{N}}(\bar{H}) = 1$. Since $\bar{N} > \bar{H} \cong A_5$, it follows that $\bar{N} \cong S_5$.
(b) Assume that $C_{\bar{N}}(\bar{H}) \neq 1$. It follows that $\bar{N} = C_{\bar{N}}(\bar{H}) \times \bar{H}$. Therefore, $C_{\bar{N}}(\bar{H}) \cong C_{\bar{N}}(\bar{H})\bar{H}/\bar{H} = \bar{N}/\bar{H} \cong \mathbb{Z}_p$. One has $\bar{N} = C_{\bar{N}}(\bar{H}) \times \bar{H} \cong \mathbb{Z}_p \times A_5$.

If $\bar{N} \cong S_5$ or $\mathbb{Z}_p \times A_5$. Then $\bar{N}$ has more than three classes of nonnormal nonnilpotent proper subgroups of the same order. It follows that N has more than three classes of nonnormal nonnilpotent proper subgroups of the same order, a contradiction.

Thus we have $G = H$. It follows that $G \cong A_5$ or $SL_2(5)$. $\square$

Proof of Theorem 1.4.   As a direct corollary of Lemma 2.1, we have $G \cong A_5$. $\square$

3.   Proof of Theorem 1.5

Proof.   Let K be a minimal nonsolvable subgroup of G. One has that $K/\Phi(K)$ is a minimal nonabelian simple group.

Assume that $\Phi(K)$ has non-prime-power order. Then for every proper subgroup H of K such that $H > \Phi(K)$, H is a nonnormal proper subgroup of K of non-prime-power order. Since any nonsolvable group has at least seven classes of nontrivial subgroups of the same order by Theorem 1.6, it follows that K has at least seven classes of nonnormal proper subgroups of the same non-prime-power order, a contradiction.

Thus $\Phi(K)$ has prime-power order. It is easy to see that K has no normal proper subgroups of non-prime-power order. By the hypothesis and Theorem 1.1, we have that K has exactly three or four classes of proper subgroups of the same non-prime-power order.

Assume that K has exactly three classes of proper subgroups of the same non-prime-power order. By Theorem 1.1, $K \cong A_5$. Since G has exactly four classes of nonnormal proper subgroups of the same non-prime-power order and A_5 has exactly three classes of nonnormal proper subgroups of the same non-prime-power order, one has $K < G$. Let M be a subgroup of G such that K is a maximal subgroup of M. If $K \trianglelefteq M$, one has $M \cong S_5$ or $A_5 \times \mathbb{Z}_p$ for some prime p. It is easy to see that both S_5 and $A_5 \times \mathbb{Z}_p$ have more than four classes of nonnormal proper subgroups of the same non-prime-power order, it follows that M has more than four classes of nonnormal proper subgroups of the same non-prime-power order, a contradiction. Next assume that K is

not normal in M. If M has no normal proper subgroups of non-prime-power order, then M has exactly four classes of proper subgroups of the same non-prime-power order. By Theorem 1.2, one has $M \cong PSL(2,7)$ or $PSL(2,8)$. Obviously, A_5 cannot be a maximal subgroup of $PSL(2,7)$ or $PSL(2,8)$, a contradiction. If M has a normal proper subgroup of non-prime-power order, say T. Obviously, $T \cap K = 1$. It follows that $M = T \rtimes K \cong T \rtimes A_5$. For every nontrivial subgroup A of K, $T \rtimes A$ is a nonnormal proper subgroup of M of non-prime-power order. It follows that M has more than four classes of nonnormal proper subgroups of the same non-prime-power order, also a contradiction.

Thus K has exactly four classes of proper subgroups of the same non-prime-power order. By Theorem 1.2, one has $K \cong PSL_2(7)$ or $PSL_2(8)$.

We claim that $G = K$.

Otherwise, assume that $K < G$. Let N be a subgroup of G such that K is a maximal subgroup of N. By the hypothesis, one must have $K \trianglelefteq N$. Obviously, $C_N(K) \cap K \trianglelefteq K$. Since $K \cong PSL_2(7)$ or $PSL_2(8)$ is a nonabelian simple group, one has $C_N(K) \cap K = 1$. By N/C-theorem, $N/C_N(K) = N_N(K)/C_N(K) \lesssim \mathrm{Aut}(K) = PGL(2,7)$ or $P\Gamma L(2,8)$.

(ⅰ) Assume that $C_N(K) = 1$. Since K is a maximal subgroup of N, it follows that $N \cong PGL(2,7)$ or $P\Gamma L(2,8)$.

(ⅱ) Assume that $C_N(K) \neq 1$. It follows that $N = C_N(K) \times K$. Therefore, $C_N(K) \cong C_N(K)K/K = N/K \cong \mathbb{Z}_p$. One has $N = C_N(K) \times K \cong \mathbb{Z}_p \times PSL_2(7)$ or $\mathbb{Z}_p \times PSL_2(8)$.

Since $PGL(2,7)$, $P\Gamma L(2,8)$, $\mathbb{Z}_p \times PSL_2(7)$ and $\mathbb{Z}_p \times PSL_2(8)$ have more than four classes of nonnormal proper subgroups of the same non-prime-power order, it follows that N has more than four classes of nonnormal proper subgroups of the same non-prime-power order, a contradiction.

Thus we have $G = K \cong PSL_2(7)$ or $PSL_2(8)$. $\qquad \square$

4. Proof of Theorem 1.7

Lemma 4.1 *Any group G with at most three classes of nonnormal nontrivial subgroups of the same prime-power order is solvable.*

Proof. Assume that G is nonsolvable. Let G be a counterexample of minimal order. It follows that G is a minimal nonsolvable group. Then $G/\Phi(G)$ is a minimal nonabelian simple group. Obviously, $G/\Phi(G)$ has at least

four classes of nontrivial subgroups of the same prime-power order. Since $G/\Phi(G)$ is a nonabelian simple group, it follows that G has at least four classes of nonnormal nontrivial subgroups of the same prime-power order, a contradiction. Therefore, the counterexample does not exist and so G is solvable. $\square$

The proof of Theorem 1.7 follows from the following two lemmas.

Lemma 4.2   *Let G be a group.*

(1) *Suppose that G has at most six classes of nonnormal nontrivial subgroups of the same order, then G is solvable.*
(2) *Suppose that G is a nonsolvable group with exactly seven classes of nonnormal nontrivial subgroups of the same order, then $G \cong A_5$.*
(3) *Suppose that G has exactly eight classes of nonnormal nontrivial subgroups of the same order, then G is solvable.*

Proof.   By $\tau_1(G)$ we denote the number of classes of nonnormal nontrivial subgroups of G of the same prime-power order and by $\tau_2(G)$ we denote the number of classes of nonnormal proper subgroups of G of the same non-prime-power order. Then G has exactly $\tau_1(G) + \tau_2(G)$ classes of nonnormal nontrivial subgroups of the same order. If G is nonsolvable, by Lemma 4.1 and Theorem 1.3, one has $\tau_1(G) \geq 4$ and $\tau_2(G) \geq 3$.

(1) Suppose that $\tau_1(G) + \tau_2(G) \leq 6$. One must have either $\tau_1(G) \leq 3$ or $\tau_2(G) \leq 2$. Then G is solvable.

(2) Suppose that G is a nonsolvable group with $\tau_1(G) + \tau_2(G) = 7$. One must have $(\tau_1(G), \tau_2(G)) = (4, 3)$. By Theorem 1.4, $G \cong A_5$.

(3) Suppose that $\tau_1(G) + \tau_2(G) = 8$. Assume that G is nonsolvable. One must have $(\tau_1(G), \tau_2(G)) = (5, 3)$ or $(4, 4)$. If $(\tau_1(G), \tau_2(G)) = (5, 3)$. By Theorem 1.4, $G \cong A_5$. However, $\tau_1(A_5) = 4 \neq 5$, a contradiction. If $(\tau_1(G), \tau_2(G)) = (4, 4)$. By Theorem 1.5, $G \cong PSL_2(7)$ or $PSL_2(8)$. However, $\tau_1(PSL_2(7)) = 5 \neq 4$ and $\tau_1(PSL_2(8)) = 6 \neq 4$, also a contradiction. Thus G is solvable. $\square$

Lemma 4.3   *Suppose that G is a nonsolvable group with exactly nine classes of nonnormal nontrivial subgroups of the same order, then $G \cong PSL_2(7)$ or $SL_2(5)$.*

Proof.   (1) Suppose that G is a nonabelian simple group. By the hypothesis, G has exactly nine classes of nontrivial subgroups of the same order. It

follows that $G \cong PSL_2(7)$ by Theorem 1.6 (4).

(2) Suppose that G is not a simple group. Let $G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_{i-1} \rhd G_i \rhd \cdots \rhd G_{s-1} \rhd G_s = 1$ be a composition series from G to G_s, where G_{i-1}/G_i is a simple group for every $1 \leq i \leq s$. Since G is nonsolvable, there exists some i for $1 \leq i \leq s$ such that G_{i-1}/G_i is a nonabelian simple group. Let j be the smallest number such that G_{j-1}/G_j is a nonabelian simple group. Obviously, G_{j-1}/G_j has no normal nontrivial subgroups. By the hypothesis, G_{j-1}/G_j has at most nine classes of nontrivial subgroups of the same order. Moreover, by Theorem 1.6, G_{j-1}/G_j might have exactly seven or nine classes of nontrivial subgroups of the same order, and then $G_{j-1}/G_j \cong A_5$ or $PSL_2(7)$.

(i) Suppose that $G_{j-1}/G_j \cong PSL_2(7)$.

(a) Suppose that $G_j = 1$. It follows that $G_{j-1} \cong PSL_2(7)$. Since G is not a simple group, one has $j > 1$. By the choice of j, $G_{j-2}/G_{j-1} \cong \mathbb{Z}_p$ for some prime p. Obviously, $C_{G_{j-2}}(G_{j-1}) \cap G_{j-1} \lhd G_{j-1}$. Since G_{j-1} is a nonabelian simple group, one has $C_{G_{j-2}}(G_{j-1}) \cap G_{j-1} = 1$. By N/C-theorem, $G_{j-2}/C_{G_{j-2}}(G_{j-1}) = N_{G_{j-2}}(G_{j-1})/C_{G_{j-2}}(G_{j-1}) \lesssim \mathrm{Aut}(G_{j-1}) = \mathrm{Aut}(PSL_2(7)) = PGL_2(7)$. If $C_{G_{j-2}}(G_{j-1}) = 1$. Since $G_{j-2} > G_{j-1}$, one has $G_{j-2} \cong PGL_2(7)$. If $C_{G_{j-2}}(G_{j-1}) \neq 1$, one has $G_{j-2} = G_{j-1} \times C_{G_{j-2}}(G_{j-1})$. Then $C_{G_{j-2}}(G_{j-1}) \cong G_{j-1}C_{G_{j-2}}(G_{j-1})/G_{j-1} = G_{j-2}/G_{j-1} \cong \mathbb{Z}_p$. It follows that $G_{j-2} \cong PSL_2(7) \times \mathbb{Z}_p$.

Whenever $G_{j-2} \cong PGL_2(7)$ or $PSL_2(7) \times \mathbb{Z}_p$, one has that G_{j-2} has more than nine classes of nonnormal nontrivial subgroups of the same order, a contradiction.

(b) Suppose that $G_j \neq 1$. Assume that G_j has non-prime-power order. Then every proper subgroup H of G_{j-1} satisfying $H > G_j$ is a nonnormal proper subgroup of G_{j-1} of non-prime-power order. It follows that G_{j-1} has at least nine classes of nonnormal proper subgroups of the same non-prime-power order by Theorem 1.6 (4). Moreover, by Lemma 4.1, G_{j-1} has at least four classes of nonnormal nontrivial subgroups of the same prime-power order. It follows that G_{j-1} has at least thirteen classes of nonnormal nontrivial subgroups of the same order, a contradiction. Assume that G_j has prime-power order. It is easy to see that G_{j-1} has more than five classes of nonnormal proper subgroups of the same non-prime-power order. It also follows that G_{j-1} has more than nine classes of nonnormal nontrivial subgroups of the same order, a contradiction.

(ii) Suppose that $G_{j-1}/G_j \cong A_5$.

(c) Suppose that $G_j = 1$. Then $G_{j-1} \cong A_5$. Also, since G is not a simple group, one has $j > 1$. By the choice of j, $G_{j-2}/G_{j-1} \cong \mathbb{Z}_p$ for some prime p. Arguing as in part (i) (a), we can get that $G_{j-2} \cong S_5$ or $A_5 \times \mathbb{Z}_p$. However, both S_5 and $A_5 \times \mathbb{Z}_p$ have more than nine classes of nonnormal nontrivial subgroups of the same order, which implies that G_{j-2} has more than nine classes of nonnormal nontrivial subgroups of the same order, a contradiction.

(d) Suppose that $G_j \neq 1$. If G_j has non-prime-power order, then every proper subgroup K of G_{j-1} satisfying $K > G_j$ is a nonnormal proper subgroup of G_{j-1} of non-prime-power order. It follows that G_{j-1} has at least seven classes of nonnormal proper subgroups of the same non-prime-power order by Theorem 1.6. Moreover, by Lemma 4.1, G_{j-1} has at least four classes of nonnormal nontrivial subgroups of the same prime-power order. It follows that G_{j-1} has at least eleven classes of nonnormal nontrivial subgroups of the same order, a contradiction. Next assume that G_j has prime-power order. It is easy to see that G_{j-1} has at least six classes of nonnormal proper subgroups of the same non-prime-power order if $|G_j|$ is a p-power for $p \geq 3$. Since G_{j-1} must have at least four classes of nonnormal nontrivial subgroups of the same prime-power order, it follows that G_{j-1} has at least ten classes of nonnormal nontrivial subgroups of the same order, a contradiction. Therefore, $|G_j|$ is a 2-power. Assume that $|G_j| = 2^n$, where $n \geq 1$.

(d$_1$) Suppose that $n \geq 2$. If G_j has a nontrivial subgroup G_t such that $G_t \trianglelefteq G_{j-1}$, it is easy to see that G_{j-1} has at least seven classes of nonnormal proper subgroups of the same non-prime-power order, which implies that G_{j-1} has least eleven classes of nonnormal nontrivial subgroups of the same order, a contradiction. If G_j has no nontrivial normal subgroups of G_{j-1}, it is easy to see that G_{j-1} has at least five classes of nonnormal proper subgroups of the same non-prime-power order. Moreover, arguing as in proof of Lemma 4.1, we can get that G_{j-1} has at least five classes of nonnormal nontrivial subgroups of the same prime-power order. It follows that G_{j-1} has at least ten classes of nonnormal nontrivial subgroups of the same order, also a contradiction.

(d$_2$) Suppose that $n = 1$. It follows that G_{j-1} might be isomorphic to $SL_2(5)$ or $A_5 \times \mathbb{Z}_2$. Obviously, $G_{j-1} \not\cong A_5 \times \mathbb{Z}_2$ since $A_5 \times \mathbb{Z}_2$ has more than nine classes of nonnormal nontrivial subgroups of the same order. Therefore, $G_{j-1} \cong SL_2(5)$. If $G > G_{j-1}$, then $j > 1$. By the choice of j, G_{j-2}/G_{j-1}

$\cong$ $\mathbb{Z}_p$ for some prime p. Obviously, $G_{j-2}/G_{j-1} \cong (G_{j-2}/Z(G_{j-1}))/(G_{j-1}/Z(G_{j-1}))$. Since $G_{j-1}/Z(G_{j-1}) \cong A_5$, arguing as in part (i) (a), we can get that $G_{j-2}/Z(G_{j-1}) \cong S_5$ or $A_5 \times \mathbb{Z}_p$, which implies that G_{j-2} has more than nine classes of nonnormal nontrivial subgroups of the same order, a contradiction. Therefore, $G = G_{j-1} \cong SL_2(5)$. $\qquad\square$

Lemmas 4.2 and 4.3 combined together give Theorem 1.7. $\qquad\square$

References

[1]   Gallian J. A. and Moulton D., *The classification of minimal non-prime-power order groups*. Arch. Math. (Basel), **59** (1992), 521–524.

[2]   Shi J. and Zhang C., *Finite groups with given quantitative non-nilpotent subgroups*. Comm. Algebra, **39** (2011), 3346–3355.

[3]   Shi J. and Zhang C., *Finite groups in which the number of classes of non-nilpotent proper subgroups of the same order is given*. Chin. Ann. Math. (Ser. A), **32** (2011), 687–692 (in Chinese).

[4]   Shi J., Zhang C. and Liang D., *Finite groups with some subgroups of given order*. Algebra Colloq., **20** (2013), 457–462.

[5]   Shi J. and Zhang C., *Finite groups with given quantitative non-nilpotent sub-groups II*. Comm. Algebra, **42** (2014), 4248–4252.

Jiangtao SHI
School of Mathematics and Information Science
Yantai University
Yantai 264005, China
E-mail: shijt@math.pku.edu.cn

Cui ZHANG
1. School of Mathematics and Information Science
   Yantai University, Yantai 264005, China
2. University of Primorska
   FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
E-mail: cuizhang2008@gmail.com

Dengfeng LIANG
Department of Mathematics
Beijing Technology and Business University
Beijing 100048, China
E-mail: dengfengliang@163.com